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Wojciech Ślepak

ON MULTIVALUED INTEGRAL MEAN

Abstract. In the present note we prove that if a set valued function $F : [0,b] \to n(Y)$, where $n(Y)$ denote all nonempty subsets of Banach space, is convex or starshaped then the multifunction defined by the formula

$$M_F(x) := \frac{1}{x} \int_0^x F(t)\,dt, \qquad x \in (0,b],$$

mapping $(0,b]$ into the family of all convex subsets of Y is, respectively, convex or starshaped. We give conditions under which the subadditivity of F implies the subadditivity of M_F.

Introduction

G. H. Toader in [5] considered an integral mean of the form

$$m_f(x) := \frac{1}{x} \int_0^x f(t)\,dt, \qquad m_f(0) = 0,$$

where $f : [0,b] \to \mathbb{R}$ is a continuous function such that $f(0) = 0$. He proved that convexity, starshapedness and subadditivity of f imply, respectively, convexity starshapedness and subadditivity of m_f. In [6] G. H. Toader deals with "m-convex" functions.

In this paper we give the set-valued versions some of the results from [5] and [6]. Our main theorem says that the subadditivity of measurable and integrably bounded set-valued function $F : [0,b] \to cc(Y)$ implies the subadditivity of M_F. Here $cc(Y)$ stands for the family of all nonempty compact and convex subsets of the Banach space Y.

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1. Some definitions and auxiliary results

For a Banach space Y we denote by $n(Y)$, $c(Y)$ and $cc(Y)$, respectively, the class of all nonempty subsets of Y, the class of all nonempty convex subsets of Y, and the class of all nonempty convex and compact subsets of Y.

DEFINITION 1. (see [1]). A set-valued function $F : [0,b] \to n(Y)$ is called Borel measurable, if its graph, i.e. the set $\{(t,x) : x \in F(t)\}$, is a Borel subset of $[0,b] \times Y$.

DEFINITION 2. (see [1]). A set-valued function $F : [0,b] \to n(Y)$ is said to be integrably bounded, if there exists a single–valued integrable function $h : [0,b] \to Y$ such that $\|x\| \leq h(t)$ for all x and t such that $x \in F(t)$.

Let us mention that the Aumann integral of F is the set of integrals of all integrable selections of F.

REMARK 1. (see [1]). If a set-valued function $F : [0,b] \to n(Y)$ is Borel measurable and integrably bounded then the Aumann integral of F is nonempty.

DEFINITION 3. Let $F : [0,b] \to n(Y)$ be Borel-measurable and integrably bounded set-valued function. We say that F has a property "P" in the mean that the set-valued function $M_F : (0,b] \to c(Y)$ defined by

$$M_F(x) := \frac{1}{x} \int_0^x F(t)\,dt, \qquad x \in (0,b], \tag{1}$$

has the property "P". In the sequel for "P" stand "convexity", "starshapedness" and "subadditivity".

REMARK 2. Let us note that making the change of variable $t = xu$ in (1) the function M_F defined by (1) can be written as

$$M_F(x) = \int_0^1 F(xu)\,du.$$

REMARK 3. Let us also note that if $F_1, F_2 : [0,b] \to n(Y)$ are integrably bounded and Borel-measurable multifunctions then

$$\int_c^d F_1(t)\,dt + \int_c^d F_2(t)\,dt \subset \int_c^d (F_1 + F_2)(t)\,dt$$

for all $c, d \in [0,b]$, $c < d$.

DEFINITION 4. Let Y be a Banach space and $m \in [0,1]$. A set-valued function $F : [0,b] \to n(Y)$ is called *m-convex* if

$$tF(x) + m(1-t)F(y) \subset F(tx + m(1-t)y)$$

for all $x, y \in [0,b]$ and $t \in [0,1]$.

Note that for $m = 1$ the *m-convexity* reduces to the convexity of F. For $m = 0$ we have the starshapedness of F i.e.

$$tF(x) \subset F(tx) \quad \text{for all } x \in [0,b],\ t \in [0,1].$$

The *m-convex* single-valued functions were investigated by G. H. Toader in [6].

Denote respectively by K_m and S the sets of multifunction:

$$K_m := \{F \in n(Y)^{[0,b]} : F \text{ is } m\text{-convex and } \{0\} \in F(0)\},$$
$$S := \{F \in n(Y)^{[0,b]} : x, y,\ x + y \in [0,b] \Longrightarrow F(x+y) \subset F(x) + F(y)\}.$$

LEMMA 1. *If for an arbitrary m, $0 \leqslant m \leqslant 1$, a multifunction $F : [0,b] \to n(Y)$ is m–convex and $\{0\} \in F(0)$ then F is starshaped.*

Proof. By the *m-convexity* of F,

$$F(tx) = F(tx + m(1-t)0) \supset tF(x) + m(1-t)F(0)$$
$$\supset tF(x) + m(1-t)\{0\} = tF(x)$$

for arbitrary $x \in [0,b]$ and $t \in [0,1]$, which was to be shown.

A set-valued function $G : [0,b] \to n(Y)$ is called decreasing if for all $x, y \in [0,b]$

$$x < y \Longrightarrow G(x) \supset G(y).$$

LEMMA 2. *Let Y be a Banach space. A set valued function $F : [0,b] \to n(Y)$ is starshaped if and only if the multifunction $(0,b] \ni x \to \frac{F(x)}{x}$ is decreasing.*

Proof. Let F be starshaped. Then $tF(y) \subset F(ty)$ for all $y \in [0,b]$ and $t \in [0,1]$. Taking arbitrary $x, y \in (0,b]$, $0 < x < y$, we have $t := \frac{x}{y} \in (0,1)$ and therefore $\frac{x}{y}F(y) \subset F(x)$, which shows that the function $(0,b] \ni x \to \frac{F(x)}{x}$ is decreasing.

Suppose that $(0,b] \ni x \to \frac{F(x)}{x}$ is decreasing. Then for every $t \in (0,1]$ and $x \in (0,b]$; $\frac{F(tx)}{tx} \supset \frac{F(x)}{x}$, i.e. $F(tx) \supset tF(x)$ which shows that F is starshaped.

2. Main results

We begin with showing that the classes K_m, $m \in [0,1]$, are monotonic with respect to m.

THEOREM 1. *If $0 \leq n \leq m \leq 1$ and $0 \in F(0)$, then*

$$K_1 \subset K_m \subset K_n \subset K_0 \subset S.$$

Proof. Suppose that a set-valued function $F : [0, b] \to n(Y)$ is m-convex. Then from Lemma 1 we have

$$F(tx + n(1-t)y) = F(tx + m(1-t)\frac{n}{m}y)$$
$$\supset tF(x) + m(1-t)F(\frac{n}{m}y) \supset tF(x) + n(1-t)F(y)$$

for all x, $y \in [0, b]$ and $t \in [0, 1]$ which shows that $K_m \subset K_n$.

It remains to show that $K_0 \subset S$. Take x, y such that $x + y \in [0, b]$ and $F \in K_0$. If either $x = 0$ or $y = 0$ the proof is trivial. Suppose that $x, y > 0$. From Lemma 2 we have

$$F(x+y) = (x+y)\frac{F(x+y)}{x+y} \subset \frac{xF(x+y)}{x+y} + \frac{yF(x+y)}{x+y}$$
$$\subset \frac{xF(x)}{x} + \frac{yF(y)}{y} = F(x) + F(y)$$

which proves that $F \in S$.

Now we prove the following

THEOREM 2. *Let Y be a Banach space and $F : [0, b] \to n(Y)$ an integrably bounded and measurable set-valued function. Then if F is m-convex then M_F is m-convex.*

Proof. Let $F : [0, b] \to n(Y)$ be m–convex set-valued function i.e.

$$F(tx + m(1-t)y) \supset tF(x) + m(1-t)F(y),$$

for all x, $y \in [0, b]$ and $t \in [0, 1]$. From Remark 3 we have

$$M_F(tx + m(1-t)y) = \int_0^1 F(txu + m(1-t)yu)\, du$$
$$\supset \int_0^1 [tF(xu) + m(1-t)F(yu)]\, du$$
$$\supset \int_0^1 tF(xu)\, du + \int_0^1 m(1-t)F(yu)\, du$$
$$= t\int_0^1 F(xu)\, du + m(1-t)\int_0^1 F(yu)\, du$$
$$= tM_F(x) + m(1-t)M_F(y)$$

for all x, $y \in [0, b]$, $t \in [0, 1]$, which means that M_F is m–convex.

Putting $m = 0$ in Theorem 2 gives the following

COROLLARY 1. *Let Y be a Banach space. If $F : [0,b] \to n(Y)$ is star-shaped set-valued function, then so is M_F.*

Putting $m = 1$ in Theorem 2 gives the following

COROLLARY 2. *Let Y be a Banach space. If $F : [0,b] \to n(Y)$ is convex set-valued function, then so is M_F.*

In the proof of the next theorem we need some remarks.

REMARK 4. Because Lebesgue measure is complete, the Borel measurability of the set valued function F is equivalent to the following condition: for every Borel set B,

$$F^{-}(B) := \{x \in [0,b] : F(x) \cap B \neq \emptyset\} \in \mathcal{L}$$

where $\mathcal{L}$ denotes the σ–algebra of all subsets of interval $[0,b]$ which are measurable in the Lebesgue sense (cf. [2]). It is easy to verify that if F is of the form $F(x) := [f(x), g(x)]$, where $f, g : [0,b] \to \mathbb{R}$, $f \leq g$, the measurability of F implies measurability of f and g. Moreover if F is integrably bounded function then f and g are integrable (in the Lebesgue sense).

REMARK 5. If a multifunction $F : [0,b] \to cc(\mathbb{R})$ has the form $F(x) := [f(x), g(x)]$, $f, g : [0,b] \to \mathbb{R}$, $f \leq g$, and f and g are integrable functions, then it is easy to check that subadditivity of F implies subadditivity of M_F.

Now we prove the following

THEOREM 3. *Let Y be a Banach space and suppose that $F : [0,b] \to cc(Y)$ is an integrably bounded and Borel–measurable set-valued function. Then the subadditivity of F implies the subadditivity of M_F.*

Proof. Let us take an arbitrary continuous linear functional $y^* : Y \to \mathbb{R}$ and consider a set-valued function $y^* \circ F : [0,b] \to cc(\mathbb{R})$ defined by

$$(y^* \circ F)(x) := y^*(F(x)), \qquad x \in [0,b].$$

Since for every $x \in [0,b]$, $F(x)$ is convex and compact, the continuity of y^* implies that

$$(y^* \circ F)(x) := [\varphi_{y^*}(x), \psi_{y^*}(x)],$$

where $\varphi_{y^*}, \psi_{y^*} : [0,b] \to \mathbb{R}$ and $\varphi_{y^*} \leq \psi_{y^*}$.

The Borel measurability and bounded integrability of F imply that so is $y^* \circ F$. From Remark 4, φ_{y^*} and ψ_{y^*} are integrable functions. From Remark 5 the function

$$(y^* \circ M_F)(x) = y^*\Big(\int_0^1 F(xu)\, du\Big) = \int_0^1 y^*(F(xu))\, du$$

is subadditive i.e.

(2) $x, y, x + y \in (0, b] \Longrightarrow$

$$(y^* \circ M_F)(x + y) \subset (y^* \circ M_F)(x) + (y^* \circ M_F)(y).$$

Since F is integrably bounded function and its values are compact subsets of Y, it follows that the sets $M_F(x)$, $x \in (0, b]$, are compact (see [1], Theorem 4). Hence the sets $(y^* \circ M_F)(x)$, $x \in (0, b]$, are also compact.

From (2) and the linearity of y^* we have

(3) $(y^* \circ M_F)(x + y) \subset (y^* \circ M_F)(x) + (y^* \circ M_F)(y) \subset y^*(M_F(x) + M_F(y)).$

for all $x, y \in [0, b]$ and $x + y \in (0, b]$.

We shall show that the function M_F must be subadditive. For an indirect proof suppose that there are $x, y \in (0, b]$ such that $x + y \in (0, b]$ and

$$M_F(x + y) \not\subset M_F(y) + M_F(x).$$

Then we could find a point $p \in M_F(x + y)$ such that $p \notin M_F(x) + M_F(y)$. The set $\{p\}$ is compact and convex, and the set $M_F(x) + M_F(y)$, being the algebraic sum of two compact and convex sets, is closed and convex.

Now by the separation theorem (c.f. Rolewicz [4], p. 98, Corollary 2.5.11) there exist $c \in \mathbb{R}$ and $\varepsilon > 0$ such that

$$y^*(p) \geq c + \varepsilon \quad \text{and} \quad \sup_{q \in M_F(x) + M_F(y)} y^*(q) \leqslant c,$$

but this contradicts to the inclusion (3), and the proof is completed.

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DEPARTMENT OF MATHEMATICS
TECHNICAL UNIVERSITY
Willowa 2
43-309 BIELSKO-BIAŁA, POLAND

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