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SEMI-RIEMANNIAN TRANSVERSAL MAPS

Abstract. A generalization of semi-Riemannian submersions to semi-Riemannian transversal maps is given. Also a fundamental equation of a regular, normal semi-Riemannian transversal map is obtained.

1. Introduction

A major flaw in Riemannian and semi-Riemannian geometry (as compared to other subjects) is a shortage of suitable kind of functions from one manifold to another that will compare their geometric properties. A class of such functions in Riemannian geometry was introduced by Fischer [1] and are called Riemannian maps. Later on, this concept was generalized to semi-Riemannian geometry by the author [2] and these functions are called semi-Riemannian maps.

A kind of such functions that compare the geometric properties of semi-Riemannian manifolds are semi-Riemannian submersions [4], [5]. Here we will generalize these functions from semi-Riemannian submersions to semi-Riemannian transversal maps to a semi-Riemannian foliation. Also, we will obtain a fundamental equation of a regular, normal semi-Riemannian transversal map to a semi-Riemannian foliation.

2. Preliminaries

Let E_1 and E_2 be inner product spaces (possibly indefinite) with inner products h_1 and h_2, respectively, and let $f : E_1 \to E_2$ be a linear map. The *transpose* ${}^t f : E_2 \to E_1$ of f is defined by $h_1({}^t f x, y) = h_2(x, fy)$, where $x \in E_2$ and $y \in E_1$. The *indefinite square norm* $\|f\|^2$ of f with respect to inner products h_1 and h_2 is defined by $\|f\|^2 = tr({}^t f \circ f)$.

Let (M_2, g_2) be a semi-Riemannian manifold with metric g_2 and let $\phi =$

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$\{N_\alpha\}_{\alpha\in A}$ be a foliation on M_2 such that leaves of ϕ are (immersed) semi-Riemannian submanifolds $(N_\alpha, g_{2_{N_\alpha}})$ with induced metric $g_{2_{N_\alpha}}$ of constant index for all N_α. Then we will call (ϕ, g_{2_ϕ}) a *semi-Riemannian foliation* on (M_2, g_2). Also let $T(M_2,\phi) = \bigcup_{\alpha\in A}\bigcup_{p\in N_\alpha} T_pN_\alpha$ be the *tangent bundle* of (ϕ, g_{2_ϕ}) and let $T(M_2,\phi)^\perp = \bigcup_{\alpha\in A}\bigcup_{p\in N_\alpha} T_pN_\alpha^\perp$ be the *normal bundle* of (ϕ, g_{2_ϕ}).

Let (M_1, g_1) be a semi-Riemannian manifold with metric g_1 and let $f:(M_1,g_1)\to(M_2,g_2)$ be a smooth function. Also let (ϕ, g_{2_ϕ}) be a semi-Riemannian foliation on (M_2, g_2). Recall that f is called *transversal* to (ϕ, g_{2_ϕ}) if $f_{*_{p_1}}(T_{p_1}M_1) + T_{f(p_1)}(M_2,\phi) = T_{f(p_1)}M_2$ for every $p_1\in M_1$. Note that then $f^{-1}(N_\alpha)$, $\alpha\in A$, are the leaves of a foliation on M_1 denoted by $f^*(\phi)$.

3. Transversal semi-Riemannian maps

Let $f:(M_1,g_1)\to(M_2,g_2)$ be a smooth function and let $p_2 = f(p_1)$. Define a linear function $f^\perp_{*_{p_1}}: T_{p_1}M_1 \to T_{p_2}(M_2,\phi)^\perp$ by $f^\perp_{*_{p_1}}x = (f_{*_{p_1}}x)^\perp$, where $(f_{*_{p_1}}x)^\perp$ is the component of $f_{*_{p_1}}x$ in $T_{p_2}(M_2,\phi)^\perp$.

Now define

$$L_1(p_1) = (\ker f^\perp_{*_{p_1}}) \cap (\ker f^\perp_{*_{p_1}})^\perp \subseteq T_{p_1}M_1,$$
$$L_2(p_2) = (\mathrm{range}\, f^\perp_{*_{p_1}}) \cap (\mathrm{range}\, f^\perp_{*_{p_1}})^\perp \subseteq T_{p_2}M_2.$$

Note that $L_1(p_1)$ is the degenerate space of the restrictions of metric $g_{1_{p_1}}$ to $(\ker f^\perp_{*_{p_1}})$ and $(\ker f^\perp_{*_{p_1}})^\perp$. Also $L_2(p_2)$ is the degenerate space of the restriction of metric $g_{2_{p_2}}$ to $(\mathrm{range}\, f^\perp_{*_{p_1}})$. Also define

$$A_1(p_1) = (\ker f^\perp_{*_{p_1}}) + (\ker f^\perp_{*_{p_1}})^\perp, A_2(p_2) = (\mathrm{range}\, f^\perp_{*_{p_1}}).$$

Next we will define some quotient spaces out of $V(p_1) = (\ker f^\perp_{*_{p_1}})$, $H(p_1) = (\ker f^\perp_{*_{p_1}})$, $A_1(p_1)$ and $A_2(p_2)$:

(1) $\overline{V}(p_1) = V(p_1)/L_1(p_1)$ and $\pi_1 : V(p_1)\to\overline{V}(p_1)$ is the natural projection,

(2) $\overline{H}(p_1) = H(p_1)/L_1(p_1)$ and $\pi_1 : H(p_1)\to\overline{H}(p_1)$ is the natural projection,

(3) $\overline{A}_1(p_1) = A(p_1)/L_1(p_1)$ and $\pi_1 : A_1(p_1)\to\overline{A}_1(p_1)$ is the natural projection, (note that, π_1 of (1) and (2) is the restriction of π_1 of this case to $V(p_1)$ and $H(p_1)$, respectively),

(4) $\overline{A}_2(p_2) = A_2(p_2)/L_2(p_2)$ and $\pi_2 : A_2(p_2)\to\overline{A}_2(p_2)$ is the natural projection.

Also we can define (nonsingular) inner products in $\overline{V}(p_1)$, $\overline{H}(p_1)$, $\overline{A}(p_1)$ and $\overline{A}_2(p_1)$ as follows:

(1′) in $\overline{V}(p_1)$, let $\overline{g}_{p_1}(\overline{x},\overline{y}) = g_{1_{p_1}}(x,y)$, where $x,y \in V(p_1)$ with $\pi_1(x) = \overline{x}, \pi_1(y) = \overline{y}$;

(2′) in $\overline{H}(p_1)$, let $\overline{g}_{1_{p_1}}(\overline{x},\overline{y}) = g_{1_{p_1}}(x,y)$, where $x,y \in H(p_1)$ with $\pi_1(x) = \overline{x}$, $\pi_1(y) = \overline{y}$;

(3′) in $\overline{A}_1(p_1)$, let $\overline{g}_{1_{p_1}}(\overline{x},\overline{y}) = g_{1_{p_1}}(x,y)$, where $x,y \in A_1(p_1)$ with $\pi_1(x) = \overline{x}$, $\pi_1(y) = \overline{y}$ (note that the inner products $\overline{g}_{1_{p_1}}$ in (1′) and (2′) are the restrictions of $\overline{g}_{1_{p_1}}$ of this case to $\overline{V}(p_1)$ and $\overline{H}(p_1)$, respectively);

(4′) in $\overline{A}_2(p_2)$, let $\overline{g}_{2_{p_2}}(\overline{x},\overline{y}) = g_{2_{p_2}}(x,y)$, where $x,y \in A_2(p_2)$ with $\pi_2(x) = \overline{x}$, $\pi_2(y) = \overline{y}$.

Finally, if $f : (M_1, g_1) \to (M_2, g_2)$ is a smooth function, define a map

$$\overline{f}^{\perp}_{*_{p_1}} : \overline{H}(p_1) \to \overline{A}_2(p_1) \text{ by } \overline{f}^{\perp}_{*_{p_1}} \overline{x} = \pi_2(f^{\perp}_{*_{p_1}} x),$$

where $x \in H(p_1)$ with $\pi_1(x) = \overline{x}$ (it is easy to check that $\overline{f}^{\perp}_{*_{p_1}}$ is well-defined).

Remarks 3.1. a) Observe that $\overline{A}_1(p_1) = \overline{V}(p_1) \oplus \overline{H}(p_1)$.

b) Note that $\overline{f}^{\perp}_{*p_1}$ may not be either injective or surjective. In fact,

$$\operatorname{rank} \overline{f}^{\perp}_{*p_1} \leq \operatorname{rank} f^{\perp}_{*p_1} - \dim L_1(p_1).$$

DEFINITION 3.2. Let $f : (M_1, g_1) \to (M_2, g_2)$ be a smooth function and let (ϕ, g_{2_ϕ}) be a semi-Riemannian foliation on (M_2, g_2). The nondegenerate transversal rank of f with respect to (ϕ, g_{2_ϕ}) at $p_1 \in M_1$ is defined to be $\operatorname{rank} \overline{f}^{\perp}_{*p_1}$.

Note that $(T_{p_1}M_1, g_{p_1})$ and $(T_{p_2}(M_2,\phi)^{\perp}, g_{2_{p_2}})$ are possibly indefinite inner product spaces as well as $(\overline{H}(p_1), \overline{g}_{p_1})$ and $(\overline{A}_2(p_2), \overline{g}_{p_2})$. Hence we can talk about indefinite square norms $\|f^{\perp}_{*p_1}\|^2$ and $\|\overline{f}^{\perp}_{*p_1}\|^2$.

THEOREM 3.3. *Let $f : (M_1, g_1) \to (M_2, g_2)$ be a smooth function and let (ϕ, g_{2_ϕ}) be a semi-Riemannian foliation on (M_2, g_2). Then $\|f^{\perp}_{*p_1}\|^2 = \|\overline{f}^{\perp}_{*p_1}\|^2$ at each $p_1 \in M_1$.*

Proof. Let $\{x_1, \ldots, x_l\}$ be an orthonormal basis for a (nondegenerate) complementary space to $L_1(p_1)$ in $V(p_1)$ and $\{y_1, \ldots, y_m\}$ be an orthonormal basis for a (nondegenerate) complementary space to $L_1(p_1)$ in $H(p_1)$. Also let $\{z_1, w_1, \ldots, z_k, w_k\}$ be an orthonormal basis for $(\operatorname{span}\{x_1, \ldots, x_l, y_1, \ldots, y_m\})^{\perp}$ such that $u_i = z_i + w_i \in L_1(p_1)$ for $i = l, \ldots, k$. Then $\{x_1, \ldots, x_l, y_1, \ldots, y_m, z_1, w_1, \ldots, z_k, w_k\}$ is an orthonormal basis for $T_{p_1}M_1$

and hence

$$\|f_{*_{p_1}}^{\perp}\|^2 = \sum_{i=1}^{k} g_{1_{p_1}}(z_i, z_i) g_{2_{p_2}}(f_{*_{p_1}}^{\perp} z_i, f_{*_{p_1}}^{\perp} z_i) + \sum_{i=1}^{k} g_{1_{p_1}}(w_i, w_i) g_{2_{p_2}}(f_{*_{p_1}}^{\perp} w_i, f_{*_{p_1}}^{\perp} w_i) + \sum_{i=1}^{k} g_{1_{p_1}}(y_i, y_i) g_{2_{p_2}}(f_{*_{p_1}}^{\perp} y_i, f_{*_{p_1}}^{\perp} y_i).$$

But since $u_i = z_i + w_i \in L_1(p_1) \subseteq V(p_1)$, we have $0 = f_{*_{p_1}}^{\perp} u_i = f_{*_{p_1}}^{\perp} z_i + f_{*_{p_1}}^{\perp} w_i$ and $0 = g_{1_{p_1}}(u_i, u_i) = g_{1_{p_1}}(z_i, z_i) + g_{1_{p_1}}(w_i, w_i)$. Hence, $f_{*_{p_1}}^{\perp} z_i = -f_{*_{p_1}}^{\perp} w_i$ and $g_{1_{p_1}}(z_i, z_i) = -g_{1_{p_1}}(w_i, w_i)$. Thus

$$\|f_{*_{p_1}}^{\perp}\|^2 = \sum_{i=1}^{m} g_{1_{p_1}}(y_i, y_i) g_{2_{p_2}}(f_{*_{p_1}}^{\perp} y_i, f_{*_{p_1}}^{\perp} y_i) = \sum_{i=1}^{m} \overline{g}_{1_{p_1}}(\overline{y}_i, \overline{y}_i) \overline{g}_{2_{p_2}}(\overline{f}_{*_{p_1}}^{\perp} \overline{y}_i, \overline{f}_{*_{p_1}}^{\perp} \overline{y}_i) = \|\overline{f}_{*_{p_1}}^{\perp}\|^2,$$

where $\pi_1(y_i) = \overline{y}_i$. ∎

DEFINITION 3.4. Let $f : (M_1, g_1) \to (M_2, g_2)$ be a smooth function and (ϕ, g_{2_ϕ}) be a semi-Riemannian foliation on (M_2, g_2). Then f is said to satisfy the generalized transversal eikonal equation at $p_1 \in M_1$ with respect to (ϕ, g_{2_ϕ}) *if* $\|f_{*_{p_1}}^{\perp}\|^2 = \operatorname{rank} \overline{f}_{*_{p_1}}^{\perp}$.

Remark 3.5. Note that, if f satisfies the generalized transversal eikonal equation at each $p_1 \in M_1$ with respect to (ϕ, g_{2_ϕ}), then, since $\|f_{*_{p_1}}^{\perp}\|^2$ is continuous on M_1, rank $\overline{f}_{*_{p_1}}^{\perp}$ is constant on the connected components of M_1.

Next we will define certain kind of functions which are solutions of the generalized transversal eikonal equation.

DEFINITION 3.6. Let $f : (M_1, g_1) \to (M_2, g_2)$ be a smooth function and (ϕ, g_{2_ϕ}) be a semi-Riemannian foliation on (M_2, g_2). Then f is called a transversal semi-Riemannian map with respect to (ϕ, g_{2_ϕ}) at $p_1 \in M_1$ if $\overline{f}_{*_{p_1}}^{\perp} : \overline{H}(p_1) \to \overline{A}_2(p_2)$ is an (into) isometry. The function f is called a transversal semi-Riemannian map on M_1 with respect to (ϕ, g_{2_ϕ}), if it is a transversal semi-Riemannian map with respect to (ϕ, g_{2_ϕ}) at each $p_1 \in M_1$.

THEOREM 3.7. *If $f : (M_1, g_1) \to (M_2, g_2)$ is a transversal semi-Riemannian map with respect to (ϕ, g_{2_ϕ}) at $p_1 \in M_1$, then f satisfies the generalized transversal eikonal equation with respect to (ϕ, g_{2_ϕ}) at $p_1 \in M_1$.*

Proof. First we will show that $\overline{F}_{p_1} = {}^t\overline{f}^{\perp}_{*_{p_1}} \circ \overline{f}^{\perp}_{*_{p_1}} : \overline{H}(p_1) \to \overline{H}(p_1)$ is the identity map. Indeed, note that for every $\overline{x}, \overline{y} \in \overline{H}(p_1)$ we can write

$$\overline{g}_{1_{p_1}}(\overline{F}_{p_1}\overline{x}, \overline{y}) = \overline{g}_{2_{p_2}}(\overline{f}^{\perp}_{*_{p_1}}\overline{x}, \overline{f}^{\perp}_{*_{p_1}}\overline{y}) = \overline{g}_{2_{p_2}}(\overline{x}, \overline{y}).$$

Hence $\overline{F}_{p_1} = \overline{\mathrm{id}}$ and it follows from Theorem 3.3 that

$$\|f^{\perp}_{*_{p_1}}\|^2 = \|\overline{f}^{\perp}_{*_{p_1}}\|^2 = \operatorname{tr}\overline{F}_{p_1} = \dim \overline{H}(p_1) = \operatorname{rank}\overline{f}^{\perp}_{*_{p_1}}. \qquad \blacksquare$$

Remark 3.8. Let $f : (M_1, g_1) \to (M_2, g_2)$ be a transversal semi-Riemannian map with respect to (ϕ, g_{2_ϕ}) at $p_1 \in M_1$. Then, since $\overline{f}^{\perp}_{*_{p_1}}$ is injective (yet may not be surjective), $\operatorname{rank}\overline{f}^{\perp}_{*_{p_1}} = \operatorname{rank} f^{\perp}_{*_{p_1}} - \dim L_1(p_1)$. Furthermore, if f is transversal semi-Riemannian map on M_1 with respect to (ϕ, g_{2_ϕ}), then, by Theorem 3.7 and Remark 3.5, since $\operatorname{rank} f^{\perp}_*$ is constant on connected components of M_1, $\dim \overline{H}(p_1) = \operatorname{rank}\overline{f}^{\perp}_*$ at each $p_1 \in M_1$, and hence constant on the connected components of M_1.

5. Semi–Riemannian transversal maps

DEFINITION 4.1. Let $f : (M_1, g_1) \to (M_2, g_2)$ be a smooth function and (ϕ, g_{2_ϕ}) be a semi-Riemannian foliation on (M_2, g_2). If

a) f is transversal to (ϕ, g_{2_ϕ}),

b) f is a transversal semi-Riemannian map on M_1 with respect to (ϕ, g_{2_ϕ}), then f is called a semi-Riemannian transversal map to (ϕ, g_{2_ϕ}).

Remark 4.2. Let $f : (M_1, g_1) \to (M_2, g_2)$ be a semi-Riemannian transversal map to (ϕ, g_{2_ϕ}), where M_1 is connected and $f(M_1)$ is a submanifold of M_2. Then, since $\operatorname{rank} f^{\perp}_{*_{p_1}} = \dim T_{p_2}(M_2, \phi)^{\perp}$ at each $p_1 \in M_1$, $\dim L_1(p_1) = \operatorname{rank} f^{\perp}_{*_{p_1}} - \operatorname{rank}\overline{f}^{\perp}_{*_{p_1}}$ is constant for all $p_1 \in M_1$. Hence, $L_1 = \bigcup_{p_1 \in M_1} L_1(p_1)$ is a smooth vector bundle over M_1, since $(\ker f^{\perp}_*)$ and $(\ker f^{\perp}_*)^{\perp}$ are smooth vector bundles over M_1. Consequently,

$$V = \bigcup_{p_1 \in M_1} V(p_1), H = \bigcup_{p_1 \in M_1} H(p_1) \text{ and } A_1 = \bigcup_{p_1 \in M_1} A_1(p_1)$$

are smooth vector bundles over M_1 and hence

$$\overline{V} = \bigcup_{p_1 \in M_1} \overline{V}(p_1), \overline{H} = \bigcup_{p_1 \in M_1} \overline{H}(p_1) \text{ and } \overline{A}_1 = \bigcup_{p_1 \in M_1} \overline{A}_1(p_1)$$

are smooth vector bundles over M_1 with $\overline{A}_1 = \overline{H} \oplus \overline{V}$. On the other hand, since

$$\operatorname{range} f^{\perp}_* = T(M_2, \phi)^{\perp}\Big|_{\mathrm{image}(f)} = A_2 = \bigcup_{p_2 \in \mathrm{image}(f)} A_2(p_2)$$

is nondegenerate, $\overline{A}_2 = \bigcup_{p_2 \in \mathrm{image}(f)} \overline{A}_2(p_1)$ is canonically isometric to A_2. Hence, range $\overline{f}_*^{\perp}$ is a nondegenerate smooth subbundle af A_2 and let us denote it by $B = \mathrm{range}\, \overline{f}_*^{\perp}$.

Remark 4.3. Let $f : (M_1, g_1) \to (M_2, g_2)$ be a semi-Riemannian transversal map to (ϕ, g_{2_ϕ}), where M_1 is connected and $f(M_1)$ is a submanifold of M_2. Note that, if rank $L_1 \neq 0$ (resp., rank $L_1 = 0$), then leaves of $f^*(\phi)$ are degenerate submanifolds (resp., (nondegenerate) semi-Riemannian submanifolds) of (M_1, g_1). In particular, if ϕ is the points of M_2, then f becomes a semi-Riemannian submersion [4].

DEFINITION 4.4. Let $f : (M_1, g_1) \to (M_2, g_2)$ be a smooth function and (ϕ, g_{2_ϕ}) be a semi-Riemannian foliation on (M_2, g_2). Then f is called normal to (ϕ, g_{2_ϕ}), if $f_{*_{p_1}} = f^{\perp}_{*_{p_1}}$ at each $p_1 \in M_1$.

PROPOSITION 4.5. *Let $f : (M_1, g_1) \to (M_2, g_2)$ be a normal semi-Riemannian transversal map to (ϕ, g_{2_ϕ}), where M_1 is connected and $f(M_1)$ is a submanifold of M_2. If $A_1 = (\ker f_*^{\perp}) + (\ker f_*^{\perp})^{\perp}$ is integrable, then $B = \mathrm{range}\, \overline{f}_*^{\perp}$ is integrable.*

Proof. First note that $f_*(A_1) = B$. Now let $X, Y \in \Gamma B$ and let $\widetilde{X}, \widetilde{Y} \in \Gamma A_1$ be lifts of X, Y to A_1. Then, since $f_*([\widetilde{X}, Y]) = [X, Y] \circ f$ and $[\widetilde{X}, \widetilde{Y}] \in \Gamma A_1$, it follows that $[X, Y] \in \Gamma B$. That is B is integrable.

5. Regular normal semi-Riemannian transversal maps

DEFINITION 5.1. A semi-Riemannian transversal map $f : (M_1, g_1) \to (M_2, g_2)$ to (ϕ, g_{2_ϕ}), where M_1 is connected, is called regular, if A_1 is a totally geodesic distribution.

Let f be a semi-Riemannian transversal map to (ϕ, g_{2_ϕ}). Then leaves of $f^*(\phi)$ may be degenerate submanifolds of (M_1, g_1). In particular, if f is regular, then these submanifolds are stationary and irrotational degenerate submanifolds of (M_1, g_1) which give rise of Gaus–Codazzi equations (see [3]). In fact, we have the following result.

PROPOSITION 5.2. *Let $f : (M_1, g_1) \to (M_2, g_2)$ be a semi-Riemannian transversal map to (ϕ, g_{2_ϕ}), where M_1 is connected. If f is regular, then A_1 is integrable and leaves of $f^*(\phi)$ are stationary, irrotational submanifolds of M_1, provided that they are degenerate.*

Proof. Note that, if A_1 is totally geodesic, then clearly A_1 is integrable, and $\overset{1}{\nabla}_x Z \in \Gamma A_1$ for every $X \in \Gamma V$, $Z \in \Gamma A_1$, where $\overset{1}{\nabla}$ is the Levi-Civita connection of M_1. Hence, it also follows from [3, Prop. 4.1.15] that leaves of $f^*(\phi)$ are stationary, irrotational submanifolds of (M_1, g_1). ■

Now we will obtain an equation relating the geometric properties of (M_1, g_1) and (M_2, g_2).

Let $f : (M_1, g_1) \to (M_2, g_2)$ be a regular normal semi-Riemannian transversal map to (ϕ, g_{2_ϕ}), where M_1 is connected and $f(M_1)$ is a submanifold of M_2. Let $\overset{1}{\nabla}$ be the Levi–Civita connection of (M_1, g_1) and $\overset{2}{D}$ be the collection of the Levi–Civita connections of the integral manifolds N_2 of B, induced by the semi-Riemannian structure of (M_2, g_2). Also let H' be a geometric realization of $\overline{H}$, that is, H' is a complement of L_1 in H. (Hence $\pi_1 : H' \to \overline{H}$ is an isometry. Note also that $f_*^\perp (= f_*) : H' \to B$ is an isometry too). Also let V' be a geometric realization of $\overline{V}$, that is, V' is a complement of L_1 in V. Hence $A_1 = H' \oplus L_1 \oplus V'$ is an orthogonal direct sum decomposition of A_1. Thus, if $U \in \Gamma A_1$, then $U = U^\perp + U^\circ + U^T$, where $U^\perp$, U° and U^T are the unique components of U in H', L_1 and V', respectively. Finally note that, since f is normal, if $X \in \Gamma B$, then there exists a unique lift $\widetilde{X} \in \Gamma H'$ of X, that is, $f_*^\perp \widetilde{X} (= f_* \widetilde{X}) = X \circ f$.

LEMMA 5.3. *Let $f : (M_1, g_1) \to (M_2, g_2)$ be a regular semi-Riemannian transversal map to (ϕ, g_{2_ϕ}), where M_1 is connected and $f(M_1)$ is a submanifold of M_2. Also let $\widetilde{X} \in \Gamma H'$ and $\widetilde{Y} \in \Gamma H'$ be lifts of $X \in \Gamma B$ and $Y \in \Gamma B$, respectively. Then*

(a) $g_1(\widetilde{X}, \widetilde{Y}) = g_2(X, Y) \circ f$,

(b) $[\widetilde{X}, \widetilde{Y}]^\perp = [X, Y]^\sim$,

(c) $(\overset{1}{\nabla}_{\widetilde{X}} \widetilde{Y})^\perp = (\overset{2}{D}_X Y)^\sim$,

(d) $(\overset{1}{\nabla}_{\widetilde{X}} \widetilde{Y})^T = \frac{1}{2}[\widetilde{X}, \widetilde{Y}]^T$.

Proof. (a) Obvious.

(b). Since $[\widetilde{X}, \widetilde{Y}]$ is f-related to $[X, Y]$, so $[\widetilde{X}, \widetilde{Y}]^\perp$ is f-related to $[X, Y]$.

(c). It suffices to show that $g_1(\overset{1}{\nabla}_{\widetilde{X}} \widetilde{Y}, \widetilde{Z}) = g_1((\overset{2}{D}_X Y)^\sim, \widetilde{Z})$ for every $\widetilde{Z} \in \Gamma H'$ being f-related to $Z \in \Gamma B$. This follows by expanding both sides in the Koszul formula. In fact, using (a) and (b), we have

$$\begin{aligned} g_1(\overset{1}{\nabla}_{\widetilde{X}} \widetilde{Y}, \widetilde{Z}) &= \frac{1}{2}[\widetilde{X} g_1(\widetilde{Y}, \widetilde{Z}) + \widetilde{Y} g_1(\widetilde{X}, \widetilde{Z}) - \widetilde{Z} g_1(\widetilde{X}, \widetilde{Y}) \\ &\quad + g_1([\widetilde{X}, \widetilde{Y}], \widetilde{Z}) - g_1([\widetilde{X}, \widetilde{Z}], \widetilde{Y}) - g_1([\widetilde{Y}, \widetilde{Z}], \widetilde{X})] \\ &= \frac{1}{2}[X g_2(Y, Z) + Y g_2(X, Z) - Z g_2(X, Y) \\ &\quad + g_2([X, Y], Z) - g_2([X, Z], Y) - g_2([Y, Z], X)] \circ f \\ &= g_2(\overset{2}{D}_X Y, Z) \circ f = g_1((\overset{2}{D}_X Y)^\sim, \widetilde{Z}). \end{aligned}$$

Hence, since $g_1|_{H'}$ is nondegenerate, the claim follows.

(d). Again from the Koszul formula, for $U \in \Gamma V'$, we have

$$\begin{aligned} g_1(\overset{1}{\nabla}_{\widetilde{X}}\widetilde{Y}, U) &= \frac{1}{2}[\widetilde{X}g_1(\widetilde{Y}, U) + \widetilde{Y}g_1(\widetilde{X}, U) - Ug_1(\widetilde{X}, \widetilde{Y}) \\ &\quad + g_1([\widetilde{X}, \widetilde{Y}], U) - g_1([\widetilde{X}, U], \widetilde{Y}) - g_1([\widetilde{Y}, U], \widetilde{X})] \\ &= \frac{1}{2}g_1([\widetilde{X}, \widetilde{Y}], U), \end{aligned}$$

since $g_1(\widetilde{X}, \widetilde{Y}) = g_2(X, Y) \circ f$ and U is f-related to zero vector field, so $Ug_1(\widetilde{X}, \widetilde{Y}) = 0$, and since $U \perp \widetilde{X}$, $U \perp \widetilde{Y}$ and U is f-related to zero vector field, so first and last two terms vanish. Thus, (d) holds. ∎

THEOREM 5.4. *Let $f : (M_1, g_1) \to (M_2, g_2)$ be a regular, semi-Riemannian transversal map to (ϕ, g_{2_ϕ}), where M_1 is connected and $f(M_1)$ is a submanifold of M_2. If $\widetilde{X}, \widetilde{Y} \in \Gamma H'$ are lifts from B spanning nondegenerate planes, then*

$$K_{N_2}(f_*\widetilde{X}, f_*\widetilde{Y}) = K_{M_1}(\widetilde{X}, \widetilde{Y}) + \frac{3}{4}g_1([\widetilde{X}, \widetilde{Y}]^T, [\widetilde{X}, \widetilde{Y}]^T)/Q(\widetilde{X}, \widetilde{Y}),$$

where K_{N_2} is the sectional curvature in the induced structure of integral manifolds N_2 of B, K_{M_1} is the sectional curvature in the structure of (M_1, g_1) and

$$Q(\widetilde{X}, \widetilde{Y}) = g_1(\widetilde{X}, \widetilde{X})g_1(\widetilde{Y}, \widetilde{Y}) - g_1(\widetilde{X}, \widetilde{Y})^2.$$

Proof. Let $\widetilde{X}, \widetilde{Y} \in \Gamma H'$ be lifts of $X, Y \in \Gamma B$. Then by Lemma 5.3,

$$\overset{1}{\nabla}_{\widetilde{X}}\widetilde{Y} = (\overset{2}{D}_X Y)^{\sim} + \frac{1}{2}[\widetilde{X}, \widetilde{Y}]^T + U',$$

where $U' \in \Gamma L_1$. Again by Lemma 5.3, for $U \in \Gamma V'$, since U is f-related to zero vector field,

$$\begin{aligned} (*)\quad g_1(\overset{1}{\nabla}_U\widetilde{X}, \widetilde{Y}) &= g_1(\overset{1}{\nabla}_{\widetilde{X}}U, \widetilde{Y}) + g_1([U, \widetilde{X}], \widetilde{Y}) \\ &= -g_1(U, \overset{1}{\nabla}_{\widetilde{X}}\widetilde{Y}) = -g_1(U, (\overset{1}{\nabla}_{\widetilde{X}}\widetilde{Y})^T) = -\frac{1}{2}g_1([\widetilde{X}, \widetilde{Y}]^T, U). \end{aligned}$$

Also, in particular, if $U' \in \Gamma L_1$, then $g_1(\overset{1}{\nabla}_{U'}\widetilde{X}, \widetilde{Y}) = 0$.

Now, by Lemma 5.3, since

$$\widetilde{X}g_1(\overset{1}{\nabla}_{\widetilde{Y}}\widetilde{Z}, \widetilde{W}) = (Xg_2(\overset{2}{D}_Y Z, W)) \circ f,$$

where $\widetilde{Z}, \widetilde{W} \in \Gamma H'$ are the lifts of $Z, W \in \Gamma B$, we have

$$g_1(\overset{1}{\nabla}_{\widetilde{X}}\overset{1}{\nabla}_{\widetilde{Y}}\widetilde{Z}, \widetilde{W}) = \widetilde{X}g_1(\overset{1}{\nabla}_{\widetilde{Y}}\widetilde{Z}, \widetilde{W}) - g_1(\overset{1}{\nabla}_{\widetilde{Y}}\widetilde{Z}, \overset{1}{\nabla}_{\widetilde{X}}\widetilde{W})$$

$$= (Xg_2(\overset{2}{D}_Y Z, W)) \circ f - g_1((\overset{2}{D}_Y X)^{\sim} + \frac{1}{2}[\widetilde{Y}, \widetilde{Z}]^T + U', (\overset{2}{D}_X W)^{\sim} + \frac{1}{2}[\widetilde{X}, \widetilde{W}]^T + U'')$$

$$= g_2(\overset{2}{D}_X \overset{2}{D}_Y Z, W) \circ f - \frac{1}{4} g_1([\widetilde{Y}, \widetilde{Z}]^T, [\widetilde{X}, \widetilde{W}]^T),$$

where $U', U'' \in \Gamma L_1$.

Similarly,

$$g_1(\overset{1}{\nabla}_{\widetilde{Y}} \overset{1}{\nabla}_{\widetilde{X}} \widetilde{Z}, \widetilde{W}) = g_2(\overset{2}{D}_Y \overset{2}{D}_X Z, W) \circ f - \frac{1}{4} g_1([\widetilde{X}, \widetilde{Z}]^T, [\widetilde{Y}, \widetilde{W}]^T).$$

Also, since $g_1(\overset{1}{\nabla}_{U'} \widetilde{Z}, \widetilde{W}) = 0$ for $U' \in \Gamma L_1$ and by $(*)$,

$$g_1(\overset{1}{\nabla}_{[\widetilde{X}, \widetilde{Y}]} \widetilde{Z}, \widetilde{W}) = g_1(\overset{1}{\nabla}_{[\widetilde{X}, \widetilde{Y}]^{\perp}} \widetilde{Z}, \widetilde{W}) + g_1(\overset{1}{\nabla}_{[\widetilde{X}, \widetilde{Y}]^T} \widetilde{Z}, \widetilde{W}) + g_1(\overset{1}{\nabla}_{U'} \widetilde{Z}, \widetilde{W})$$

$$= g_2(\overset{2}{D}_{[X,Y]} Z, W) \circ f - \frac{1}{2} g_1([\widetilde{Z}, \widetilde{W}]^T, [\widetilde{X}, \widetilde{Y}]^T),$$

where $U' \in \Gamma L_1$.

Thus, if $\overset{1}{R}$ is the curvature tensor of M_1 and $\overset{2}{R}$ is the curvature tensor of the inducted structure on N_2, then

$$g_1(\overset{1}{R}(\widetilde{X}, \widetilde{Y})\widetilde{Z}, \widetilde{W}) = g_1(\overset{1}{\nabla}_{\widetilde{X}} \overset{1}{\nabla}_{\widetilde{Y}} \widetilde{Z}, \widetilde{W}) - g_1(\overset{1}{\nabla}_{\widetilde{Y}} \overset{1}{\nabla}_{\widetilde{X}} \widetilde{Z}, \widetilde{W}) - g_1(\overset{1}{\nabla}_{[\widetilde{X}, \widetilde{Y}]} \widetilde{Z}, \widetilde{W})$$

$$= g_2(\overset{2}{D}_X \overset{2}{D}_Y Z, W) \circ f - \frac{1}{4} g_1([\widetilde{Y}, \widetilde{Z}]^T, [\widetilde{X}, \widetilde{W}]^T) - g_2(\overset{2}{D}_Y \overset{2}{D}_X Z, W) \circ f$$

$$+ \frac{1}{4} g_1([\widetilde{X}, \widetilde{Z}]^T, [\widetilde{Y}, \widetilde{W}]^T) - g_2(\overset{2}{D}_{[X,Y]} Z, W) \circ f + \frac{1}{2} g_1([\widetilde{Z}, \widetilde{W}]^T, [\widetilde{X}, \widetilde{Y}]^T)$$

$$= g_2(\overset{2}{R}(X, Y)Z, W) \circ f - \frac{1}{4} g_1[\widetilde{Y}, \widetilde{Z}]^T, [\widetilde{X}, \widetilde{W}]^T)$$

$$+ \frac{1}{4} g_1([\widetilde{X}, \widetilde{Z}]^T, [\widetilde{Y}, \widetilde{W}]^T) + \frac{1}{2} g_1([\widetilde{Z}, \widetilde{W}]^T, [\widetilde{X}, \widetilde{Y}]^T).$$

Hence, by setting $\widetilde{Z} = \widetilde{Y}$, $\widetilde{X} = \widetilde{W}$ and dividing both sides by $Q(\widetilde{X}, \widetilde{Y})$, the Theorem follows.

Remark 5.5. Notice that the choice of V' is dummy. Indeed, since $\pi_1([\widetilde{Z}, \widetilde{W}]^T) = (\pi_1([\widetilde{Z}, \widetilde{W}]))^T$, where $(\pi_1([\widetilde{Z}, \widetilde{W}]))^T$ is the component of $\pi_1([\widetilde{Z}, \widetilde{W}])$ in $\overline{V}$ (and similarly for $[\widetilde{X}, \widetilde{Y}]^T$), so we have

$$g_1([\widetilde{Z}, \widetilde{W}]^T, [\widetilde{X}, \widetilde{Y}]^T) = \overline{g}_1((\pi_1([\widetilde{Z}, \widetilde{W}])^T, (\pi_1([\widetilde{X}, \widetilde{Y}]))^T).$$

Hence, Theorem 5.4 states that

$$K_{N_2}(f_*\widetilde{X}, f_*\widetilde{Y})$$
$$= K_{M_1}(\widetilde{X}, \widetilde{Y}) + \frac{3}{4}\overline{g}_1((\pi_1([\widetilde{X}, \widetilde{Y}]))^T, (\pi_1([\widetilde{X}, \widetilde{Y}]))^T)/Q(\widetilde{X}, \widetilde{Y}).$$

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