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φ-FUNCTIONS OF N VARIABLES

In this note, Young's inequality $uv \le \varphi(u) + \varphi^*(v)$, where φ and φ^* are convex φ-functions complementary in the sense of Young, is generalized to the case when φ is a function of N variables. Some inequalities given in [1] and [3] are extended to the case of φ-functions of N variables.

Let $N > 1$ be a natural number and $R_+ = [0, \infty)$. Let φ be a real function on R_+^N denoted by

$$\varphi_{u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_N}(u_i) = \varphi(u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_N),$$

for fixed $u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N \in R_+$.

φ is called a φ-function if it satisfies the following conditions:

1. $\varphi(0, \ldots, 0) = 0$,
2. for every $i \in \{1, \ldots, N\}$ and fixed $u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N$ the function $\varphi_{u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_N}(u_i)$ is non-decreasing, continuous and

$$\lim_{u_i \to \infty} \varphi_{u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_N}(u_i) = \infty.$$

Let us denote

$$(1) \quad \varphi^*(u_1, \ldots, u_N) = \sup_{x_1,\ldots,x_N \ge 0} \{u_1x_1 + \ldots + u_Nx_N - \varphi(x_1, \ldots, x_N)\}.$$

One can easily see that the function φ^* is a convex function, i.e., it satisfies the condition $\varphi^*(\lambda u + (1 - \lambda)v) \le \lambda\varphi^*(u) + (1 - \lambda)\varphi^*(v)$ for every real λ, $0 \le \lambda \le 1$, and arbitrary $u = (u_1, \ldots, u_N)$, $v = (v_1, \ldots, v_N)$ from R_+^N. It follows from convexity of φ^* that it is continuous. Without difficulty one can show that the function φ^* is a φ-function. For instance, we prove that for arbitrary fixed $u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N \ge 0$ we have $\lim_{u_i \to \infty} \varphi^*_{u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_N}(u_i) = \infty$.

In what follows let $u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N \ge 0$ and the real $M > 0$ be given. We choose arbitrary $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_N \ge 0$ and we take $v_i > 0$

such that

$$u_1v_1 + \ldots + u_{i-1}v_{i-1} + u_{i+1}v_{i+1} + \ldots + u_Nv_N \leq \varphi(v_1, \ldots, v_N).$$

Let us denote

$$m = [M+\varphi(v_1,\ldots,v_N)-(u_1v_1+\ldots+u_{i-1}v_{i-1}+u_{i+1}v_{i+1}+\ldots+u_Nv_N)]\frac{1}{v_i}.$$

Then for $u_i \geq m$ we get

$$\begin{aligned}
\varphi^*_{u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_N}(u_i)& \\
&\geq \varphi^*_{u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_N}(m) \\
&= \sup_{x_1,\ldots,x_N\geq 0}\{u_1x_1 + \ldots + u_{i-1}x_{i-1} + mx_i \\
&\quad + u_{i+1}x_{i+1} + \ldots + u_Nx_N - \varphi(x_1,\ldots,x_N)\} \\
&\geq u_1v_1 + \ldots + u_{i-1}v_{i-1} + mv_i + u_{i+1}v_{i+1} + \ldots \\
&\quad + u_Nv_N - \varphi(v_1,\ldots,v_N) = M.
\end{aligned}$$

From (1) it follows the Young's inequality

$$u_1v_1 + \ldots + u_Nv_N \leq \varphi(u_1,\ldots,u_N) + \varphi^*(v_1,\ldots,v_N)$$

which is true for arbitrary $u_1,\ldots,u_N,v_1,\ldots,v_N \geq 0$.

For a φ-function φ let us denote $\varphi_i(u_i) = \varphi(0,\ldots,0,u_i,0,\ldots,0)$. Assume that for $i = 1,\ldots,N$ the functions φ_i have derivatives φ_i' continuous, strictly increasing and such that $\varphi_i'(0) = 0$ and $\lim_{u_i\to\infty}\varphi_i'(u_i) = \infty$. Let p_i^* denote the inverse functions to φ_i'. Because the functions

$$\varphi_i(u_i) = \int_0^{u_i}\varphi_i'(t)\,dt \quad \text{and} \quad \int_0^{u_i}p_i^*(t)dt$$

are complementary in the sense of Young, then from [4] (Theorem 13.6) it follows

$$\begin{aligned}
\int_0^{u_i}p_i^*(t)\,dt &= \sup_{x_i\geq 0}\{u_ix_i - \varphi_i(x_i)\} \\
&= \sup_{x_i\geq 0}[\sup_{x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_N\geq 0}\{u_ix_i - \varphi(0,\ldots,0,x_i,0,\ldots,0) \\
&\quad + \varphi(0,\ldots,0,x_i,0,\ldots,0) - \varphi(x_1,\ldots,x_N)\}] \\
&= \sup_{x_1,\ldots,x_N\geq 0}\{0\cdot x_1 + \ldots + 0\cdot x_{i-1} \\
&\quad + u_ix_i + 0\cdot x_{i+1} + \ldots + 0\cdot x_N - \varphi(x_1,\ldots,x_N)\} \\
&= \varphi^*(0,\ldots,0,u_i,0,\ldots,0).
\end{aligned}$$

Therefore for every $i = 1, \dots, N$ we have

$$\varphi^*(0, \dots, 0, u_i, 0, \dots, 0) = \int_0^{u_i} p_i^*(t)\, dt.$$

From (1) it follows that

$$\varphi(u_1, \dots, u_N) \geq \sup_{x_1, \dots, x_N \geq 0} \{u_1 x_1 + \dots + u_N x_N - \varphi^*(x_1, \dots, x_N)\}.$$

We will show that

$$(2) \quad \varphi(u_1, \dots, u_N) = \sup_{x_1, \dots, x_N \geq 0} \{u_1 x_1 + \dots + u_N x_N - \varphi^*(x_1, \dots, x_N)\}.$$

For this let us suppose, that there exist $u_1, \dots, u_N \geq 0$ such that

$$(3) \quad \varphi(u_1, \dots, u_N) > \sup_{x_1, \dots, x_N \geq 0} \{u_1 x_1 + \dots + u_N x_N - \varphi^*(x_1, \dots, x_N)\}.$$

Since the functions p_i^* are increasing and $\lim_{t \to \infty} p_i^*(t) = \infty$, $i = 1, \dots, N$, so for given $u_1, \dots, u_N \geq 0$ there exist $y_1, \dots, y_N \geq 0$ such that we have

$$\int_0^{y_i} [u_1 + \dots + u_N - p_i^*(t)] dt \leq 0 \quad \text{and} \quad u_1 + \dots + u_N - p_i^*(t) \leq 0, i = 1, \dots, N,$$

for $t \geq y_i$. Let $R = \sqrt{N} \max\{y_1, \dots, y_N\}$. We shall prove that for arbitrary $x_1, \dots, x_N$ satisfying $x_1^2 + \dots + x_N^2 > R^2$ we have

$$f(x_1, \dots, x_N) = u_1 x_1 + \dots + u_N x_N - \varphi^*(x_1, \dots, x_N) \leq 0.$$

In what follows let be $x_0 = \max\{x_1, \dots, x_N\}$. Then $x_0 = x_i$ for any x_i, $i = 1, \dots, N$, and $x_i > \frac{R}{\sqrt{N}} \geq y_i$. Hence

$$\begin{aligned}
f(x_1, \dots, x_N) &= u_1 x_1 + \dots + u_N x_N - \varphi^*(x_1, \dots, x_N) \\
&\leq u_1 x_i + \dots + u_N x_i - \varphi^*(0, \dots, 0, x_i, 0, \dots, 0) \\
&= \int_0^{x_i} [u_1 + \dots + u_N - p_i^*(t)]\, dt \\
&= \int_0^{y_i} [u_1 + \dots + u_N - p_i^*(t)]\, dt + \int_{y_i}^{x_i} [u_1 + \dots + u_N - p_i^*(t)]\, dt \\
&\leq 0.
\end{aligned}$$

Therefore for every $x_1, \dots, x_N$ satisfying $x_1^2 + \dots + x_N^2 > R^2$ the function f is negative, what implies that f can be nonnegative only in the compact set $\{(x_1, \dots, x_N) \in R_+^N : x_1^2 + \dots + x_N^2 \leq R^2\}$. Since φ^* is continuous, also the function

$$f(x_1, \dots, x_N) = u_1 x_1 + \dots + u_N x_N - \varphi^*(x_1, \dots, x_N)$$

is continuous. Therefore, there exist $v_1, \ldots, v_N \geq 0$ such that

$$\sup_{x_1,\ldots,x_N \geq 0} \{f(x_1, \ldots, x_N)\} = u_1 v_1 + \ldots + u_N v_N - \varphi^*(v_1, \ldots, v_N).$$

From this and (3) we get

$$\begin{aligned}\varphi(u_1, \ldots, u_N) > & \sup_{x_1,\ldots,x_N \geq 0} \{u_1 x_1 + \ldots + u_N x_N - \varphi^*(x_1, \ldots, x_N)\} \\ &= u_1 v_1 + \ldots + u_N v_N - \varphi^*(v_1, \ldots, v_N)\end{aligned}$$

and

$$\varphi^*(v_1, \ldots, v_N) > u_1 v_1 + \ldots + u_N v_N - \varphi(u_1, \ldots, u_N).$$

On the second hand from (1) it follows

$$\varphi^*(v_1, \ldots, v_N) \leq u_1 v_1 + \ldots + u_N v_N - \varphi(u_1, \ldots, u_N)$$

which gives a contradiction.

The φ-functions φ and φ^* satisfying (1) and (2) are called complementary in the sense of Young, analogously to φ-functions of one and two variables.

EXAMPLE 1. Let $\varphi(u_1, \ldots, u_N) = \frac{1}{p}(u_1^p + \ldots + u_N^p)$, with $p > 1$. Then

$$\varphi^*(u_1, \ldots, u_N) = \frac{1}{q}(u_1^q + \ldots + u_N^q) \quad \text{with } \frac{1}{p} + \frac{1}{q} = 1.$$

EXAMPLE 2. Let $\varphi(u_1, \ldots, u_N) = \frac{1}{p}(u_1^2 + \ldots + u_N^2)^{\frac{p}{2}}$, with $p > 1$. Then

$$\varphi^*(u_1, \ldots, u_N) = \frac{1}{q}(u_1^2 + \ldots + u_N^2)^{\frac{q}{2}} \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1.$$

THEOREM 1. *Let φ denote a φ-function of class C^1. If for every $i \in \{1, \ldots, N\}$ and fixed $u_1, \ldots, u_{i-1},\ u_{i+1}, \ldots, u_N \in R_+$ the function $\varphi_{u_1,\ldots,u_{i-1},\ u_{i+1},\ldots,u_N}(u_i)$ is convex, then*

$$\sum_{i=1}^{N} u_i \partial_i \varphi(u_1, \ldots, u_N) \leq N \varphi(2u_1, \ldots, 2u_N),$$

($\partial_i \varphi$ denotes partial derivative of φ with respect to u_i).

Proof. For every $i \in \{1, \ldots, N\}$ and for fixed $u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_N \in R_+$ the function $\partial_i \varphi(u_1, \ldots, u_N)$ is non-decreasing with respect to u_i because the function $\varphi_{u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_N}(u_i)$ is convex. We can calculate

$$\begin{aligned}&\sum_{i=1}^{N} u_i \partial_i \varphi(u_1, \ldots, u_N) \\ &\qquad \leq \sum_{i=1}^{N} \int_{u_i}^{2u_i} \partial_i \varphi(u_1, \ldots, u_{i-1}, t, u_{i+1}, \ldots, u_N)\, dt\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i=1}^{N} \int_{0}^{2u_i} \partial_i \varphi(u_1, \ldots, u_{i-1}, t, u_{i+1}, \ldots, u_N)\, dt \\
&= \sum_{i=1}^{N} [\varphi_{u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_N}(2u_i) - \varphi_{u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_N}(0)] \\
&\le N\varphi(2u_1, \ldots, 2u_N).
\end{aligned}
$$

THEOREM 2. *Let φ denote a φ-function of class C^2 such that the differential of second order $d^2\varphi$ is nonnegative. Let for every $u_1, \ldots, u_{i-1}, u_{i+1}, \ldots$ $\ldots, u_N \in R_+$ the function $\varphi_{u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_N}(u_i)$ be convex and let $\varphi_i'(0) = 0$ for every $i \in \{1, \ldots, N\}$. Then for every differentiable functions $f_1, \ldots, f_N$ from R into R we have*

$$
\frac{d}{dt}\varphi(|f_1(t)|, \ldots, |f_N(t)|) \le \varphi(|f_1'(t)|, \ldots, |f_N'(t)|) + N\varphi(2|f_1(t)|, \ldots, 2|f_N(t)|).
$$

Proof. Using Taylor's formula and the assumption $d^2\varphi \ge 0$, we get

$$
\begin{aligned}
&\varphi(x_1, \ldots, x_N) \\
&\quad = \varphi(u_1, \ldots, u_N) + \sum_{i=1}^{N} \partial_i \varphi(u_1, \ldots, u_N)(x_i - u_i) \\
&\qquad + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \partial_{i,j}^2 \varphi(\xi_1, \ldots, \xi_N)(x_i - u_i)(x_j - u_j) \\
&\quad \ge \sum_{i=1}^{N} \partial_i \varphi(u_1, \ldots, u_N)(x_i - u_i).
\end{aligned}
$$

Hence $\varphi(x_1, \ldots, x_N) - \sum_{i=1}^{N} \partial_i \varphi(u_1, \ldots, u_N)(x_i - u_i) \ge 0$. Using Theorem 1, we get

$$
\begin{aligned}
&\sum_{i=1}^{N} \partial_i \varphi(u_1, \ldots, u_N) x_i - \varphi(x_1, \ldots, x_N) \\
&\qquad \le \sum_{i=1}^{N} \partial_i \varphi(u_1, \ldots, u_N) x_i - \varphi(x_1, \ldots, x_N) \\
&\qquad\quad + \Big[\varphi(x_1, \ldots, x_N) - \sum_{i=1}^{N} \partial_i \varphi(u_1, \ldots, u_N)(x_i - u_i)\Big] \\
&\qquad = \sum_{i=1}^{N} \partial_i \varphi(u_1, \ldots, u_N) u_i \\
&\qquad \le N\varphi(2u_1, \ldots, 2u_N).
\end{aligned}
$$

From this

$$\begin{aligned}&\varphi^*(\partial_1\varphi(u_1,\dots,u_N),\dots,\partial_N\varphi(u_1,\dots,u_N))\\ &= \sup_{x_1,\dots,x_N\geq 0}\{\sum_{i=1}^{N}\partial_i\varphi(u_1,\dots,u_N)x_i-\varphi(x_1,\dots,x_N)\}\leq N\varphi(2u_1,\dots,2u_N).\end{aligned}$$

We can extend the domain of φ-function φ taking $\varphi(\varepsilon_1 u_1,\dots,\varepsilon_n u_N) = \varphi(u_1,\dots,u_N)$, where $\varepsilon_i=\pm 1$. From the Young inequality it follows

$$\begin{aligned}&\frac{d}{dt}\varphi(|f_1(t)|,\dots,|f_N(t)|)\\ &= \frac{d}{dt}\varphi(f_1(t),\dots,f_N(t))\\ &= \sum_{i=1}^{N}\partial_i\varphi(f_1(t),\dots,f_N(t))f_i'(t)\\ &\leq \sum_{i=1}^{N}\partial_i\varphi(|f_1(t)|,\dots,|f_N(t)|)|f_i'(t)|\\ &\leq \varphi(|f_1'(t)|,\dots,|f_N'(t)|)\\ &\quad+\varphi^*(\partial_1\varphi(|f_1(t)|,\dots,|f_N(t)|),\dots,\partial_N\varphi(|f_1(t)|,\dots,|f_N(t)|))\\ &\leq \varphi(|f_1'(t)|,\dots,|f_N'(t)|)+N\varphi(2|f_1(t)|,\dots,2|f_N(t)|).\end{aligned}$$

In the sequel let us take the notation

$$A=\{\alpha=(\alpha_1,\dots,\alpha_n):\alpha_i=0 \text{ or } \alpha_i=1; i=1,\dots,n\}.$$

The inequality $|t|\geq|s|$ for $t=(t_1,\dots,t_n)$ and $s=(s_1,\dots,s_n)$ means that $|t_i|\geq|s_i|$ for $i=1,\dots,n$. For given $r>0$ we denote the sets

$$\begin{aligned}&J_k=\{t_k: s_k\leq t_k\leq s_k+r \text{ if } s_k\geq 0; s_k-r\leq t_k\leq s_k \text{ if } s_k<0\},\\ &C(s,r)=J_1\times J_2\times\dots\times J_n.\end{aligned}$$

THEOREM 3. *Let the φ-function φ satisfy the assumption of Theorem* 2 *and let Ω be an open set in R^n such that for any $r>0$ and every $s\in\overline{\Omega}$ at least one set $C(s,r)$ is included in $\overline{\Omega}$. If functions $f_1,\dots,f_N:\overline{\Omega}\to R$ have continuous derivatives $D^\alpha f_i$ $i=1,\dots,N$ for $\alpha\in A$, then for every $s\in\overline{\Omega}$ we have*

$$\begin{aligned}&\varphi(|f_1(s)|,\dots,|f_N(s)|)\\ &\quad\leq\sum_{\alpha\in A}\left(N+\frac{1}{r}\right)^{n-|\alpha|}\int_{\Omega}\varphi(2^{n-|\alpha|}|D^\alpha f_1(t)|,\dots,2^{n-|\alpha|}|D^\alpha f_N(t)|)\,dt.\end{aligned}$$

P r o o f. The proof runs analogously as that of Theorem 3 from [1].

THEOREM 4. *Let the φ-function φ satisfies the assumption of Theorem 2. If functions $f_1, \ldots, f_N : R^n \to R$ have continuous derivatives $D^\alpha f_i, i = 1, \ldots, N$ $\alpha \in A$, then for every $s \in R^n$ the inequality*

$$\varphi(|f_1(s)|, \ldots, |f_N(s)|) \\ \leq \sum_{\alpha \in A} N^{n-|\alpha|} \int_{|s| \leq |t|} \varphi(2^{n-|\alpha|}|D^\alpha f_1(t)|, \ldots, 2^{n-|\alpha|}|D^\alpha f_N(t)|)\, dt$$

holds.

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