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REMARKS ON A PAPER BY BEG, LATIF AND MINHAS ABOUT FIXED POINTS

1. Introduction

In their paper [1] Beg, Latif and Minhas established a Meir-Keeler [9] type fixed point theorem for a pair (f, g) of maps on a locally convex topological vector space. In Section 2 of this note we show that under the assumptions of their Theorem 2 [1] the maps f and g coincide so, in fact, Theorem 2 [1] is a theorem for a single map. Further, we discuss the continuity of maps satisfying Meir-Keeler type conditions in a topological vector space. In Section 3, using the convergence principle of Leader [6], we improve Theorem 2 [1] by establishing a necessary and sufficient condition for the existence of a contractive fixed point for maps on a uniform space. Finally, in Section 4 we clarify Corollary 4 [1], which appears to deal with the empty class of maps. We give a reformulation of this corollary obtaining a fixed point result for some expanding maps on a uniform space.

For a discussion of several variants of a Meir-Keeler condition, see our paper [4]. We also invite the reader to study our article [5], in which we present that some fixed point theorems (including the Banach Contraction Principle) in uniform spaces are subsumed by the Knaster-Tarski ordering theorem (cf., e.g., [2], p. 14).

The sets of all nonnegative reals and nonnegative integers are denoted by $\mathbb{R}_+$ and $\mathbb{Z}_+$, respectively. Following Leader [7] we say that a selfmap f of a topological space E has *a contractive fixed point* if there is a $w \in E$ such that $fw = w$ and $\lim_{n\to\infty} f^n x = w$ for all $x \in E$, where f^n denotes the nth iterate of f.

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2. Remark on a Theorem of Beg, Latif and Minhas

Throughout this section the letter E denotes a Hausdorff locally convex topological vector space over the field $\mathbf{R}$ with a calibration Γ, that is, Γ is a family of continuous seminorms on E inducing the topology of E. The following Meir-Keeler type fixed point theorem was established in [1].

THEOREM 1 (Beg-Latif-Minhas). *Let a space E be sequentially complete. Let f and g be selfmaps of E such that given an $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in X$ and $p \in \Gamma$,*

$$\epsilon \leq p(x-y) < \epsilon + \delta \quad \textit{implies} \quad \textit{that} \quad p(fx - gy) < \epsilon. \tag{1}$$

If f is continuous, then f has a fixed point. If both f and g have fixed points, then each of them has a unique fixed point and these two points coincide.

PROPOSITION 1. *Let D be an open subset of E and $f, g : D \mapsto E$. Assume that given $p \in \Gamma$ and $x, y \in D$,*

$$p(x-y) > 0 \quad \textit{implies} \quad \textit{that} \quad p(fx - gy) < p(x-y). \tag{2}$$

If f is sequentially continuous, then $f = g$.

Proof. Fix a point $x_0 \in D$. Let $p \in \Gamma$ and $p \neq 0$. Then there is a $y_p \in E$ such that $p(y_p) > 0$. Define $x_n := x_0 + \frac{1}{n} y_p$ for $n \in \mathbb{N}$. By hypothesis, $x_n \in D$ for sufficiently large n. Clearly, $p(x_n - x_0) > 0$ so by (2), $p(fx_n - gx_0) < p(x_n - x_0)$. By continuity of f, letting $n \to \infty$ we infer that $p(fx_0 - gx_0) = 0$. This forces $fx_0 = gx_0$, since E is a Hausdorff space. ∎

COROLLARY 1. *Under the assumptions of Theorem 1, the maps f and g coincide.*

Proof. By Proposition 1, it suffices to show that (1) implies (2). Let $p \in \Gamma$, $x_0, y_0 \in X$ and $p(x_0 - y_0) > 0$. Define $\epsilon := p(x_0 - y_0)$. There is a $\delta > 0$ such that (1) holds. Clearly, $\epsilon \leq p(x_0 - y_0) < \epsilon + \delta$ so by (1), $p(fx_0 - gy_0) < \epsilon = p(x_0 - y_0)$. Therefore, (2) is satisfied. ∎

So in the sequel we shall consider condition (1) with $f = g$. Since (1) implies (2), it can be easily verified that (1) with $f = g$ is equivalent to the following condition:

given an $\epsilon > 0$, there is a $\delta > 0$ such that for all $x, y \in E$ and $p \in \Gamma$,

$$0 < p(x-y) < \epsilon + \delta \quad \text{implies that} \quad p(fx - gy) < \epsilon. \tag{3}$$

Now if Γ is a singleton, that is, E is a normed space, then (3) implies that f is continuous. More generally, we have the following

PROPOSITION 2. *Let (E, d) be a metric space, $D \subseteq E$ and $f : D \mapsto E$. Assume that given an $\epsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in D$,*

$$0 < d(x,y) < \epsilon + \delta \quad \textit{implies} \quad \textit{that} \quad d(fx, fy) < \epsilon.$$

Then f is continuous.

Proof. Observe that the above condition easily implies that f is nonexpansive, hence continuous. ∎

Unfortunately, Proposition 2 cannot be extended to the case, in which E is a topological vector space as is shown in Example 1 below. So in Corollary 3 [1] the authors should add the assumption that f is continuous if they want to deduce this result from Theorem 1.

EXAMPLE 1. Let $E := \mathbb{R}^2$ and $\Gamma := \{p_1, p_2\}$, where $p_1(x,y) := |x|$ and $p_2(x,y) := |y|$ for $x, y \in \mathbb{R}$. Let $D := \{(0,y) : y \in \mathbb{R}\}$. Consider the following Dirichlet type function:

$$f(0,y) := (1,0) \quad \text{if } y \text{ is rational, and} \quad f(0,y) := (0,0) \quad \text{for otherwise.}$$

Then (3) is trivially satisfied since $p_1((0,y_1) - (0,y_2)) = 0$ and $p_2(f(0,y_1) - f(0,y_2)) = 0$ for all $y_1, y_2 \in \mathbb{R}$, but f is discontinuous even at each point of D.

3. An iff fixed point criterion in uniform spaces

Throughout this section the letter E denotes a sequentially complete Hausdorff uniform space with a family Γ of pseudometrics inducing the uniform structure of E. So in particular, the results of this section can be applied in the case, in which E is a nonempty closed subset of a sequentially complete locally convex Hausdorff topological vector space.

We start with recalling the following convergence principle of Leader [6].

THEOREM 2 (Leader). *Let $q : \mathbb{Z}_+^2 \mapsto \mathbb{R}_+$ be such that for all $k, m, n \in \mathbb{Z}_+$,*

$$q(m,n) \le q(m,k) + q(k,k) + q(k,n). \tag{4}$$

Then the following conditions are equivalent

(i) $\lim_{m,n\to\infty} q(m,n) = 0$.

(ii) *Given an $\epsilon > 0$, there exist an $r \in \mathbb{N}$ and $\delta > 0$ such that for all $m, n \in \mathbb{Z}_+$,*

$$q(m,n) < \epsilon + \delta \quad \textit{implies} \quad \textit{that} \quad q(m+r, n+r) < \epsilon.$$

As an immediate consequence we obtain the following Meir-Keeler type fixed point theorem for maps on a uniform space, which improves Theorem 2 [1]. Its metric counterpart was established in [7].

THEOREM 3. *Let f be a sequentially continuous selfmap of E. The following conditions are equivalent*

(i) *f has a contractive fixed point.*

(ii) *Given* $p \in \Gamma$, $\epsilon > 0$ *and* $x, y \in E$, *there exist* $\delta > 0$ *and* $r \in \mathbb{N}$ *such that for all* $m, n \in \mathbb{Z}_+$,

$$p(f^m x, f^n y) < \epsilon + \delta \quad \textit{implies} \quad \textit{that} \quad p(f^{m+r} x, f^{n+r} y) < \epsilon.$$

In particular, condition (ii) holds if given $p \in \Gamma$ *and* $\epsilon > 0$, *there exist* $\delta > 0$ *and* $r \in \mathbb{N}$ *such that for all* $x, y \in E$,

$$p(x, y) < \epsilon + \delta \quad \textit{implies} \quad \textit{that} \quad p(f^r x, f^r y) < \epsilon. \tag{5}$$

Proof. Observe that (i) easily implies (ii), since given $x, y \in E$ the sequences $(f^n x)_{n=1}^{\infty}$ and $(f^n y)_{n=1}^{\infty}$ are equivalent-Cauchy, i.e., $\lim_{m,n\to\infty} p(f^m x, f^n y) = 0$ for all $p \in \Gamma$. To prove (ii)$\Rightarrow$(i) fix $x, y \in E$ and $p \in \Gamma$. For $m, n \in \mathbb{Z}_+$ define $q(m, n) := p(f^m x, f^n y)$. Then q satisfies (4) so by hypothesis and Th. 2, $\lim_{m,n\to\infty} q(m, n) = 0$. In particular, $\lim_{n\to\infty} q(n, n) = 0$ for each $p \in \Gamma$, which means that $(f^n x)_{n=1}^{\infty}$ and $(f^n y)_{n=1}^{\infty}$ are equivalent. Moreover, if we put $y := x$ then $\lim_{m,n\to\infty} q(m, n) = 0$ for each $p \in \Gamma$ means that $(f^n x)_{n=1}^{\infty}$ is Cauchy. Thus, by completeness and Hausdorfness, there is a unique point $w \in E$ such that $f^n x \to w$ for all $x \in E$. Then $w = fw$, since f is sequentially continuous. ∎

REMARK 1. We emphasize that condition (5) is weaker than (1) (with $f = g$) for the following two reasons. Firstly, a real δ in (5) may depend on a pseudometric $p \in \Gamma$, whereas in (1) δ is to be chosen universally for all seminorms $p \in \Gamma$. Secondly, an integer r in (5) may vary with ϵ, while in (1) $r = 1$ for all $\epsilon > 0$. Moreover, it follows from Theorem 3 that in Theorem 1 a map f need not be defined on the whole space E, but on a nonempty closed subset of E.

REMARK 2. Another iff fixed point criterion involving a behaviour of the map $(x, y) \mapsto d(f^n x, f^n y)$ (the letter d denotes a metric) is given in [3].

As an application of Theorem 3 we give the following result, which is a generalization of Matkowski's Theorem 1.2 [8] (cf. also Theorem 3.2 in [2], p. 12). We emphasize that this result cannot be deduced from Theorem 1 even if E is a metric space (see Example 1 in [4]).

COROLLARY 2. *For* $p \in \Gamma$, *let* $\phi_p : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *be nondecreasing and such that* $\lim_{n\to\infty} \phi_p^n(t) = 0$ *for* $t \in \mathbb{R}_+$. *Let* f *be a selfmap of* E *such that*

$$p(fx, fy) \le \phi_p(p(x, y)), \quad \textit{for all} \quad x, y \in E \quad \textit{and} \quad p \in \Gamma. \tag{6}$$

Then f *has a contractive fixed point.*

Proof. Since $\phi_p(t) < t$ for $t > 0$ (see [2], p. 12), (6) forces the continuity of f. We show that (6) implies (5). Fix $p \in \Gamma$ and $\epsilon > 0$. Choose, e.g., $\delta := 1$. Then by hypothesis, given $x, y \in E$,

$$p(x, y) < \epsilon + 1 \quad \text{implies that} \quad p(f^n x, f^n y) \le \phi_p^n(\epsilon + 1) < \epsilon,$$

for n large enough (such an n depends only on ϵ). So (5) is verified and Theorem 3 applies. ■

4. On some expanding maps in uniform spaces

The letter E denotes here a uniform space with the same properties as in the previous section. The following condition was used in Corollary 4 [1]:

$$\text{given an} \quad \epsilon > 0 \quad \text{there exists a} \quad \delta > 0 \quad \text{such that for} \quad x, y \in E,$$
$$p(x,y) < \epsilon \quad \text{implies that} \quad \epsilon \le p(fx, fy) < \epsilon + \delta.$$

Unfortunately, there is no map f, which would satisfy the above condition. To see it, fix an $x \in E$, and put $\epsilon := 1$ and $y := x$. Then $p(x,y) < 1$, but the inequality $p(fx, fy) \ge 1$ does not hold. We may propose the following reformulation of Corollary 4 [1].

COROLLARY 3. *Let f be a surjective selfmap of E such that given $p \in \Gamma$ and $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in E$,*

$$p(x,y) \ge \epsilon \quad \textit{implies} \quad \textit{that} \quad p(fx, fy) \ge \epsilon + \delta. \tag{7}$$

Then f has a unique fixed point.

Proof. We show that f is one-to-one. If $x \ne y$, then by Hausdorfness of E, there is a $p \in \Gamma$ such that $p(x,y) > 0$. Put $\epsilon := p(x,y)$. By (7), there is a $\delta > 0$ such that $p(fx, fy) \ge \epsilon + \delta > 0$, which implies that $fx \ne fy$. Let f^{-1} be the inverse of f. By hypothesis, f^{-1} is a selfmap of E. Moreover, (7) easily implies that (5) holds with $r = 1$ and f^{-1} substituted for f. Finally, it is easy to observe that (5) with $r = 1$ implies the continuity of a map involved. So by Theorem 3, f^{-1} has a unique fixed point w. Clearly, w is also a unique fixed point of f. ■

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