DEMONSTRATIO MATHEMATICA
Vol. XXXI No 2 1998

Gabriela Kohr

CERTAIN SUFFICIENT CONDITIONS OF INJECTIVITY IN $\mathbf{C}^n$

In this paper we obtain some sufficient conditions of injectivity for continuous mappings defined on some domains in $\mathbf{C}^n$ and we give some interesting consequences and examples.

1. Introduction

Let $\mathbf{C}^n$ denote the space of n complex variables $z = (z_1, \ldots, z_n)'$ with the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w}_j$ and the usual norm $\|z\| = \langle z, z\rangle^{1/2}$. Let I be the identity in the set of all continuous linear operators A from $\mathbf{C}^n$ into $\mathbf{C}^n$ with the standard operator norm $\|A\| = \sup\{\|Aw\| : \|w\| = 1\}$. Also, let B be the Euclidean unit ball in $\mathbf{C}^n$, which in the case $n = 1$ is denoted by U and it is called the unit disc. The symbol $'$ denotes the transpose of vectors and matrices and if $A = [a_{ij}]_{1\le i,j\le n}$, $a_{ij} \in \mathbf{C}$, then we denote by $\overline{A}$ the conjugate of matrix A, i.e. $\overline{A} = [\overline{a}_{ij}]_{1\le i,j\le n}$. On the other hand, if $z = (z_1, \ldots, z_n)' \in \mathbf{C}^n$, then $\overline{z} = (\overline{z}_1, \ldots, \overline{z}_n)'$.

If D is a domain in $\mathbf{C}^n$ and $f = (f_1, \ldots, f_n) : D \to \mathbf{C}^n$, then we say that $f \in C^1(D)$ is for each $j, k = 1, \ldots, n$, the functions $u_j = \mathrm{Re} f_j$, $v_j = \mathrm{Im} f_j$, have all first order partial derivatives with respect to the real variables $x_k = \mathrm{Re} z_k$, $y_k = \mathrm{Im} z_k$ and they are continuous on D. It is well known that if $f \in C^1(D)$ and

$$J_r f = \det \frac{\partial(u_1, v_1, \ldots, u_n, v_n)}{\partial(x_1, y_1, \ldots, x_n, y_n)} \neq 0,$$

where $u_j = \mathrm{Re} f_j$, $v_j = \mathrm{Im} f_j$, $j = 1, \ldots, n$, then f is a locally diffeomorphism of C^1 class on D. Also, if f is univalent (injective) on D and $J_r f(z) > 0$, $z \in D$, then f is a diffeomorphism from D onto $f(D)$.

1991 *Mathematics Subject Classification*: 32H, 30C45, 32H99.

For $f \in C^1(D)$, let us

$$D_z f(u) = \left[\frac{\partial f_j}{\partial z_k}(u)\right]_{1\le j,k\le n} \quad \text{and} \quad D_{\bar z} f(u) = \left[\frac{\partial f_j}{\partial \bar z_k}(u)\right]_{1\le j,k\le n},$$

for all $u \in D$. If $f \in H(D)$, where $H(D)$ means the set of holomorphic mappings from D into $\mathbf{C}^n$, then $D_{\bar z}f(u) = 0$, for all $u \in D$ and $D_z f(u)$ is the Fréchet derivative of f at u, denoted by $Df(u)$. Also, if $f \in H(D)$, we say that f is biholomorphic on D, if its inverse f^{-1} does exist, is holomorphic on a domain Ω and $f^{-1}(\Omega) = G$.

P.T. Mocanu [Mo1] gave several sufficient conditions of univalence for functions of C^1 class defined on some domains in $\mathbf{C}$ and very recently M. Cristea ([Cr1], [Cr2]) obtained other general topological conditions of injectivity. These conditions generalized some known criteria of univalence due to Alexander [Al], Noshiro [No], Warshawski [Wa], Ozaki [Oz], Kaplan [Ka] and Wolf [Wo]. An extension of Kaplan's results was given by T.J. Suffridge [Su1], considering holomorphic mappings defined on some convex domains in $\mathbf{C}^n$ and in a Banach space.

Very recently M. Kohr-Ile, G. Kohr [Ko-I-Ko] obtained other extensions of Alexander-Noshiro-Warshawski and Kaplan's results for functions of C^1 class defined on some convex domains on the complex plane. Note that, these criteria of univalence are extensions of some new results due to E. Janiec [Ja], A. Lecko and J. Stankiewicz [Le-St]. Also, some sufficient conditions of univalence for mappings of C^1 class, defined on some domains in $\mathbf{C}^n$, were obtained by the authors (see for example [Ko1], [Ko2], [Ko-Li]).

On the other hand, recently P.T. Mocanu [Mo2] obtained some very nice and interesting conditions of univalence for functions of C^1 class defined on some domains in $\mathbf{C}$ and also, M. Cristea [Cr3] using a topological reason, extended these results for functions which are only continuous on $\mathbf{C}$.

In this paper, we are going to present other conditions of injectivity for mappings which are continuous on some domains (not necessarily convex) in $\mathbf{C}^n$. Note that, some of our results (for example Theorems 2.1 and 2.2), could be easy extended on a Hilbert space.

For our purpose, let us the following definition.

DEFINITION 1.1. Let D be a domain in $\mathbf{C}^n$ and let $\varphi \in [0, \pi)$. We say that D is φ-angular convex if it is possible to join each pair of distinct points $z_1, z_2 \in D$ by a pair of straight line segments $[z_1, z_3]$ and $[z_3, z_2]$ lying in D such that

$$|\arg\langle z_2 - z_3, z_3 - z_1\rangle| \le \varphi. \tag{1.1}$$

Obviously, each zero-angular convex domain in $\mathbf{C}^n$ is convex.

For $0 < \varphi \le \pi$, we denote by C_φ the cone $\{z \in \mathbf{C} : |\arg z| < \varphi\}$.

If $D \subset \mathbf{C}^n$ is open, $\lambda \in \mathbf{C}^n$, $\|\lambda\| = 1$, $f : D \to \mathbf{C}^n$ is a mapping and $z \in D$, we set

$$M(f,z;\lambda) = \Big\{ w \in \mathbf{C}_\infty : \exists\, (t_p) \subset \mathbf{R}, t_p \to 0,\ \frac{1}{t_p}\langle f(z+t_p\lambda) - f(z), \lambda\rangle \to w,\ p \to \infty \Big\}.$$

Note that, if $f \in C^1(D)$, then

$$M(f,z;\lambda) = \{\langle [D_z f(z)]\lambda + [D_{\bar z} f(z)]\overline{\lambda}, \lambda\rangle\}$$

and if $f \in H(D)$, then, of course, $M(f,z;\lambda) = \{\langle Df(z)\lambda, \lambda\rangle\}$.

2. Main results

THEOREM 2.1. *Let D be a φ-angular convex domain in $\mathbf{C}^n$, where $0 \le \varphi < \pi$ and let $f : D \to \mathbf{C}^n$ be a continuous mapping, such that $M(f,z;\lambda)$ is compact, for all $z \in D$ and $\lambda \in \mathbf{C}^n$, $\|\lambda\| = 1$.*

If

$$|\arg w| < \frac{\pi - \varphi}{2}, \quad w \in M(f,z;\lambda),\ z \in D,\ \lambda \in \mathbf{C}^n,\ \|\lambda\| = 1,$$

then f is injective on D.

Proof. Let $z_1, z_2 \in D$, with $z_1 \neq z_2$. Since D is φ-angular convex, there exists $z_3 \in D$ such that $z_1 \neq z_3$, $z_2 \neq z_3$, $[z_1,z_3] \cup [z_2,z_3] \subset D$ and $|\arg\langle z_2 - z_3, z_3 - z_1\rangle| \le \varphi$.

Let $\psi = \frac{\pi-\varphi}{2}$ and $\lambda = \frac{z_1 - z_3}{\|z_1 - z_3\|}$. Also, let $q : [0,1] \to \mathbf{C}^n$, given by $q(t) = (1-t)z_1 + tz_3$, for all $t \in [0,1]$, then $q(t) \in D$, $t \in [0,1]$.

If $t \in [0,1]$, using the fact that $M(f, q(t); \lambda)$ is a compact set included in the open set C_ψ, then we can find an $\varepsilon_t > 0$ such that

$$\langle f(q(t)) - f(q(s)), q(t) - q(s)\rangle \in C_\psi, \quad \text{for} \quad |t-s| < \varepsilon_t.$$

On the other hand, since $[0,1]$ is compact, we can find $0 = t_0 < t_1 < \ldots < t_p = 1$ such that there exists $\alpha_i \in (t_{i-1} - \varepsilon_{t_{i-1}}, t_{i-1} + \varepsilon_{t_{i-1}}) \cap (t_i - \varepsilon_{t_i}, t_i + \varepsilon_{t_i})$ for $i = 1,\ldots,p$. We let $s_0 = t_0$, $s_1 = \alpha_1$, $s_2 = t_1$, $s_3 = \alpha_2, \ldots, s_{2k} = t_k$ for $k = 0,1,\ldots,p$, $s_{2k+1} = \alpha_{k+1}$, for $k = 0,1,\ldots,p-1$ and we denote by

$$Z_k = \left\langle f(q(s_{k+1})) - f(q(s_k)), \frac{q(s_{k+1}) - q(s_k)}{\|q(s_{k+1}) - q(s_k)\|^2} \right\rangle,$$

for $k = 0,1,\ldots,2p-1$.

It is clear that $q(s_{k+1}) - q(s_k) = (s_{k+1} - s_k)(z_3 - z_1)$ and also,

$$|\arg Z_k| = |\arg\langle f(q(s_{k+1})) - f(q(s_k)), q(s_{k+1}) - q(s_k)\rangle| < \psi,$$

hence $Z_k \in C_\psi$, for $k = 0, 1, \ldots, 2p-1$. Now, since C_ψ is a convex set and

$$\sum_{k=0}^{2p-1} (s_{k+1} - s_k) = 1,$$

it follows that

$$\sum_{k=0}^{2p-1} (s_{k+1} - s_k) Z_k \in C_\psi,$$

thus

$$\frac{1}{\|z_3 - z_1\|^2} \langle f(z_3) - f(z_1), z_3 - z_1 \rangle = \sum_{k=0}^{2p-1} (s_{k+1} - s_k) Z_k \in C_\psi,$$

i.e.

$$|\arg\langle f(z_3) - f(z_1), z_3 - z_1 \rangle| < \psi.$$

Therefore, if $A = \langle f(z_3) - f(z_1), z_3 - z_1 \rangle$, then $A \in C_\psi$, so $|\arg A| < \psi$ and $A \neq 0$.

Using same kinds of arguments as before, we deduce that $B \in C_\psi$, where

$$B = \langle f(z_2) - f(z_3), z_2 - z_3 \rangle,$$

hence $|\arg B| < \psi$ and $B \neq 0$.

Note that, the following relation holds:

$$\begin{aligned} \langle f(z_2) - f(z_1), (z_3 - z_1)A \rangle &= \langle f(z_2) - f(z_3), (z_3 - z_1)A \rangle \\ &\quad + \langle f(z_3) - f(z_1), (z_3 - z_1)A \rangle \\ &= |A|^2 \left[1 + \frac{1}{A} \langle f(z_2) - f(z_3), z_3 - z_1 \rangle \right]. \end{aligned}$$

On the other hand, since

$$\begin{aligned} \left| \arg \frac{\langle f(z_2) - f(z_3), z_3 - z_1 \rangle}{A} \right| &\leq |\arg A| + |\arg\langle f(z_2) - f(z_3), z_3 - z_1 \rangle| \\ &\leq |\arg A| + |\arg B| + |\arg\langle z_2 - z_3, z_3 - z_1 \rangle| \\ &< \pi - \varphi + \varphi = \pi, \end{aligned}$$

then,

$$1 + \frac{1}{A} \langle f(z_2) - f(z_3), z_3 - z_1 \rangle \neq 0,$$

therefore,

$$\langle f(z_2) - f(z_1), (z_3 - z_1)A \rangle \neq 0,$$

which implies that $f(z_1) \neq f(z_2)$. This completes the proof.

Note that, in the case $n = 1$ this result was recently obtained by M. Cristea [Cr3].

If $f \in C^1(D)$ in Theorem 2.1, we obtain the following condition of univalence.

COROLLARY 2.1. *Let D be a φ-angular convex domain in $\mathbf{C}^n$, where $0 \le \varphi < \pi$ and let $f \in C^1(D)$.*
Assume that

$$|\arg\langle[D_z f(z)](v) + [D_{\bar{z}} f(z)](\bar{v}), v\rangle| < \frac{\pi - \varphi}{2}, \tag{2.1}$$

for all $z \in D$ and $v \in \mathbf{C}^n$, $\|v\| = 1$, then f is univalent (injective) on D.

COROLLARY 2.2. *Let $0 \le \varphi < \pi$ and $D \subset \mathbf{C}^n$ be a φ-angular convex domain. Let $f \in H(D)$ which satisfies the following relation*

$$|\arg\langle Df(z)(v), v\rangle| < \frac{\pi - \varphi}{2},$$

for all $z \in D$ and $v \in \mathbf{C}^n$, $\|v\| = 1$, then f is univalent on D.

For $n = 1$ in Corollary 2.1, we deduce the following result due to P.T. Mocanu [Mo2].

COROLLARY 2.3 [Mo2]. *Let $D \subset \mathbf{C}$ be a φ-angular convex domain with $\varphi \in [0, \pi[$. If $f \in C^1(D)$ satisfies*

$$|\arg f'_\theta(z)| < \frac{\pi - \varphi}{2}, \tag{2.2}$$

for all $z \in D$ and $\theta \in [0, 2\pi[$, where $f'_\theta(z) = \frac{\partial f}{\partial z} + e^{-2i\theta}\frac{\partial f}{\partial \bar{z}}$, then f is univalent on D.

Proof. Let $z \in D$ and $v \in \mathbf{C}$, $|v| = 1$.
Then

$$\begin{aligned}[D_z f(z)](v) + [D_{\bar{z}} f(z)](\bar{v}) &= \frac{\partial f}{\partial z}(z)v + \frac{\partial f}{\partial \bar{z}}(z)\bar{v} \\ &= v\left[\frac{\partial f}{\partial z}(z) + e^{-2i\arg v}\frac{\partial f}{\partial \bar{z}}(z)\right] = v f'_\theta(z),\end{aligned}$$

where $\theta = \arg v$. If the condition (2.2) holds, then

$$|\arg\langle v f'_\theta(z), z\rangle| = |\arg f'_\theta(z)| < \frac{\pi - \varphi}{2},$$

hence the condition (2.1) holds. Therefore, using the result of Corollary 2.1, we conclude that f is univalent on D.

A direct application of Theorem 2.1 is the following sufficient condition of injectivity.

THEOREM 2.2. *Let $D \subset \mathbf{C}^n$ be a domain, let $f : D \to \mathbf{C}^n$ be a continuous mapping and let $g : D \to \mathbf{C}^n$ be a homeomorphism such that $g(D)$ is φ-angular convex domain in $\mathbf{C}^n$, with $0 \le \varphi < \pi$. If for every $a \in D$ there* *exists* $\varepsilon_a > 0$ *such that*

$$|\arg\langle f(z) - f(a), g(z) - g(a)\rangle| < \frac{\pi - \varphi}{2},$$

if $\|z - a\| < \varepsilon_a$, $z \neq a$, *then* f *is injective on* D.

Proof. Let denote by $\Delta = g(D)$, then Δ is a φ-angular convex domain and if $h(w) = (f \circ g^{-1})(w)$, $w \in \Delta$, then h is continuous on Δ and it satisfies the following relation:

$$|\arg\langle h(w) - h(b), w - b\rangle| < \frac{\pi - \varphi}{2},$$

for all $b \in \Delta$, and $w \in \Delta$ closed to b, $w \neq b$. Using a similar proof as in Theorem 2.1, we deduce that h is injective on Δ, hence f is injective on D, too.

Another consequence of Corollary 2.1 is the following result.

COROLLARY 2.4. *Let* $0 \leq \varphi < \pi$, *let* $D \subset \mathbf{C}^n$ *be a domain, let* $f \in C^1(D)$ *and* $g \in H(D)$, *such that* g *is biholomorphic on* D *and* $g(D)$ *is a* φ*-angular convex domain in* $\mathbf{C}^n$.

If

$$(2.3) \qquad |\arg\langle [D_z f(z)][Dg(z)]^{-1}(v) + [D_{\overline{z}} f(z)][\overline{Dg(z)}]^{-1}(\overline{v}), z\rangle| < \frac{\pi - \varphi}{2},$$

for all $z \in D$ *and* $v \in \mathbf{C}^n$, $\|v\| = 1$, *then* f *is univalent on* D.

Proof. If we denote by $\Delta = g(D)$, then Δ is φ-angular convex domain and if $h(w) = (f \circ g^{-1})(w)$, for all $w \in \Delta$, then $h \in C^1(\Delta)$ and by a straightforward calculation, we obtain

$$D_w h(w) = [D_z f(z)][Dg(z)]^{-1},$$
$$D_{\overline{w}} h(w) = [D_{\overline{z}} f(z)]\overline{[Dg(z)]}^{-1},$$

for all $w \in \Delta$ and $z = g^{-1}(w)$. Now, if the condition (2.3) holds, then the following condition holds too:

$$|\arg\langle [D_w h(w)](\alpha) + [D_{\overline{w}} h(w)](\overline{\alpha}), \alpha\rangle| < \frac{\pi - \varphi}{2},$$

for all $w \in \Delta$ and $\alpha \in \mathbf{C}^n$, $\|\alpha\| = 1$, therefore from Corollary 2.1, we deduce that h is univalent on Δ, so f is univalent on D, too. This completes the proof.

REMARK. Note that, if in Corollary 2.4 we change the assumption that g is biholomorphic, by the following: $g \in C^1(D)$, such that g is injective on D and $J_r g(z) > 0$, $z \in D$, and also if $g(D)$ is φ-angular convex domain, with $0 \leq \varphi < \pi$, then we can obtain another sufficient condition of injectivity for

$f \in C^1(D)$. Note that, if $g(z) \equiv z$, $z \in D$, in Corollary 2.4, then we obtain the result of Corollary 2.1.

Also, if in Corollary 2.4 $g(D)$ is a convex domain, we obtain a result, recently deduced by the author (see [Ko2]).

We finish this paper with some simple examples, which are direct applications of the above results.

EXAMPLE 2.1. Let $0 \leq \varphi < \pi$, let $D \subset \mathbf{C}^n$ be a domain, let $h \in H(D)$ and $g \in H(D)$ such that g is biholomorphic on D and $g(D)$ is a φ-angular convex domain in $\mathbf{C}^n$.

If

$$(2.4) \qquad |\arg\langle [Dh(z)][Dg(z)]^{-1}(v) + \overline{v}, v\rangle| < \frac{\pi - \varphi}{2},$$

for all $z \in D$ and $v \in \mathbf{C}^n$, $\|v\| = 1$, then f is univalent on D, where $f(z) = h(z) + \overline{g(z)}$, $z \in D$.

Proof. Since $g, h \in H(D)$, then $f \in C^1(D)$ and it satisfies the following relations

$$D_z f(z) = Dh(z) \quad \text{and} \quad D_{\overline{z}} f(z) = [\overline{Dg(z)}].$$

Next, it is sufficient to see that the relation (2.4) is equivalent to the following:

$$|\arg\langle [D_z f(z)][Dg(z)]^{-1}(v) + [D_{\overline{z}} f(z)][\overline{Dg(z)}]^{-1}(\overline{v}), v\rangle| < \frac{\pi - \varphi}{2},$$

for all $z \in D$ and $v \in \mathbf{C}^n$, $\|v\| = 1$, therefore, applying the result of Corollary 2.4, we conclude f is univalent on D.

For example, if we consider $h(z) = \lambda g(z)$, where $\lambda \in \mathbf{C}$, then, from Example 2.1, we obtain:

EXAMPLE 2.2. Let $0 \leq \varphi < \pi$, let $D \subset \mathbf{C}^n$ be a domain and let $g \in H(D)$ such that g is biholomorphic on D and $g(D)$ is a φ-angular convex domain in $\mathbf{C}^n$. If $\lambda \in \mathbf{C}$ and if

$$\operatorname{Re}\lambda - \operatorname{tg}\frac{\varphi}{2}|\operatorname{Im}\lambda| > \operatorname{tg}\frac{\varphi}{2} + 1,$$

then f is univalent on D, where $f(z) = \lambda g(z) + \overline{g(z)}$, $z \in D$.

Proof. Let $h(z) = \lambda g(z)$, $z \in D$, then $h \in H(D)$ and $Dh(z) = \lambda Dg(z)$, $z \in D$, hence the relation (2.4) is equivalent to the following

$$|\arg[\langle \overline{v}, v\rangle + \lambda]| < \frac{\pi - \varphi}{2},$$

for all $v \in \mathbf{C}^n$, $\|v\| = 1$.

It is obvious to see that the above inequality is equivalent to the following

$$\text{Re}[\lambda + \langle \overline{v}, v\rangle] > \text{tg}\,\frac{\varphi}{2}|\text{Im}[\lambda + \langle \overline{v}, v\rangle]|, \tag{2.5}$$

for all $v \in \mathbf{C}^n$, $\|v\| = 1$.

Therefore, if

$$\text{Re}\lambda - \text{tg}\,\frac{\varphi}{2}|\text{Im}\lambda| > \text{tg}\,\frac{\varphi}{2} + 1,$$

then the relation (2.5) holds and taking into account Example 2.1, we conclude that f is univalent on D, where $f(z) = \lambda g(z) + \overline{g(z)}$, $z \in D$.

Finally, we construct an example of a univalent mapping on a $\frac{\pi}{2}$- angular convex domain, by starting from some holomorphic functions of one variable, which satisfy the following requirements.

EXAMPLE 2.3. Let D be a $\frac{\pi}{2}$-angular convex domain in $\mathbf{C}^n$ and let $f_1(z_1), \ldots, f_n(z_n)$ be holomorphic functions (of one variable) on $\mathbf{C}$, which satisfy

$$\sum_{j=1}^{n} |v_j|^2 \text{Re} f_j'(z_j) > \left|\sum_{j=1}^{n} |v_j|^2 \text{Im} f_j'(z_j)\right|,$$

for all $z = (z_1, \ldots, z_n)' \in D$ and $v_1, \ldots, v_n \in \mathbf{C}$, $\sum_{j=1}^{n} |v_j|^2 = 1$, then h is univalent on D, where

$$h(z) = (f_1(z_1), \ldots, f_n(z_n))', \quad z = (z_1, \ldots, z_n)' \in D.$$

Proof. It suffices to observe that the assumption from hypothesis is equivalent to the following

$$\text{Re}\langle Dh(z)(v), v\rangle > |\text{Im}\langle Dh(z)(v), v\rangle|, \quad z \in D,\ v \in \mathbf{C}^n,\ \|v\| = 1,$$

i.e.

$$|\arg\langle Dh(z)(v), v\rangle| < \frac{\pi}{4}, \quad z \in D,\ v \in \mathbf{C}^n,\ \|v\| = 1.$$

Therefore, applying the result of Corollary 2.2, we deduce that h is univalent on D.

Finally, we would like to point other excellent papers concerning univalent mappings, especially harmonic univalent mappings in the complex plane. For this aim, see [Cl-Sm], [Dor-Su], [Su2].

References

[Al] W. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Ann. Math. 17 (1915/16), 12–22.

[Cl-Sm] J. Clunie, T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A. I Math. 9 (1984), 3–25.

[Cr1] M. Cristea, *Certain sufficient conditions for univalency*, Studii şi Cerc. Matem. (Mathematical Report) 44, 1 (1992), 37–42.

[Cr2] M. Cristea, *Certain sufficient conditions for global univalence*, Mathematica (Cluj), 36(59), 2(1994), 137–144.

[Cr3] M. Cristea, *A generalization of a theorem of P.T. Mocanu*, (to appear).

[Dor-Su] M. Dorff, T. J. Suffridge, *The inner mapping radius of harmonic mappings on the disc*, Complex Variables, 33, 1-4(1997), 97–104.

[Ja] E. Janiec, *Some sufficient conditions for univalence of holomorphic functions*, Demonstratio Math. 22, 3 (1989), 717–727.

[Ka] W. Kaplan, *Close-to-convex schlicht functions*, Michigan Math. J. 1 (1952), 169–185.

[Ko1] G. Kohr, M. Kohr, *Some sufficient criteria of univalency in* $\mathbf{C}^n$, Zeszyty Nauk. Politech. Rzeszow, Matematyka 18 (1995), 65–70.

[Ko2] G. Kohr, *Certain sufficient conditions of univalency for complex mappings in the class* C^1 *on* $\mathbf{C}^n$, Studii şi Cerc. Matem. 48, 5-6(1996), 349–356.

[Ko-Li] G. Kohr, P. Liczberski, *On some criteria of injectivity for complex mappings in the class* C^1, Complex Variables Theory Appl. 32 (1997), 217–223.

[Ko-I-Ko] M. Kohr-Ile, G. Kohr, *Some sufficient conditions of univalency for complex functions in the class* C^1, Mathematica (Cluj), 37(60), 1-2(1995), 123–129.

[Le-St] A. Lecko, J. Stankiewicz, *Some sufficient conditions for univalence*, Zeszyty Nauk. Politech. Rzeszow, Matematyka 14, 101(1992), 33–38.

[Mo1] P. T. Mocanu, *Sufficient conditions of univalency for complex functions in the class* C^1, Mathematica, Rev. Anal. Numér. Théor. Approx. 10, 1(1981), 75–79.

[Mo2] P. T. Mocanu, *A sufficient condition for injectivity in the complex plane*, P.U.M.A., 6, 2, (1995), 231–238.

[No] K. Noshiro, *On the theory of schlicht functions*, J. Fac. Sci. Hokaido Imp. Univ., 2(1934), 124–155.

[Oz] S. Ozaki, *On the theory of multivalent functions*, Sci. Rep. Tokyo Bunrika Daigaku A, 40(1935), 167–188.

[Su1] T. J. Suffridge, *Starlikeness, convexity and other geometric properties of holomorphic mappings in higher dimensions*, Lecture Notes in Math. 599 (1976), 146–159.

[Su2] T. J. Suffridge, *Harmonic univalent polynomials*, Complex Variables (to appear).

[Wa] S. E. Warshawski, *On higher derivatives on the boundary in conformal mappings*, Trans. Amer. Math. Soc. 38 (1935), 310–340.

[Wo] I. Wolf, *L'integrale d'une fonction holomorphe et a partie réelle positive dans un semi-plane est univalente*, C.R. Acad. Sci., Paris 198: 13 (1934), 1209–1210.

FACULTY OF MATHEMATICS
BABEŞ-BOLYAI UNIVERSITY
1 M. Kogălniceanu Str.
3400 CLUJ-NAPOCA, ROMANIA
E-mail: gkohr@math.ubbcluj.ro

Received May 12, 1997.