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INTEGRAL INEQUALITIES OF HADAMARD TYPE FOR LOG-CONVEX FUNCTIONS

Abstract. In this paper, we derive some integral inequalities for log-convex functions which are closely connected with the classical inequality due to Hermite–Hadamard.

1. Introduction

In what follows, I will be an interval of real numbers. Recall that the mapping $f : I \to \mathbb{R}$ is said to be *convex* on I, if for all $x, y \in I$ and $t \in [0,1]$ one has the inequality

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y). \tag{1.1}$$

A function $f : I \to (0, \infty)$ is said to be *log-convex* or *multiplicatively convex* if $\log f$ is convex, or, equivalently, if for all $x, y \in I$ and $t \in [0,1]$ one has the inequality ([1], p. 7)

$$f(tx + (1-t)y) \le [f(x)]^t [f(y)]^{1-t}. \tag{1.2}$$

We note that, if f, g are convex and g is increasing, then $g \circ f$ is convex; moreover, since $f = \exp(\log f)$, it follows that a log-convex function is convex, but not conversely ([1], p. 7). This follows directly from (1.2) since by the arithmetic-geometric mean inequality, we have $[f(x)]^t[f(y)]^{1-t} \le tf(x) + (1-t)f(y)$ for all $x, y \in I$ and $t \in [0,1]$.

The following relation (see e.g. [1], p. 137) is well-known in the literature as Hermite-Hadamard's inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}, \tag{1.3}$$

where $f : I \to \mathbb{R}$ is a convex map on the interval I, and $a, b \in I$ with $a < b$.

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For some very recent results related to this classical result see the papers [3]–[11] or the books [1], [2].

Note that, if we apply (1.3) for the log-convex functions $f : I \to (0, \infty)$, we have

$$\ln f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \ln f(x)\,dx \le \frac{\ln f(a) + \ln f(b)}{2}$$

implying

$$f\left(\frac{a+b}{2}\right) \le \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \le \sqrt{f(a)f(b)} \tag{1.4}$$

which is an inequality of Hadamard type for log-convex functions.

2. Results

Let us denote by $A(a, b)$ the *arithmetic mean* of the nonnegative real numbers a, b and by $G(a, b)$ the *geometric mean* of the same numbers. Note that, by the use of these notations, Hadamard's inequality (1.3) can be written in the form

$$f(A(a,b)) \le \frac{1}{b-a}\int_a^b A(f(x), f(a+b-x))\,dx \le A(f(a), f(b)), \tag{2.1}$$

since

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.$$

We now prove a similar result for log-convex mappings and geometric means.

THEOREM 2.1. *Let $f : I \to (0, \infty)$ be a log-convex mapping on I and $a, b \in I$ with $a < b$. Then one has the inequality*

$$f(A(a,b)) \le \frac{1}{b-a}\int_a^b G(f(x), f(a+b-x))\,dx \le G(f(a), f(b)). \tag{2.2}$$

Proof. Since f is log-convex, we have for all $t \in [0, 1]$ the inequalities

$$f(ta + (1-t)b) \le [f(a)]^t [f(b)]^{1-t}, \quad f((1-t)a + tb) \le [f(a)]^{1-t}[f(b)]^t.$$

Multiplying them and taking the square roots, we obtain

$$G(f(ta + (1-t)b), f((1-t)a + tb)) \le G(f(a), f(b)) \tag{2.3}$$

for all $t \in [0, 1]$.

Integrating both members of (2.3) on $[0,1]$ over t, we have

$$\int_0^1 G(f(ta+(1-t)b), f((1-t)a+tb))\,dt \le G(f(a), f(b)).$$

If we change the variable $x := ta + (1-t)b,\ t \in [0,1]$, we get

$$\int_0^1 G(f(ta+(1-t)b),\ f((1-t)a+tb))dt = \frac{1}{b-a}\int_a^b G(f(x), f(a+b-x))\,dx$$

and the second inequality in (2.2) is proved. Now, by (1.2), for $t = 1/2$, we have $f(\frac{x+y}{2}) \le \sqrt{f(x)f(y)}$ for all $x, y \in I$. If we choose $x = ta+(1-t)b$, $y = (1-t)a + tb$, we get the inequality

$$f\left(\frac{a+b}{2}\right) \le G(f(ta+(1-t)b),\ f((1-t)a+tb)) \tag{2.4}$$

for all $t \in [0,1]$. Proceeding as above, we obtain the first inequality in (2.2). This proves the theorem.

COROLLARY 2.2. *With the above assumptions and if f is nondecreasing on I, we have the inequality*

$$f(G(a,b)) \le \frac{1}{b-a}\int_a^b G(f(x), f(a+b-x))\,dx \le G(f(a), f(b)). \tag{2.5}$$

The following result offers another inequality of Hadamard type for convex functions

COROLLARY 2.3. *Let $f : I \to \mathbb{R}$ be a convex function I and $a, b \in I$ with $a < b$. Then one has the inequalities*

$$f\left(\frac{a+b}{2}\right) \le \ln\left[\frac{1}{b-a}\int_a^b \exp\left[\frac{f(x)+f(a+b-x)}{2}\right]dx\right] \le \frac{f(a)+f(b)}{2}. \tag{2.6}$$

Proof. Define the mapping $g : I \to (0,\infty)$, $g(x) = \exp f(x)$ which is clearly log-convex on I. If we now apply Theorem 2.1, we get

$$\exp f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \sqrt{\exp f(x)\cdot \exp f(a+b-x)}\,dx$$
$$\le \sqrt{\exp f(a)\cdot \exp f(b)}$$

implying

$$\exp f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \exp\left[\frac{f(x)+f(a+b-x)}{2}\right] dx$$
$$\leq \exp\left(\frac{f(a)+f(b)}{2}\right).$$

Applying then the mapping log, we deduce the desired result (2.6).

COROLLARY 2.4. *Let $f: I \to \mathbb{R}$ be a convex function on the interval I, and $a, b \in I$ with $a < b$. If f is symmetrical with respect to the point $\frac{a+b}{2} \in I$, then one has the inequality*

$$f\left(\frac{a+b}{2}\right) \leq \ln\left[\frac{1}{b-a}\int_a^b \exp f(x)\, dx\right] \leq f(a). \tag{2.7}$$

The proof follows easily from Theorem 2.1 and we omit the details.

THEOREM 2.5. *Let $f: I \to (0, \infty)$ be a log-convex mapping on I and $a, b \in I$ with $a < b$. Then one has the inequalities*

$$f\left(\frac{a+b}{2}\right) \leq \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\, dx\right] \tag{2.8}$$
$$\leq \frac{1}{b-a}\int_a^b G(f(x), f(a+b-x))\, dx$$
$$\leq \frac{1}{b-a}\int_a^b f(x)\, dx \leq L(f(a), f(b))$$

where $L(p,q)$ is the logarithmic mean of the strictly positive real numbers p, q, i.e.,

$$L(p,q) = \frac{p-q}{\ln p - \ln q}, \quad \text{if } p \neq q \quad \text{and} \quad L(p,p) = p.$$

Proof. The first inequality of (2.8) is that of (1.4). Integrating the equality

$$G(f(x), f(a+b-x)) = \exp[\ln(G(f(x), f(a+b-x)))]$$

on $[a, b]$ and using the well-known Jensen's integral inequality for the convex mapping $\exp(\cdot)$, we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b G(f(x), f(a+b-x))\,dx \\
&= \frac{1}{b-a}\int_a^b \exp[\ln(G(f(x), f(a+b-x)))]\,dx \\
&\geq \exp\left[\frac{1}{b-a}\int_a^b \ln[G(f(x), f(a+b-x))]\,dx\right] \\
&= \exp\left[\frac{1}{b-a}\int_a^b \left(\frac{\ln f(x) + \ln(f(a+b-x))}{2}\right)dx\right] \\
&= \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right],
\end{aligned}$$

since obviously

$$\int_a^b \ln f(x)\,dx = \int_a^b \ln f(a+b-x)\,dx.$$

So the second inequality in (2.8) is proved.

By the *arithmetic mean-geometric mean* inequality, we have that

$$G(f(x), f(a+b-x)) \leq \frac{f(x)+f(a+b-x)}{2}, \quad x \in [a,b],$$

from which, by integration, we get

$$\frac{1}{b-a}\int_a^b G(f(x), f(a+b-x))\,dx \leq \frac{1}{b-a}\int_a^b f(x)\,dx$$

and the third inequality of (2.8) is proved.

To prove the last inequality, we observe that, by the log-convexity of f,

$$f(ta+(1-t)b) \leq [f(a)]^t[f(b)]^{1-t} \tag{2.9}$$

for all $t \in [a,b]$. Integrating (2.9) over t in $[0,1]$, we have

$$\int_0^1 f(ta+(1-t)b)\,dt \leq \int_0^1 [f(a)]^t[f(b)]^{1-t}\,dt.$$

As

$$\int_0^1 f(ta+(1-t)b)\,dt = \frac{1}{b-a}\int_a^b f(x)\,dx$$

and

$$\int_0^1 [f(a)]^t [f(b)]^{1-t}\,dt = f(b)\int_0^1 \left(\frac{f(a)}{f(b)}\right)^t dt$$

$$= f(b)\left[\frac{\left(\frac{f(a)}{f(b)}\right)^t}{\ln\left(\frac{f(a)}{f(b)}\right)}\right]\Bigg|_0^1 = f(b)\left[\frac{\frac{f(a)}{f(b)}-1}{\ln\left(\frac{f(a)}{f(b)}\right)}\right] = L(f(a), f(b)),$$

the Theorem 2.5 is proved.

COROLLARY 2.6. *Let $f : I \to \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. Then one has the inequality*

$$\exp f\left(\frac{a+b}{2}\right) \le \exp\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right) \tag{2.10}$$

$$\le \frac{1}{b-a}\int_a^b \exp\left[\frac{f(x)+f(a+b-x)}{2}\right] dx$$

$$\le \frac{1}{b-a}\int_a^b \exp f(x)\,dx \le E(f(a), f(b))$$

where E is the exponential mean, i.e.

$$E(p,q) = \frac{\exp p - \exp q}{p-q} \quad \text{for } p \ne q \quad \text{and} \quad E(p,p) = \exp p.$$

Proof. The mapping $g : I \to (0,\infty)$, $g(x) = \exp f(x)$ is log-convex. Now, if we apply Theorem 2.5 for the mapping g, we easily deduce the inequality (2.10). We omit the details.

Remark 2.7. Note that the inequality

$$\exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] \tag{2.11}$$

$$\le \frac{1}{b-a}\int_a^b G(f(x), f(a+b-x))\,dx \le \frac{1}{b-a}\int_a^b f(x)\,dx$$

holds for every strictly positive and integrable mapping $f : [a,b] \to \mathbb{R}$ and the inequality

$$\exp\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right) \le \frac{1}{b-a}\int_a^b \exp\left[\frac{f(x)+f(a+b-x)}{2}\right] dx \tag{2.12}$$

$$\leq \frac{1}{b-a}\int_a^b \exp f(x)\,dx$$

holds for every map f on $[a,b]$, $f:[a,b]\to\mathbb{R}$ integrable. Taking into account that the above two inequalities hold, we can assert that for every map $f:[a,b]\to(0,\infty)$ integrable on $[a,b]$ we have the inequalities

$$\begin{aligned}(2.13)\qquad \exp\left[\frac{1}{b-a}\int_a^b \ln f(x)\,dx\right] &\leq \frac{1}{b-a}\int_a^b G(f(x),f(a+b-x))\,dx\\ &\leq \frac{1}{b-a}\int_a^b f(x)\,dx\\ &\leq \ln\left[\frac{1}{b-a}\int_a^b \exp A(f(x),f(a+b-x))\,dx\right]\\ &\leq \ln\left[\frac{1}{b-a}\int_a^b \exp f(x)\,dx\right],\end{aligned}$$

which is of interest by itself.

3. Applications

1). Let $p\geq 1$ and $a,b\in[0,\infty)$ with $a<b$. Then one has the inequalities

$$(3.1)\qquad \left(\frac{a+b}{2}\right)^p \leq \ln\left[\frac{1}{b-a}\int_a^b \exp\left[\frac{x^p+(a+b-x)^p}{2}\right]dx\right] \leq \frac{a^p+b^p}{2},$$

$$\begin{aligned}(3.2)\qquad \exp\left(\frac{a+b}{2}\right)^p &\leq \exp\left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]\\ &\leq \frac{1}{b-a}\int_a^b \exp\left[\frac{x^p+(a+b-x)^p}{2}\right]dx\\ &\leq \frac{1}{b-a}\int_a^b \exp x^p\,dx \leq \frac{\exp b^p-\exp a^p}{b^p-a^p},\end{aligned}$$

where (3.1) follows from Corollary 2.3 and (3.2) from Corollary 2.6 applied for the convex mapping $f:[0,\infty)\to[0,\infty)$, $f(x)=x^p$. We omit the details.

2). Consider $a,b \in (0,\infty)$ with $0 < a < b$ and the convex mapping $f:(0,\infty)\to\mathbb{R}$, $f(x) = -\ln x$. Applying Corollary 2.3, we get

$$-\ln\left(\frac{a+b}{2}\right) \le \ln\left[\frac{1}{b-a}\int_a^b \exp\left(-\left[\frac{\ln x+\ln(a+b-x)}{2}\right]\right)dx\right] \le -\frac{\ln a+\ln b}{2}$$

which is equivalent to

$$\frac{a+b}{2} \ge \left(\frac{1}{b-a}\int_a^b \frac{dx}{\sqrt{x(a+b-x)}}\right)^{-1} \ge \sqrt{ab} \tag{3.3}$$

i.e. a refinement of the $A-G$ inequality. If we apply Corollary 2.6 for the same mapping, we get

$$\exp\left[-\ln\left(\frac{a+b}{2}\right)\right] \le \exp\left[-\frac{1}{b-a}\int_a^b \ln x\,dx\right]$$
$$\le \frac{1}{b-a}\int_a^b \exp\left[-\frac{\ln x+\ln(a+b-x)}{2}\right]dx$$
$$\le \frac{1}{b-a}\int_a^b \exp[-\ln x]\,dx \le \frac{\exp(-\ln b)-\exp(-\ln a)}{-\ln b+\ln a}.$$

Having

$$\frac{1}{b-a}\int_a^b \ln x\,dx = \ln\left[\left(\frac{b^b}{a^a}\right)^{1/(b-a)}\cdot\frac{1}{e}\right],$$

we deduce from the above inequality,

$$\frac{2}{a+b} \le e\left(\frac{a^a}{b^b}\right)^{1/(b-a)} \le \frac{1}{b-a}\int_a^b \frac{dx}{\sqrt{x(a+b-x)}} \le \frac{\ln b-\ln a}{b-a} \le \frac{b-a}{ba(\ln b-\ln a)} \tag{3.4}$$

or, equivalently,

$$A(a,b) \ge I(a,b) \ge \left(\frac{1}{b-a}\int_a^b \frac{dx}{\sqrt{x(a+b-x)}}\right)^{-1} \ge L(a,b) \ge \frac{ba(\ln b-\ln a)}{b-a}, \tag{3.5}$$

where $A(a,b)=\frac{a+b}{2}$-arithmetic mean, $I(a,b)=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}$-identric mean $L(a,b)=\frac{b-a}{\ln b-\ln a}$-logarithmic mean.

3). Consider $a,b\in(0,\infty)$ with $0<a<b$ and the convex mapping $f:(0,\infty)\to(0,\infty)$, $f(x)=\frac{1}{x}$. Applying Corollary 2.3, we get

$$\frac{2}{a+b}\le\ln\left[\frac{1}{b-a}\int_a^b\exp\left[\frac{\frac{a+b}{2}}{x(a+b-x)}\right]dx\right]\le\frac{\frac{1}{a}+\frac{1}{b}}{2}$$

which is equivalent to

$$A(a,b)\ge\left(\ln\left[\frac{1}{b-a}\int_a^b\exp\left[\frac{\frac{a+b}{2}}{x(a+b-x)}\right]dx\right]\right)^{-1}\ge H(a,b), \tag{3.6}$$

where $H(a,b)=2(a^{-1}+b^{-1})^{-1}$ is the harmonic mean of $a,b>0$. Now, if we apply Corollary 2.6 to f we easily obtain the inequality

$$\exp\left(\frac{2}{a+b}\right)\le\left(\frac{b}{a}\right)^{1/(b-a)}\le\frac{1}{b-a}\int_a^b\exp\left[\frac{\frac{a+b}{2}}{x(a+b-x)}\right]dx$$
$$\le\frac{1}{b-a}\int_a^b\exp\left(\frac{1}{x}\right)dx\le\frac{\exp(\frac{1}{a})-\exp(\frac{1}{b})}{a^{-1}-b^{-1}}.$$

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