DEMONSTRATIO MATHEMATICA
Vol. XXXI No 2 1998

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FILLING A CUBE WITH SUBSEQUENCES OF BOXES

Let C be a subset of Euclidean d-space E^d and let (C_n) be a finite or infinite sequence of d-dimensional convex bodies. We say that (C_n) *permits a filling* of C, if there exist rigid motions $\sigma_1, \sigma_2, \ldots$ such that $C \setminus \bigcup \sigma_n C_n$ has Lebesgue measure 0 and sets $\sigma_n C_n$, where $n = 1, 2, \ldots$, have pairwise disjoint interiors and are subsets of C. Obviously, if (C_n) is finite and permits a filling of C, then $\bigcup \sigma_n C_n = C$. We say that C *can be filled with a subsequence* of (C_n) if there exists a finite or infinite subsequence (C_{n_i}) of the sequence (C_n) such that (C_{n_i}) permits a filling of C, and we say that the bodies $C_{n_1}, C_{n_2}, \ldots$ *are used for the filling.*

The problem is to find a possibly large class of bodies and a possibly small number v such that each sequence of bodies from this class of total volume not smaller than v contains a subsequence which permits a filling of C. This question is a particular version of the general problem of finding conditions under which a given container can be tiled by a given colection of solids (comp. [2], [3]).

We can interpret a filling with subsequences as covering a floor with tiles. We are given a container of tiles, and we know only that in the container there are tiles of a fixed shape and that the total area of the tiles is greater than or equal to v. We want to choose from the container as many tiles as it is necessary in order to cover the floor with them. The question is, what shape should the tiles have and how big should the number v be in order to be able to cover the floor with some tiles from the container. For instance, consider a covering a unit-square floor with square-shaped tiles. To cover this floor we cannot use a container with tiles of side length $\frac{2}{3}$. On the other hand, for the covering we can use a container in which we have tiles of sides of the form 2^{-1} or 3^{-1} of total area equal to 2. From the container we can

1991 *Mathematics Subject Classification*: 52C17.
Key words and phrases: packing, covering, filling, tiling, box.

choose identical tiles of total area equal to 1 and we can cover the floor with them.

Let $q \geq 2$ be an integer. A *box* is a rectangular parallelotope, and a box is *q-standard* if its edge lengths are of the form q^{-t}, where $t \in \{0, 1, \ldots\}$. Denote by $I^d(s)$ the cube $\{(x_1, \ldots, x_d) : 0 \leq x_j \leq s \text{ for } j = 1, \ldots, d\}$. By an *integer cube* we mean the cube $I^d(s)$, where s is a positive integer. In Section 1 we consider a filling of the integer cube with subsequences of sequences of q-standard boxes. We give an estimate of the total volume of the sequences containing a subsequence which permits a filling of $I^d(s)$. In Section 2 we show that this estimate is the best possible.

1. Filling with subsequences of sequences of standard boxes

Let B be a box in E^d. Denote by $w_1(B), \ldots, w_d(B)$ the edge lengths of B and assume that $w_1(B) \leq \ldots \leq w_d(B)$. We will also write w_j instead of $w_j(B)$, for short. Let $q \geq 2$ be an integer. By a *q-regular box* we mean a box R with $w_j(R) = q^{-t-1}$ for $j \leq k$ and $w_j(R) = q^{-t}$ for $j = k+1, \ldots, d$, where t is a non-negative integer and $k \in \{0, \ldots, d-1\}$. Observe that each q-regular box is q-standard.

We begin with two lemmas concerning filling with q-regular boxes.

LEMMA 1. *Let t be a non-negative integer. Every sequence of q-regular boxes of edge lengths not greater than q^{-t} and of total volume equal to q^{-td} permits a filling of $I^d(q^{-t})$.*

Proof. Consider a sequence (R_n) of q-regular boxes of edge lengths not greater than q^{-t} and of total volume equal to q^{-td}. Without loss of generality we can assume that $\text{Vol}(R_1) \geq \text{Vol}(R_2) \geq \ldots$. By a *proper position* in $I^d(q^{-t})$ for a regular box R from our sequence we mean the position

$$\{(x_1, \ldots, x_d);\ a_j w_j(R) \leq x_j \leq (a_j + 1) w_j(R) \ \text{ for } \ j = 1, \ldots, d\},$$

where $a_j \in \{0, \ldots, q^{-t} w_j^{-1}(R) - 1\}$ for $j = 1, \ldots, d$. A proper position for a box R_k from the sequence (R_n) is *free* if no interior point of this position is covered by boxes $\sigma_1 R_1, \ldots, \sigma_{k-1} R_{k-1}$. We can fill $I^d(q^{-t})$ by putting boxes $R_1, R_2, \ldots$ succesively, in free proper positions. After putting each box R_k, the uncovered part of our cube is the union of a number of q-regular boxes each of which is congruent to R_{k+1} and have disjoint interiors. Consequently, our sequence can be packed in $I^d(q^{-t})$. Thus, we obtain a filling of $I^d(q^{-t})$. ∎

LEMMA 2. *Let $t \geq 0$ and $z \geq 1$ be integers and let (T_n) be a sequence of q-regular boxes of edge lengths not greater than q^{-t}. Let the total volume of (T_n) be equal to v. If $v \geq zq^{-td}$, then there exists a subsequence of (T_n) which permits a filling of z disjoint cubes, each of edge length q^{-t}. Moreover, if $v > zq^{-td}$, then a finite subsequence of (T_n) permits a filling of the z cubes.*

Proof. Observe that if $v > zq^{-td}$, then (T_n) contains a finite subsequence (T_{n_i}) of total volume equal to zq^{-td}. If $v = zq^{-td}$, then we take (T_n) as (T_{n_i}). We can divide boxes from (T_{n_i}) into z groups such that the volume of boxes in each of the groups is equal to q^{-td}. By Lemma 1 we conclude that boxes from each of these groups permit a filling of a cube of edge length q^{-t}. Consequently, (T_{n_i}) permits a filling of z disjoint cubes of edge length q^{-t}. ∎

Let t be a non-negative integer and let $k \in \{0, \dots, d-1\}$. We say that a q-standard box S *is of type* (t,k), if $w_j(S) < q^{-t}$ for $j \leq k$ and $w_j(S) = q^{-t}$ for $j = k+1, \dots, d$. Observe, that each box of type $(t,0)$ is a cube. By an *r_q-set* we mean a union of a number of q-regular boxes with mutually disjoint interiors. We admit here the infinite number of q-regular boxes. Also, we regard the empty set as an r_q-set.

In the next lemma we will show that every sequence of q-standard boxes contains a subsequence which permits a filling of some r_q-set of a sufficiently large volume. Obviously, the total volume of boxes used for the filling of an r_q-set is equal to the volume of this set. We will estimate the total volume of boxes not used for the filling.

LEMMA 3. *Let $t_0 \geq 0$, $d \geq 2$ and $q \geq 2$ be integers. Put*

$$v_d(q) = \frac{q \cdot q^2 \cdot \ldots \cdot q^d}{(q^2-1)(q^3-1) \cdot \ldots \cdot (q^d-1)} - q.$$

Assume than (S_n) is a sequence of d-dimensional q-standard boxes of edge lengths not greater than q^{-t_0}. Moreover, let the total volume v of (S_n) be greater than or equal to $v_d(q)q^{-t_0 d}$. Then there exists some r_q-set Q whose volume is greater than $v - v_d(q)q^{-t_0 d}$ and there exists a subsequence (S_{n_j}) of (S_n) such that (S_{n_j}) permits a filling of Q.

Proof. We will proceed by induction with respect to the dimension d.

Let $d = 2$. Consider a sequence (S_n) of q-standard rectangles of edge lengths not greater than q^{-t_0} of total area equal to v, where $v \geq v_2(q)q^{-2t_0}$.

Let t be an integer not smaller than t_0 and let (S_{n_i}) be the subsequence of all non-regular rectangles from (S_n) with $w_2 = q^{-t}$. All the rectangles from (S_{n_i}) can be divided into a number of groups such that the total area of rectangles in one of the group is smaller than q^{-2t-1} and such that the total area of rectangles in each of the remaining groups is equal to q^{-2t-1}. Obviously, boxes with $w_2 = q^{-t}$ of total area equal to q^{-2t-1} permit a filling of a q-regular rectangle with $w_1 = q^{-t-1}$ and $w_2 = q^{-t}$. Consequently, we can fill some r_q-set with a subsequence of (S_{n_i}) so that the total area of rectangles not used for the filling is smaller than q^{-2t-1}.

Clearly, the subsequence of q-regular rectangles from (S_n) permits a filling of some r_q-set. In (S_n) we have non-regular boxes with $w_2 = q^{-t}$ for $t = t_0, t_0 + 1, \ldots$. Observe that the union of r_q-sets with disjoint interiors is an r_q-set. Hence, a subsequence of (S_n) permits a filling of some r_q-set so that the total area of the rectangles not used for the filling is smaller than

$$\sum_{t=t_0}^{\infty} q^{-2t-1} = \frac{q}{q^2-1} q^{-2t_0} = v_2(q) q^{-2t_0}.$$

Thus, the area of this r_q-set is greater than $v - v_2(q)q^{-2t_0}$.

Let $d \geq 2$. Assume that for each $m = 2, \ldots, d-1$ and for each non-negative integer t every sequence of m-dimensional q-standard boxes of total volume v greater than or equal to $q^{-tm} v_m(q)$ contains a subsequence which permits a filling of some m-dimensional r_q-set whose volume is greater than $v - q^{-tm} v_m(q)$.

Consider a sequence $(T_n^{(d)})$ of q-standard d-dimensional boxes of type (t, k), where t is a non-negative integer and $k \in \{0, \ldots, d-1\}$. If $k = 0$, then all the boxes in $(T_n^{(d)})$ are cubes. Hence $(T_n^{(d)})$ permits a filling of an r_q-set which is a union of a number of q-standard cubes.

Assume that $k \geq 1$. We will show that some r_q-set can be filled with a subsequence of $(T_n^{(d)})$ so that the total volume of boxes not used for the filling is smaller than $(1 + v_k(q))q^{-td-k}$. We put here $v_1(q) = 0$.

For $k = 1$ all the boxes from $(T_n^{(d)})$ have edge lengths $w_1 < q^{-t}$ and $w_2 = \ldots = w_d = q^{-t}$. All the boxes from $(T_n^{(d)})$ can be divided into a number of groups such that the total volume of boxes in one of the group is smaller than q^{-td-1} and such that the total volume of boxes in each of the remaining groups is equal to q^{-td-1}. Boxes of total volume equal to q^{-td-1} permit a filling of a q-regular box with $w_1 = q^{-t-1}$ and $w_2 = \ldots = w_d = q^{-t}$. Consequently, we can fill some r_q-set with a subsequence of $(T_n^{(d)})$ so that the total volume of boxes not used for the filling is smaller than $q^{-td-1} = (1 + v_1)q^{-td-1}$.

Let $k \geq 2$. Consider the sequence $(T_n^{(k)})$ of k-dimensional bottoms of boxes from $(T_n^{(d)})$. By a k-dimensional bottom of $T_n^{(d)}$ we mean the k-dimensional box of edge lengths $w_1(T_n^{(d)}), \ldots, w_k(T_n^{(d)})$.

Assume that the total k-dimensional volume of $(T_n^{(k)})$ is not smaller than $v_k(q)q^{-tk-k}$. By the inductive assumption we conclude that $(T_n^{(k)})$ contains a subsequence $(T_{n_i}^{(k)})$ which permits a filling of some k-dimensional r_q-set so that the total volume of boxes not used for the filling is smaller than $v_k(q)q^{-tk-k}$. Let the total k-dimensional volume of boxes from $(T_{n_i}^{(k)})$ be equal to $v^{(k)}$. There exists a non-negative integer z such that $zq^{-tk-k} \leq v^{(k)} < (z+1)q^{-tk-k}$. By Lemma 2 we conclude that we can fill z disjoint k-dimensional cubes of edge length q^{-t-1} with a subsequence of $(T_{n_i}^{(k)})$ so that the volume of boxes not used for the filling is smaller than q^{-tk-k}. If $z = 0$ here, then we regard all the boxes from $(T_{n_i}^{(k)})$ as not used for the filling. Consequently, we can fill a number of disjoint k-dimensional cubes of edge lengths q^{-t-1} with a subsequence of $(T_n^{(k)})$ so that the total volume of boxes not used for the filling is smaller than $(1 + v_k(q))q^{-tk-k}$.

Assume that the total k-dimensional volume of $(T_n^{(k)})$ is smaller than $v_k q^{-tk-k}$. Then we can also say that a number of cubes can be filled with a subsequence of $(T_n^{(k)})$ so that the volume of boxes not used for the filling is smaller than $(1+v_k(q))q^{-tk-k}$. In this case all the boxes are not used for the filling.

For $k \geq 2$ all the boxes from $(T_n^{(d)})$ have edge lengths $w_{k+1} = \ldots = w_d = q^{-t}$. Obviously, the box with $w_1 = \ldots = w_k = q^{-t-1}$ and with $w_{k+1} = \ldots = w_d = q^{-t}$ is q-regular. Consequently, some d-dimensional r_q-set can be filled with a subsequence of $(T_n^{(d)})$ so that the total volume of boxes not used for the filling is smaller than

$$(1 + v_k(q))q^{-tk-k}q^{-t(d-k)} = (1 + v_k(q))q^{-td-k}.$$

Consider a sequence (S_n) of d-dimensional q-standard boxes of edge lengths not greater than q^{-t_0}. In (S_n) we have boxes of types (t,k) for $t \geq t_0$ and for $k = 0, \ldots, d-1$. Boxes of type $(t,0)$ permit a filling of an r_q-set. For $k \geq 1$ we can fill an r_q-set with a subsequence of boxes of type (t,k) from (S_n) so that the total volume of boxes not used for the filling is smaller that $(1+v_k(q))q^{-td-k}$. This means that we can fill some r_q-set with a subsequence of (S_n) so that the total volume of boxes not used for the filling is smaller than

$$\sum_{t=t_0}^{\infty}\sum_{k=1}^{d-1}[1 + v_k(q)]q^{-td-k}$$
$$= q^{-t_0 d}\frac{q^d}{q^d - 1}[q^{-1}(1 + v_1(q)) + \ldots + q^{-d+1}(1 + v_{d-1}(q))].$$

This value is equal to $v_d(q)q^{-t_0 d}$. This means that if the total volume v of (S_n) is not smaller than $v_d(q)q^{-t_0 d}$, then (S_n) contains a subsequence which permits a filling of some r_q-set whose volume is greater than $v - v_d(q)q^{-t_0 d}$. ∎

THEOREM. *Let $q \geq 2$ and $s \geq 1$ be integers. Every sequence (S_n) of q-standard boxes of total volume greater than or equal to*

$$s^d + q\prod_{i=2}^{d} q^i/(q^i - 1) - q$$

contains a subsequence which permits a filling of $I^d(s)$. Moreover, a finite subsequence of (S_n) suffices.

Proof. Consider a sequence of q-standard boxes of total volume greater than or equal to $s^d + q\prod_{i=2}^{d} q^i/(q^i - 1) - q = s^d + v_d(q)$. By Lemma 3 we conclude that an r_q-set of the volume greater than s^d can be filled with a subsequence of our sequence. By Lemma 2 we see that this subsequence contains a finite subsequence which permits a filling of s^d disjoint unit cubes. Hence, this finite subsequence permits a filling of $I^d(s)$. ∎

REMARK 1. Two-standard boxes are considered by many authors discussing covering and packing methods (see the survey papers [1] and [6]). Obviously, not every covering is a filling. Therefore most of the covering methods are not useful for a filling. The covering method of Moon and Moser [7] is an exception. They consider a method of covering of the unit cube by boxes of edge lengths not greater than 1. This method is based on the method of covering by sequences of two-standard boxes. The first observation is that in the case when $q = 2$, Theorem can be also proved by the similar way like Theorem 2 from [7]. Another observation is as follows: in the proof of Theorem 2 from [7] each box B of edge lengths not greater than 1 contains a two-standard box S such that $\mathrm{Vol}(B)/\mathrm{Vol}(S) < 2^d$. Observe, that it is sufficient to consider two-standard boxes of edge lengths smaller than or equal to $\frac{1}{2}$. Each box B of edge lengths not greater than 1 contains a two-standard box S of edge lengths smaller than or equal to $\frac{1}{2}$ such that $\mathrm{Vol}(B)/\mathrm{Vol}(S) \leq 2^d$. We have an equality here only for $B = I^d(1)$. If we consider two-standard boxes of edge lengths not greater than $\frac{1}{2}$ in the proof of Theorem 2 from [7], then we can put there $[1 + (\frac{1}{2})^d v_d(2)]2^d = 2^d + 1.463\ldots$ instead of $[1 + v_d(2)]2^d = 2.463\ldots \cdot 2^d$.

REMARK 2. Consider a filling of an integer cube $I^d(s)$ with a subsequence (S_{n_i}) of a sequence (S_n) of q-standard boxes such that:

(i) each box $\sigma_{n_i} S_{n_i}$ has edges parallel to the axes of the coordinate system,

(ii) the projection of the edge of $\sigma_{n_i} S_{n_i}$ parallel to the j-th coordinate axis on the j-th coordinate axis is a segment whose both endpoints are multiples of the length of the edge of $\sigma_{n_i} S_{n_i}$ parallel to the j-th axis.

The filling described above is called *q-adic* (comp. [4] and [5]). From our proof we conclude that Theorem is true also if the word "filled" is replaced by the expression "q-adic filled".

In Theorem we consider q-standard boxes. A natural question is how to extend this class of boxes. The first observation, based on Theorem, is as follows:

COROLLARY. *Let $m \geq 2$ be an integer and let $q_1, \ldots, q_m$ be integers greater than 1. Assume that a sequence (Q_n) contains boxes which are q_i-standard, where $i = 1, \ldots, m$. If the total volume of (Q_n) is greater than or equal to $\sum_{i=1}^{m}[1 + v_d(q_i)]$, then the unit cube can be filled with a subsequence of (Q_n).*

Let $q_1, \ldots, q_d$ and $p_1, \ldots, p_d$ be positive integers. By a $(q_1, p_1, \ldots, q_d, p_d)$-*standard box* we mean a box with $w_j = p_j^{-1} q_j^{-m_j}$, where $m_j \in \{0, 1, \ldots\}$ for $j = 1, \ldots, d$. Arguing similarly like for q-standard boxes we conclude that there exists a number v such that every sequence of $(q_1, p_1, \ldots, q_d, p_d)$-standard boxes of total volume not smaller than v contains a subsequence which permits a filling of the unit cube. We can generalize this result. Let p and q be positive integers. By a $[p, q]$-*standard box* we mean a $(q_1, p_1, \ldots$ $\ldots, q_d, p_d)$-standard box, where $p_j \leq p$ and $q_j \leq q$ for $j = 1, \ldots, d$. Let (Q_n) be a sequence of $[p, q]$-standard boxes of total volume equal to v. Obviously, there exist positive integers $p_1, \ldots, p_d$ and $q_1, \ldots, q_d$ such that (Q_n) contains a subsequence of $(q_1, p_1, \ldots, q_d, p_d)$-standard boxes of total volume not smaller than $p^{-d} q^{-d} v$. Consequently, we obtain the following result.

PROPOSITION. *Let p and q be positive integers. Then there exists a number v such that every sequence of $[p, q]$-standard boxes of total volume greater than or equal to v contains a subsequence which permits a filling of the unit cube.*

2. A lower-bound example

In this section we will show that the value given in Theorem cannot be lowered. Observe that if we have a sequence of q-standard boxes of total volume equal to $1 + v$ which does not permit a filling of the unit cube, then by adding $s^d - 1$ cubes of edge length 1 we obtain a sequence of q-standard boxes of total volume equal to $s^d + v$ which does not permit a filling of $I^d(s)$. Consequently it is sufficient to consider the case when $s = 1$. For simplicity we will consider only the case when $q = 2$. For other q we proceed similarly.

We say that a rectangle R with $w_1(R) = b$ and $w_2(R) = h$ *is of the form* $b \times h$. Since now we will say *standard* instead of two-standard and *regular* instead of two-regular, for short.

Now we present a sequence of standard boxes of total volume arbitrarily close to $1 + v_d(2)$ such that $I^d(1)$ cannot be filled with any subsequence of this sequence. This sequence is constructed inductively.

Let $p \geq 2$ be an integer. For $d = 2$ our sequence contains the following rectangles:

(i) a square of edge length h for each $h = 2^{-1}, \ldots, 2^{-p}$,

(ii) a regular rectangle of the form $h \times 2h$ for each $h = 2^{-1}, \ldots, 2^{-p}$,

(iii) a rectangle of the form $2^{-u} \times 2^{-v}$ for each $v = 0, \ldots, p-2$ and for each $u = v+2, \ldots, p$.

The total area of this sequence is equal to $\frac{5}{3} - 2(\frac{1}{2})^p - \frac{5}{3}(\frac{1}{4})^p$. Obviously, we can choose a number p in such a way that the area of our sequence is arbitrarily close to $1 + v_2(2) = \frac{5}{3}$. We will show that the unit square cannot be filled with any subsequence of this sequence.

The total area of regular rectangles from this sequence is equal to $1-(\frac{1}{4})^p$. This means that we can fill only a part of $I^2(1)$ with the subsequence of regular rectangles from the sequence.

Consider a regular rectangle of the form $\frac{1}{2}h \times h$ contained in $I^2(1)$ which is not filled with regular rectangles from our sequence. Observe that we cannot fill any regular rectangle with a subsequence of a sequence of non-regular rectangles. If we use a number of non-regular rectangles with $w_2 = h$ for the filling of a regular rectangle of the form $\frac{1}{2}h \times h$, then we fill only a part of this rectangle with them. Let the uncovered part be a rectangle R of the form $b \times h$, where $b < h$. We show that this uncovered rectangle R cannot be filled with any subsequence of rectangles from our sequence with $w_2 = h$. Consider three cases. If $b = \frac{1}{4}h$, then a part of R can be filled with a regular rectangle of the form $b \times 2b$. A rectangle of the form $b \times 2b$ contained in R remains uncovered. This uncovered rectangle cannot be filled with any subsequence of regular rectangles from our sequence. If we use non-regular rectangles for the filling, then we encounter a similar situation as at the beginning. We have to fill a rectangle of the form $\frac{1}{2}h_0 \times h_0$, where $h_0 = \frac{1}{2}h$. If $b > \frac{1}{4}h$, then we can partially cover R with a subsequence of rectangles with $w_2 = h$ in such a way that the uncovered part is a rectangle of the form $b \times 4b$. We obtain our first case when $b = \frac{1}{4}h$. If $b < \frac{1}{4}h$, then we can partially fill R with a subsequence of non-regular rectangles in such a way that a rectangle of the form $b \times 4b$ remains uncovered. We obtain our first case.

Since the above consideration is true for each h, we conclude that the unit square cannot be filled with any subsequence of our sequence.

We proceed by induction. Let $d \geq 3$ and let $\epsilon_m > 0$ for $m = 2, \ldots, d-1$. Assume that we have a finite sequence of m-dimensional standard boxes, for each $m \in \{2,\ldots,d-1\}$, such that the m-dimensional volume of this sequence is greater than $1+v_m(2)-\epsilon_m$ and such that the unit m-dimensional cube cannot be filled with any subsequence of this sequence. This means that, for each $m \in \{2,\ldots,d-1\}$ and for each $r \in \{0,1,\ldots\}$, we can constuct a finite sequence of m-dimensional standard boxes of edge lengths smaller than or equal to 2^{-r} of total volume greater than $(1+v_m(2)-\epsilon_m)2^{-rm}$ such that the cube of edge length 2^{-r} cannot be filled with any subsequence of this sequence. Consequently, for each $j=1,\ldots,d-2$, for each $h=2^0,2^1,\ldots$ and for each $\epsilon_{d-j}>0$, we can find the following family $\mathcal{F}(j,h,\epsilon_{d-j})$ of d-dimensional standard boxes. Each box from $\mathcal{F}(j,h,\epsilon_{d-j})$ has edge lengths $w_{d-j+1}=\ldots=w_d=h$. The total volume of boxes from $\mathcal{F}(j,h,\epsilon_{d-j})$ is greater than $h^j(\frac{1}{2}h)^{d-j}(1+v_{d-j}(2)-\epsilon_{d-j})$. The $(d-j)$-dimensional cube of edge length $\frac{1}{2}h$ cannot be filled with any subsequence of the sequence of $(d-j)$-dimensional bottoms of boxes from $\mathcal{F}(j,h,\epsilon_{d-j})$. By $(d-j)$-dimensional bottom of a box B we mean a box of edge lengths $w_1(B),\ldots,w_{d-j}(B)$.

Let $p\geq 2$ be an integer. By the above assumption we can construct the following sequence of d-dimensional standard boxes. Our sequence contains:

(i) a box of the form: $w_1=\ldots=w_{d-j}=\frac{1}{2}h,\ w_{d-j+1}=\ldots=w_d=h$ for each $j=1,\ldots,d-1$ and for each $h=1,\ldots,2^{-p+1}$,

(ii) a cube of edge length 2^{-u} for each $u=1,\ldots,p$,

(iii) a box with $w_1=2^{-u}$ and with $w_2=\ldots=w_d=2^{-v}$ for each $v=0,\ldots,p-2$ and for each $u=v+2,\ldots,p$,

(iv) boxes from the family $\mathcal{F}(j,h,\epsilon_{d-j})$ for each $j=1,\ldots,d-2$ and for each $h=1,\ldots,2^{-p+1}$.

The total volume of boxes described in (i) and (ii) is equal to $1-2^{-dp}$. The total volume of boxes from (iii) is equal to

$$\frac{1}{2}\cdot\frac{2^d-2^{-d(p-2)}}{2^d-1}-\left(\frac{1}{2}\right)^p-\frac{1-2^{-(d-1)(p-2)}}{2^d-2}\left(\frac{1}{2}\right)^{p-1}.$$

The total volume of boxes described in (iv) is equal to

$$\frac{2^d}{2^d-1}(1-2^{-dp})\left[\left(\frac{1}{2}\right)^2(1+v_2(2)-\epsilon_2)+\ldots+\left(\frac{1}{2}\right)^{d-1}(1+v_{d-1}(2)-\epsilon_{d-1})\right].$$

This means that we can choose a number p and numbers $\epsilon_2,\ldots,\epsilon_{d-1}$ in such a way that the total volume of our sequence is arbitrarily close to

$$1+\frac{2^d}{2^d-1}\left[\frac{1}{2}+\left(\frac{1}{2}\right)^2(1+v_2(2))+\ldots+\left(\frac{1}{2}\right)^{d-1}(1+v_{d-1}(2))\right]=1+v_d(2).$$

Observe that boxes from (i) and (ii) cannot fill the entire unit cube. Moreover, we cannot fill any box described in (i) or (ii) with a subsequence of the sequence of boxes described in (iii) and (iv). Arguing similarly as for $d = 2$ we conclude that the unit cube cannot be filled with any subsequence of our sequence.

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Received April 7, 1997.