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A COINCIDENCE THEOREM FOR DENSIFYING MAPPINGS AND APPLICATIONS

1. Introduction

Over the past two decades there has been a great deal of work on the fixed point theorems for contractive and densifying nonlinear mappings in Banach and metric spaces (cf. [1]). Unfortunately a small number of these can be applied to real significant problems (cf. [1]). After the paper of Goebel [2], the coincidence theory of contractive nonlinear mappings develops in much the same manner as the fixed point theory (cf. [1]).

In a recent paper [3] a coincidence theorem in uniform spaces for nonlinear mappings under contractive type conditions has been proved. As a consequence of it an existence of singular solutions for ordinary and hyperbolic partial differential equations has been obtained. The main purpose of the present paper is to formulate a coincidence theorem for a class of nonlinear mappings. An application to the existence of generalized solutions of ordinary differential equations in Banach spaces is made.

2. Preliminaries and an abstract result

First we are going to prove a fixed point theorem adapting for our purposes a scheme developed in [4] and [5]. We recall some definitions and notations from these two papers.

By E we shall mean a T_2-separated locally convex linear topological space. Let $\{\chi_\alpha(\cdot)\}_{\alpha\in\mathfrak{A}}$ (where $\mathfrak{A}$ is an index set) be a family of measures of noncompactness on E and $j : \mathfrak{A} \to \mathfrak{A}$ be a mapping of the index set $\mathfrak{A}$ into itself. A mapping $\chi : 2^E \to [0,\infty]$ satisfying $\chi(\overline{\mathrm{co}}\Omega) = \chi(\Omega)$ for every subset $\Omega \subseteq E$ ($\overline{\mathrm{co}}\Omega$ being the convex closure of Ω) is called a measure of noncompactness. We suppose that every $\chi_\alpha(\cdot)$ is semiadditive ($\chi_\alpha(\Omega_1 \cup \Omega_2) = \max\{\chi_\alpha(\Omega_1), \chi_\alpha(\Omega_2)\}$, $\alpha \in \mathfrak{A}$) and translation invariant ($\chi_\alpha(x+\Omega) = \chi_\alpha(\Omega)$ for every $x \in E$, $\alpha \in \mathfrak{A}$, and $\Omega \subseteq E$). It is known (cf.

[5]) that semiadditivity implies monotonicity (i.e. $\Omega_1 \subseteq \Omega_2 \Rightarrow \chi_\alpha(\Omega_1) \leq \chi_\alpha(\Omega_2)$, $\alpha \in \mathfrak{A}$).

We make the following basic assumptions which will be denoted by (B):

(B1) E satisfies the first countability axiom.

(B2) $\chi_{j(\alpha)}(\Omega) \leq \chi_\alpha(\Omega)$ for every $\Omega \in 2^E$ and $\alpha \in \mathfrak{A}$.

Let $M \subseteq E$. The mapping $f : M \to E$ is called densifying iff it is continuous and for every noncompact set $\Omega \subseteq M$ there exists $\alpha(\Omega) \in \mathfrak{A}$ such that $\chi_{\alpha(\Omega)}(f(\Omega)) < \chi_{j(\alpha(\Omega))}(\Omega)$.

THEOREM 1. *Under the assumptions* (B) *the densifying mapping* $f : M \to M$ *has a fixed point in* M.

Proof. We first prove that

$$\chi_\alpha(\{x\} \cup \Omega) = \chi_\alpha(\Omega) \tag{1}$$

for every $\alpha \in \mathfrak{A}$, every $x \in E$ and every $\Omega \subseteq E$.

Let us choose an arbitrary $\alpha \in \mathfrak{A}$. Since χ_α is monotone (being semiadditive) we have $\chi_\alpha(\{x\}) \leq \chi_\alpha(\Omega)$ if $x \in \Omega$. Since χ_α is translation invariant, $\chi_\alpha(\{x\}) \leq \chi_\alpha(\Omega)$ holds true for every $x \in X$ and every $\Omega \subseteq E$. Now, the semiadditivity of χ_α implies:

$$\chi_\alpha(\{x\} \cup \Omega) = \max\{\chi_\alpha(\{x\}), \chi_\alpha(\Omega)\} = \chi_\alpha(\Omega)$$

and (1) is proved.

Next we are going to show that there exists a nonempty compact set $K \subseteq M$ such that $f(K) = K$. Let us choose $x \in M$ and denote $\Omega = \{f^n(x) : n = 0, 1, 2, \ldots\}$. Then $\Omega = \{x\} \cup f(\Omega)$, i.e. $\chi_\alpha(\Omega) = \chi_\alpha(f(\Omega))$ for every $\alpha \in \mathfrak{A}$ because of (1). Since f is densifying, Ω is compact. Let K be the set of its limit points and let us choose an arbitrary $y \in K$. Then $y = \lim_{k\to\infty} f^{n_k}(x)$, i.e $f(y) = \lim_{k\to\infty} f^{n_k+1}(x)$. This means that $f(K) \subseteq K$. On the other hand, let z be a limit point of the set

$$\{f^{n_k-1}(x)\}_{k=1}^{\infty} \subset \Omega \text{ (without loss of generality } n_1 \geq 1).$$

Then $z = \lim_{l\to\infty} f^{n_{k_l}-1}(x)$, which implies $f(z) = \lim_{l\to\infty} f^{n_{k_l}}(z) = y$. Since $z \in K$, we have $y = f(z) \in f(K)$, i.e. $K \subseteq f(K)$. We thus proved $f(K) = K$. Since f is densifying, K is a compact subset of M.

Let us now construct a transfinite sequence of sets $\{\Omega_\gamma\}$ in the following way:

$$\Omega_0 = M,$$

$$\Omega_\gamma = \begin{cases} \overline{\text{co}} f(\Omega_{\gamma-1}) & \text{if } \gamma \text{ has a predecessor,} \\ \bigcap_{\beta<\gamma} \Omega_\gamma & \text{if } \gamma \text{ has no predecessor.} \end{cases}$$

It is easy to verify that the following propositions hold for every γ:

(a) $\Omega_\gamma \subseteq M$,
(b) Ω_γ is closed and convex,
(c) $f(\Omega_\gamma) \subseteq \Omega_\gamma$,
(d) $K \subseteq \Omega_\gamma$.

In addition we have $\Omega_\gamma \subset \Omega_{\gamma-1}$ if γ has a predecessor and $\Omega_\gamma \subset \Omega_\beta$ for $\beta < \gamma$ if γ has no predecessor. Then there is an ordinal number γ_0 such that $\Omega_{\gamma_0+1} = \Omega_{\gamma_0}$. It follows from (a)–(d) that Ω_{γ_0} is a nonempty closed and convex subset of M.

Moreover, we have

$$\chi_\alpha(\Omega_{\gamma_0}) = \chi_\alpha(\Omega_{\gamma_0+1}) = \chi_\alpha(\overline{\text{co}} f(\Omega_{\gamma_0})) = \chi_\alpha(f(\Omega_{\gamma_0}))$$

for each $\alpha \in \mathfrak{A}$. Taking into account (B) and the fact that f is densifying, we conclude that Ω_{γ_0} is compact. Because $f(\Omega_{\gamma_0}) \subseteq \Omega_{\gamma_0}$ we can apply the Shauder–Tikhonoff fixed point theorem and thus complete the proof.

Our next aim is to prove a corollary of Theorem 1 which gives a somewhat different point of view on its result. In order to do this we recall some necessary notions and some basic relations between them.

Let A be an arbitrary set, (X, τ) be a topological space and the operators R and T map A into X. The topologies τ_R and τ_T on A are defined as

$$\tau_R = \{R^{-1}(\theta) : \theta \in \tau\}, \qquad \tau_T = \{T^{-1}(\theta) : \theta \in \tau\},$$

where τ is the family of the open subsets of X. The mapping R is said to be continuous with respect to the mapping T if for each $x \in A$ and for each neighbourhood U of Rx there exists a neighbourhood V of Tx such that $RT^{-1}(V) \subset U$ (i.e. $Ty \in V$ implies $Ry \in U$).

LEMMA 1 ([6]). *The mapping R is continuous with respect to T iff τ_T is stronger than τ_R (every R-open set is T-open).*

LEMMA 2 ([6]). *Let A be an arbitrary set, (X, τ) be a Hausdorff topological space, R and T map A into X and let R be continuous with respect to T. Let $R(A)$ and $T(A)$ be the ranges of R and T respectively. Then the mapping $H : T(A) \to R(A)$ defined by $Hx = RT^{-1}x$, $x \in T(A)$, is single-valued and continuous.*

Now we are ready to prove the corollary of Theorem 1 which can be viewed upon as a coincidence theorem.

COROLLARY 1. *Let A be an arbitrary set, R and T map A into E and $R(A)$ and $T(A)$ be the ranges of R and T respectively. Let $T(A)$ be a closed and convex subset of E, $R(A) \subseteq T(A)$ and R be continuous with respect to T. We assume that for every $\Omega \subseteq A$, such that $T(\Omega)$ is noncompact,* *there exists $\alpha(\Omega) \in \mathfrak{A}$ such that $\chi_{\alpha(\Omega)}(R(\Omega)) < \chi_{j(\alpha(\Omega))}(T(\Omega))$. Then there exists $a_0 \in \mathfrak{A}$ such that $Ra_0 = Ta_0$.*

Proof. Let us define the operator $Hx = RT^{-1}x$ for $x \in T(A)$. We have $H : T(A) \to R(A) \subseteq T(A)$ and according to Lemma 2 H is single-valued and continuous. Let $\Sigma \subseteq T(A)$ be noncompact and let $\Omega = T^{-1}(\Sigma)$, i.e. $\Omega = \{a \in A : Ta \in \Sigma\}$. Let $\alpha(\Omega) \in \mathfrak{A}$ be such that $\chi_{\alpha(\Omega)}(R(\Omega)) < \chi_{j(\alpha(\Omega))}(T(\Omega))$. Then $\chi_{\alpha(\Omega)}(H(\Sigma)) = \chi_{\alpha(\Omega)}(RT^{-1}(\Sigma)) = \chi_{\alpha(\Omega)}(R(\Omega)) < \chi_{j(\alpha(\Omega))}(T(\Omega)) = \chi_{j(\alpha(\Omega))}(\Sigma)$, i.e. $H : T(A) \to T(A)$ is densifying. According to Theorem 1 there exists $x \in T(A)$ such that $x = Hx$, i.e. $x = RT^{-1}x$. Let $a_0 \in A$ be such that $Ta_0 = x$. Then $Ra_0 = x = Ta_0$ and the corollary is proved.

Remark 1. Corollary 1 generalizes Theorem 1 on p. 45 in [6] in the case when:

(a) $\Sigma \subset E$ is compact iff $\chi_\alpha(\Sigma) = 0$ for each $\alpha \in \mathfrak{A}$ and

(b) $j : \mathfrak{A} \to \mathfrak{A}$ is surjective.

3. An application: treatment of noncontinuable solutions

In this section we use the approach proposed in [3] to treat noncontinuable solutions of ordinary differential equations in Banach spaces, i.e. solutions which "blow up". The setting of the problem follows next.

Let $\{t_k\}_{k=0}^{\infty}$ be a sequence of numbers having the properties:

1) $t_k < t_{k+1}$ for $k = 1, 2, \ldots$; 2) $\lim_{k\to\infty} t_k = +\infty$. Let $\{p_k(t)\}_{k=0}^{\infty}$ be a sequence of functions with the properties: $p_k(\cdot)$ is differentiable and monotonely increasing on $[t_k, t_{k+1})$; $p_k(t_k) = 1$; $\lim_{t\to t_{k+1}-0} p_k(t) = +\infty$.

Let Y be a Banach space and $f : [t_0, +\infty) \times Y \to Y$.

The function $x(\cdot) : [t_0, +\infty) \to Y$ is called a generalized solution of the Cauchy problem (cf. [3])

$$
\begin{aligned}
&x'(t) = f(t, x(t)) \quad \text{for } t \geq t_0 \\
&x(t_0) = x_0 \in Y
\end{aligned}
\tag{2}
$$

if it has the following properties:

a) $x(t)$ is continuous on $[t_k, t_{k+1})$ for $k = 0, 1, 2, \ldots$;

b) $x(t) = x_k + \int_{t_k}^{t} f(s, x(s))\, ds \qquad$ on $[t_k, t_{k+1})$;

c) $x_{k+1} = \lim_{t\to t_{k+1}-0} \left\{ \left[x_k + \int_{t_k}^{t} f(s, x(s))\, ds \right] / p_k(t) \right\}$ exists for $k = 0, 1, 2, \ldots$

Following [5], we introduce a family of measures of noncompactness in $C([t_0,+\infty),Y)$.

PROPOSITION 1 ([5]). *Let* $\Omega \subseteq C([t_0,+\infty),Y)$, $\Omega_t = \{f(t) : f \in \Omega\} \subseteq Y$ *and* χ_H *be the Hausdorff measure of noncompactness in* Y. *Then*

$$\widetilde{\chi}_\alpha(\Omega) = \max_{t_0 \le t \le \alpha} \chi_H(\Omega_t) \tag{3}$$

is a family of semiadditive and translation invariant measures of noncompactness in $C([t_0,+\infty),Y)$ *for* $\alpha \ge t_0$.

Remark 2. In [5] B. N. Sadovski considers the Banach space $C([a,b],Y)$ and introduces one measure of noncompactness in it as in Proposition 1.

Remark 3. The family (3) of measures of noncompactness (for $\alpha \ge t_0$) will be used in the application of the results of the previous section for proving existence of a generalized solution of the Cauchy problem (2).

THEOREM 2. *Let* p, q *and* r *be reals satisfying*

$$p \ge q, \quad p > 1, \quad q > 0, \quad r \ge 0 \tag{4}$$

and

$$p - q - pr \ge 0. \tag{5}$$

Let $p_k(t) = (t_{k+1} - t_k)^p/(t_{k+1} - t)^p$ *in* $[t_k, t_{k+1})$ *for* $k = 0,1,2,\ldots$

Let Y *be a Banach space and* $f(t,y) : [t_0,+\infty) \times Y \to Y$ *have the following properties*:

(6) a) $\|f(t,y)\|_Y \le \dfrac{\widehat{C}}{(t_{k+1}-t)^q}(1 + \|y\|_Y^r)$ *for* $t \in [t_k, t_{k+1})$ *and* $y \in Y$

where $\widehat{C}$ *is a positive constant*;

b) *if* K *is a compact subset of* $[t_k, t_{k+1})$ *for some* $k = 0,1,2,\ldots$, *and* B *is a bounded subset of* Y, *then* $f(t,y)$ *is uniformly continuous on* $K \times B$;

c) *for every subset* D *of* Y *and for every* $t \in [t_k, t_{k+1})$

$$\chi_H[f(t,D)] \le \widetilde{k} \cdot \chi_H(D) \tag{7}$$

holds true, where χ_H *is the Hausdorff measure of noncompactness in* Y *and the constant* $\widetilde{k}$ *satisfies*

$$\widetilde{k}(t_{k+1} - t_k) < 1 \quad \textit{for } k = 0,1,2,\ldots. \tag{8}$$

Then there exists a generalized solution of (2).

The proof of Theorem 2 will be divided into several steps aimed at making preparations for applying Theorem 1.

Let us first note that (4) and (5) imply $r < 1$.

Let $x_0 \in Y$ be the initial condition of the problem (2), let $\{L_k\}_{k=0}^{\infty}$ be a sequence of positive reals and let us define inductively $C_0 = 0$, $C_k = C_{k-1} + L_{k-1}(t_k - t_{k-1})$ for $k = 0, 1, 2, \ldots$. We define the set $M \subset C([t_0, +\infty), Y)$ by

$$(9)\quad \begin{aligned} M = \{x(\cdot) \in C([t_0, +\infty), Y) : \|x(t_k) - x_0\|_Y \\ \le C_k \text{ for } k = 0, 1, 2, \ldots \text{ and } x(\cdot) \\ \text{is Lipschitz continuous on } [t_k, t_{k+1}] \text{ with Lipschitz constant } L_k \\ \text{for } k = 0, 1, 2, \ldots\}. \end{aligned}$$

Let the operator $S : M \to \mathcal{F}([t_0, +\infty), Y)$ be defined as

$$(10)\qquad (Sx)(t) = \frac{1}{p_k(t)}\left[x_k + \int_{t_k}^{t} f(s, p_k(s).x(s))\, ds\right] \quad \text{for } t \in [t_k, t_{k+1})$$

where x_k for $k = 1, 2, 3, \ldots$ is defined by

$$(11)\qquad x_k = \lim_{t \to t_k - 0} \frac{1}{p_{k-1}(t)}\left[x_{k-1} + \int_{t_{k-1}}^{t} f(s, p_{k-1}(s).x(s))\, ds\right]$$

(if it exists) and $\mathcal{F}([t_0, +\infty), Y)$ is the set of all functions defined on $[t_0, +\infty)$ and taking values in Y.

Next we show that we can choose the constants $\{L_k\}_{k=0}^{\infty}$ in such a way, that $S(M) \subseteq M$ be satisfied. In this case all x_k from (11) are well defined because M consists of functions which are uniformly Lipschitz on $[t_k, t_{k+1}]$ for $k = 1, 2 \ldots$

Remark 4. For the calculations in the proof of the following Proposition 2 we make an additional assumtion: $q \neq 1$ and $pr + q \neq 1$. In the case when $q = 1$ or $pr + q = 1$ the calculations and the inequality (13) (see below) must be changed (accordingly).

PROPOSITION 2. *The numbers $\{L_k\}_{k=0}^{\infty}$ can be choosen in such a way that $S(M) \subseteq M$ be satisfied, where the set M is defined by (9) and the operator $S : M \to \mathcal{F}([t0, +\infty), Y)$ is defined by* (10).

Proof. Let, by induction , L_i for $i = 0, 1, \ldots, k-1$ be already choosen. This means that $x_k \in Y$ is well defined by (11) (the initial condition x_0 being given), that C_i for $i = 0, 1, \ldots, k$ are already known and that

$$(12)\qquad \|x_k\|_Y \le \|x_0\|_Y + C_k.$$

Since $r < 1$ we can choose L_k so large that the following inequality is satisfied:

$$\frac{p(\|x_0\|_Y + C_k)}{t_{k+1} - t_k} + \frac{\widehat{C}}{(t_{k+1} - t_k)^q}\left[\frac{2p}{|1-q|} + 1\right] \tag{13}$$
$$+\frac{\widehat{C}}{(t_{k+1} - t_k)^q}\left(\frac{2p}{|1-q-pr|} + 1\right) \cdot [\|x_0\|_Y + C_k + L_k(t_{k+1} - t_k)]^r < L_k.$$

Let $x(\cdot)$ be an arbitrary fixed element of M. Using (9), for $t \in [t_k, t_{k+1}]$ we obtain

$$\|x(t) - x(t_k)\| \leq L_k(t - t_k) \leq L_k(t_{k+1} - t_k), \text{ i.e.}$$
$$\|x(t)\|_Y \leq \|x(t_k)\|_Y + L_k(t_{k+1} - t_k) \leq \|x_0\|_Y + C_k + L_k(t_{k+1} - t_k). \tag{14}$$

In view of the mean value theorem, it is sufficient to prove that $\|[(Sx)(t)]'\|_Y \leq L_k$ for $t \in [t_k, t_{k+1}]$. Because of

$$[(Sx)(t)]' = \frac{p(t_{k+1} - t)^{p-1}}{(t_{k+1} - t_k)^p}\left[x_k + \int_{t_k}^{t} f(s, p_k(s)x(s))\, ds\right]$$
$$+ \frac{(t_{k+1} - t)^p}{(t_{k+1} - t_k)^p}.f(t, p_k(t).x(t)),$$

taking into account (4), (5), (6), (12) and (14), we have

$$\|[(Sx)(t)]'\|_Y$$
$$\leq \frac{p(t_{k+1} - t)^{p-1}}{(t_{k+1} - t_k)^p} \cdot \left[\|x_k\|_Y + \int_{t_k}^{t} \|f(s, p_k(s)x(s))\|_Y\, ds\right]$$
$$+ \frac{(t_{k+1} - t)^p}{(t_{k+1} - t_k)^p} \cdot \|f(t, p_k(t).x(t))\|_Y$$
$$\leq \frac{p(t_{k+1} - t)^{p-1}}{(t_{k+1} - t_k)^p}$$
$$\times \left[\|x_k\|_Y + \widehat{C}\int_{t_k}^{t} (t_{k+1} - s)^{-q}\left(1 + \frac{(t_{k+1} - t_k)^{pr}}{(t_{k+1} - s)^{pr}}\|x(s)\|_Y^r\right) ds\right]$$
$$+ \frac{(t_{k+1} - t)^p}{(t_{k+1} - t_k)^p} \cdot \frac{\widehat{C}}{(t_{k+1} - t)^q}\left[1 + \frac{(t_{k+1} - t_k)^{pr}}{(t_{k+1} - s)^{pr}}\|x(t)\|_Y^r\right]$$

$$\leq \frac{p\|x_k\|_Y(t_{k+1}-t)^{p-1}}{(t_{k+1}-t_k)^p} + \frac{\widehat{C}.p.(t_{k+1}-t)^{p-1}}{(t_{k+1}-t_k)^p} \cdot \int_{t_k}^{t} (t_{k+1}-s)^{-q}\, ds$$

$$+ \frac{\widehat{C}.p.(t_{k+1}-t)^{p-1}}{(t_{k+1}-t_k)^{p-pr}}$$

$$\times \int_{t_k}^{t} (t_{k+1}-s)^{-q-pr}[\|x_0\|_Y + C_k + L_k(t_{k+1}-t_k)]^r\, ds$$

$$+ \frac{\widehat{C}(t_{k+1}-t)^{p-q}}{(t_{k+1}-t_k)^p} + \frac{\widehat{C}(t_{k+1}-t)^{p-q-pr}}{(t_{k+1}-t_k)^{p-pr}}$$

$$\times [\|x_0\|_Y + C_k + L_k(t_{k+1}-t_k)]^r$$

$$= \frac{p.\|x_k\|_Y.(t_{k+1}-t)^{p-1}}{(t_{k+1}-t_k)^p} - \frac{\widehat{C}.p}{1-q} \cdot \frac{(t_{k+1}-t)^{p-q}}{(t_{k+1}-t_k)^p}$$

$$+ \frac{\widehat{C}.p}{1-q} \cdot \frac{t_{k+1}-t)^{p-1}}{(t_{k+1}-t_k)^{p+q-1}}$$

$$+ \left[- \frac{\widehat{C}.p.(t_{k+1}-t)^{p-q-pr}}{(1-q-pr)(t_{k+1}-t_k)^{p-pr}} + \frac{\widehat{C}.p.(t_{k+1}-t)^{p-1}}{(1-q-pr)(t_{k+1}-t_k)^{p+q-1}} \right]$$

$$\times [\|x_0\|_Y + C_k + L_k(t_{k+1}-t_k)]^r$$

$$+ \frac{\widehat{C}(t_{k+1}-t)^{p-q}}{(t_{k+1}-t_k)^p} + \frac{\widehat{C}(t_{k+1}-t)^{p-q-pr}}{(t_{k+1}-t_k)^{p-pr}}$$

$$\times [\|x_0\|_Y + C_k + L_k(t_{k+1}-t_k)]^r$$

$$\leq \frac{p(\|x_0\|_Y + C_k)}{(t_{k+1}-t_k)} + \frac{2\widehat{C}.p}{|1-q|} \cdot \frac{1}{(t_{k+1}-t_k)^q} + \frac{\widehat{C}}{(t_{k+1}-t_k)^q}$$

$$+ \frac{2\widehat{C}.p}{|1-q-pr|} \cdot \frac{1}{(t_{k+1}-t_k)^q}[\|x_0\|_Y + C_k + L_k(t_{k+1}-t_k)]^r$$

$$+ \frac{\widehat{C}}{(t_{k+1}-t_k)^q}[\|x_0\|_Y + C_k + L_k(t_{k+1}-t_k)]^r < L_k,$$

the last inequality being (13). Proposition 2 is thus proved.

PROPOSITION 3. *The set M, defined by* (9) *is closed and convex subset of* $C([t_0, +\infty), Y)$.

Proof. Obvious.

PROPOSITION 4. *The operator $S : M \to M$ is continuous in the topology of $C([t_0, +\infty), Y)$.*

Proof. Let the sequence $\{x_n(\cdot)\}_{n=1}^{\infty} \subset M$ tend to $\bar{x}(\cdot)$ in the topology of $C([T_0, +\infty), Y)$, which is generated by the family of seminorms $\|x(\cdot)\|_\alpha = \max_{t_0 \le t \le \alpha} \|x(t)\|_Y$ for $\alpha \ge t_0$ (since M is closed, $\bar{x}(\cdot) \in M$ holds true). We have to prove that the sequence $\{S(x_n)(\cdot)\}_{n=1}^{\infty}$ tends to $(S\bar{x})(\cdot)$ in the same topology. Since every interval $[t_0, \alpha]$ consists of finite number of intervals $[t_k, t_{k+1}]$ (plus, may be, an interval $[t_k, \alpha]$), it is sufficient to prove that $\{S(x_n)(\cdot)\}_{n=1}^{\infty}$ converges uniformly to $(S\bar{x})(\cdot)$ on every $[t_k, t_{k+1}]$.

Fix k and $\delta \in (0, t_{k+1} - t_k)$. Because of the definitions of M and $p_k(t)$, the set $\{y \in Y : y = p_k(t).x(t),\ t \in [t_k, t_{k+1} - \delta]\}$ is contained in $B = \{y \in Y : \|y\|_Y \le \gamma\}$ for sufficiently large γ. Choosing an arbitrary $\varepsilon > 0$, we can find $\delta_0 > 0$ such that $\bar{t}, \bar{\bar{t}} \in [t_k, t_{k+1} - \delta]$, $|\bar{t} - \bar{\bar{t}}| < \delta_0$ and $y_1, y_2 \in B$, $\|y_1 - y_2\|_Y \le \delta_0$ imply $\|f(\bar{t}, y_1) - f(\bar{\bar{t}}, y_2)\|_Y < \varepsilon$ because f is uniformly continuous on $[t_k, t_{k+1} - \delta] \times B$. Further on, there is an integer n_0 such that $n \ge n_0$ implies $\max_{t_k \le t \le t_{k+1} - \delta} p_k(t) \|x_n(t) - \bar{x}(t)\|_Y < \delta_0$. Then for $n \ge n_0$ we have

$$\max_{t_k \le t \le t_{k+1} - \delta} \|(Sx_n)(t) - (S\bar{x})(t)\|_Y$$

$$\le \max_{t_k \le t \le t_{k+1} - \delta} \frac{1}{p_k(t)} \left\| \int_{t_k}^{t} f(s, p_k(s).x_n(s))\, ds - \int_{t_k}^{t} f(s, p_k(s), \bar{x}(t))\, ds \right\|_Y$$

$$\le \max_{t_k \le t \le t_{k+1} - \delta} \int_{t_k}^{t} \|f(s, p_k(s).x_n(s)) - f(s, p_k(s).\bar{x}(s))\|_Y\, ds$$

$$\le \int_{t_k}^{t_{k+1} - \delta} \varepsilon\, ds = \varepsilon.(t_{k+1} - t_k - \delta).$$

The uniform convergence of $\{S(x_n)(\cdot)\}_{n=1}^{\infty}$ to $(S\bar{x})(\cdot)$ on $[t_k, t_{k+1} - \delta]$ is thus proved. Since $\delta \in (0, t_{k+1} - t_k)$ was arbitrary fixed, we conclude that $\{S(x_n)(\cdot)\}_{n=1}^{\infty}$ is pointwise converging to $(S\bar{x})(\cdot)$ on $[t_k, t_{k+1}]$. Since $\{S(x_n)(\cdot)\}_{n=1}^{\infty} \subset M$, it consists of uniformly Lipschitz on $[t_k, t_{k+1}]$ functions (with Lipschitz constant L_k). It follows directly from here and from the pointwise convergence of $\{S(x_n)(\cdot)\}_{n=1}^{\infty}$ to $(S\bar{x})(\cdot)$ on $[t_k, t_{k+1}]$ that $\{S(x_n)(\cdot)\}_{n=1}^{\infty}$ converges to $(S\bar{x})(\cdot)$ uniformly on $[t_k, t_{k+1}]$.

PROPOSITION 5. *Let Ω be a subset of M such that $\widehat{K} = \{y \in Y : y = (Sx)(t_k), x(\cdot) \in \Omega\}$ is a compact set and $\Omega_t = \{x(t) : x(\cdot) \in \Omega\} \subset Y$. Then* *for each $t \in [t_k, t_{k+1}]$ we have*

$$\chi_H\left\{\frac{1}{p_k(t)}\left[\widehat{K} + \int_{t_k}^{t} f(s, p_k(s).\Omega)\,ds\right]\right\} \leq \widetilde{k}.(t_{k+1} - t_k). \max_{t_k \leq t \leq t_{k+1}} \chi_H(\Omega_t). \tag{15}$$

where χ_H is the Hausdorff measure of noncompactness in Y.

Proof. Let $t \in [t_k, t_{k+1}]$ be fixed. Since

$$\chi_H\left\{\left[\widehat{K} + \int_{t_k}^{t} f(s, p_k(s).\Omega)\,ds\right]\right\} = \chi_H\left[\int_{t_k}^{t} f(s, p_k(s).\Omega)\,ds\right]$$

(because of the compactness of $\widehat{K}$), (15) is equivalent to

$$\frac{1}{p_k(t)} \cdot \chi_H\left[\int_{t_k}^{t} f(s, p_k(s)\Omega)\,ds\right] \leq \widetilde{k}.(t_{k+1} - t_k). \max_{t_k \leq t \leq t_{k+1}} \chi_H(\Omega_t). \tag{16}$$

Since $\Omega \subset M$, the set $\{y \in Y : y = p_k(s).x(s),\ s \in [t_k, t], x(\cdot) \in \Omega\}$ is contained in a bounded subset B of Y. The inclusion $\Omega \subset M$ implies also that Ω is equicontinuous. Since f is uniformly continuous on $[t_k, t] \times B$, the set of functions $f(p_k(s).\Omega)$ is equicontinuous too. Hence for each $\varepsilon > 0$ there exists a sufficiently large number n such that

$$\left\| \int_{t_k}^{t} f(s, p_k(s)x(s))\,ds - \frac{t - t_k}{n} \sum_{i=1}^{n} f(s_i, p_k(s_i)x(s_i)) \right\|_Y < \varepsilon \tag{17}$$

for each $x(\cdot) \in \Omega$, where $s_i = t_k + \frac{i}{n}(t - t_k)$, $i = 0, 1, 2, \ldots, n$.

Let us denote

$$\Gamma_n = \left\{ y \in Y : y = \frac{1}{n} \sum_{i=1}^{n} f(s_i, p_k(s_i).x(s_i)), x(\cdot) \in \Omega \right\}.$$

Then (17) means that the sequence of sets $\{(t - t_k).\Gamma_n\}_{n=1}^{\infty}$ tends to $\int_{t_k}^{t} f(s, p_k(s).\Omega)\,ds$ in the Hausdorff metric. Hence $\lim_{n\to\infty} \chi_H[(t - t_k)\Gamma_n] = \chi_H[\int_{t_k}^{t} f(s, p_k(s)\Omega)\,ds]$, because χ_H is continuous with respect to the Hausdorff convergence. Hence, to prove (16), we need to show that

$$\frac{1}{p_k(t)}(t - t_k)\chi_H(\Gamma_n) \leq \widetilde{k}(t_{k+1} - t_k) \max_{t_k \leq t \leq t_{k+1}} \chi_H(\Omega_t). \tag{18}$$

We shall prove that $\chi_H(\Gamma_n) \leq p_k(t)\widetilde{k} \max_{t_k \leq t \leq t_{k+1}} \chi_H(\Omega_t)$ which, because of $t - t_k < t_{k+1} - t_k$ and $p_k(t) > 0$, implies (18).

Denoting $Q_n = \bigcup_{i=1}^{n} f(s_i, p_k(s_i)\Omega_{s_i})$, we have $\chi_H(Q_n) = \max_{i=1,\dots,n} \chi_H[f(s_i, p_k(s_i)\Omega_{s_i})]$ because χ_H is semiadditive. Since $\Gamma_n \subset \overline{\text{co}}Q_n$, then taking into account (7) and the fact that $p_k(s)$ is increasing we have

$$\chi_H(\Gamma_n) \le \chi_H(\overline{\text{co}}Q_n) = \chi_H(Q_n) \le \max_{i=1,\dots,n} \widetilde{k}.\chi_H[p_k(s_i).\Omega_{s_i}]$$

$$\le \max_{i=1,\dots,n} \widetilde{k}.p_k(s_i).\chi_H(\Omega_{s_i}) \le p_k(t).\widetilde{k}. \max_{t_k \le t \le t_{k+1}} \chi_H(\Omega_t).$$

We thus proved (15) for $t \in [t_k, t_{k+1})$. To prove (15) for $t = t_{k+1}$, we note that the set of functions $S(\Omega)$ is equicontinuous because $S(\Omega) \subset M$. Hence $[S(\Omega)]_t = \{y \in Y : y = y(t), y(\cdot) \in S(\Omega)\}$ is continuous in t with respect to the Hausdorff metric. Hence we can pass to the limit in (15) when $t \to t_{k+1} (t < t_{k+1})$ and thus complete the proof of the proposition.

Remark 5. The proof of Proposition 5 follows directly the proof of Lemma 3 on p. 237 in [5]. We give the proof here for reader's convenience.

Proof of Theorem 2. Let $\mathfrak{A} = [t_0, +\infty)$, $\{\widetilde{X}_\alpha\}_{\alpha \in A}$ be defined by (3) and $j : \mathfrak{A} \to \mathfrak{A}$ be defined by $j(\alpha) = \alpha$ for each $\alpha \in \mathfrak{A}$. Since $j(\alpha) = \alpha$, (B2) is satisfied. In view of the preceeding propositions, what remains to be shown is that the operator S is densifying.

Let Ω be a noncompact subset of M. Since $\Omega \subset M$ implies equicontinuity of Ω on every interval, $[t_0, \alpha]$, $\alpha > t_0$, there is $t \ge t_0$ such that Ω_t is noncompact in Y. Let k be the integer with the property: Ω_t is compact for all $t \le t_k$ (if there is $t \in [t_0, t_1]$ such that Ω_t is noncompact, $k = 0$ by definition) and there is $t \in (t_k, t_{k+1}]$ such that Ω_t is noncompact. Then in view of (8) and (15) we have $\max_{t_k \le t \le t_{k+1}} \chi_H[S(\Omega)] < \widetilde{\chi}_{t_{k+1}}(\Omega)$. Further on (15) implies $\widetilde{\chi}_{t_k}[S(\Omega)] = 0$, because Ω_t is compact for $t \le t_k$. Hence $\widetilde{\chi}_{t_k}[S(\Omega)] < \widetilde{\chi}_{t_{k+1}}(\Omega)$, which means that $S : M \to M$ is densifying. Applying Theorem 1 we end the proof.

Remark 6. Under the additional assumption $t_{k+1} - t_k \le 1$ Theorem 2 is valid with $p_k(t) = \exp\{(t_{k+1} - t_k)^{-1}\}/\exp\{(t_{k+1} - t)^{-1}\}$. We omit the calculations, because they are long and boring, but in principal the same as in the proof of Proposition 2.

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