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ON A SYSTEM OF FUNCTIONAL EQUATIONS

1. Introduction

We study the following system

$$f_i(x) = \sum_{j=1}^{n} \sum_{k=1}^{m} a_{ijk}[x, f_j(S_{ijk}(x))] + g_i(x), \tag{1.1}$$

for $i = 1, 2, \ldots, n$ and $x \in I \subset R$, where I is a bounded or unbounded interval. The given functions $g_i : I \to I, a_{ijk} : I \times R \to R$ are continuous, f_i are unknown functions. Using Banach fixed point theorem, we prove the existence and uniqueness of solution of the system (1). The obtained solution is also stable with respect to the functions g_i.

In [1], the system (1.1) is studied with $I = [-b, b]$, $n = m = 2$, $S_{ijk}(x)$ binomials of first degree and

$$a_{ijk}(x, y) = \widetilde{a}_{ijk} y, \tag{1.2}$$

where $\widetilde{a}_{ijk}$ are real constans. The solution is approximated by a uniformly convergent recurrent sequence, and it is stable with respect to the functions g_i. In [2], [3], [4] the existence and uniqueness of solution of the functional equation

$$f(x) = a(x, f(S(x))) \tag{1.3}$$

in the functional space $BC[a, b]$ is studied.

In this paper, by using the Banach fixed point theorem, we obtain the existence, the uniqueness and also the stability of the solution of the system (1.1) with respect to the functions g_i, where $I = [a, b]$ or I is unbounded interval of R. In the case of a_{ijk} like in (1.2) and $S_{ijk}(x)$ being the functions of first degree and $g \in C^r(I; R^n)$, $I = [-b, b]$, we obtain a Maclaurin expansion of the solution of the system (1.1) until the order r. Furthermore, if $g_i(x)$ are the polynomials of degree r, then the solution of the system (1.1) is also the polynomial of degree r. The obtained result is a generalization the results in [1]. We also give the numerical calculation on some examples.

2. The theorems on existence, uniqueness and stability of solution

With $I = [a, b]$, we denote by $X = C(I; R^n)$ the Banach space of the functions $f : I \to R^n$ continuous on I with respect to the norm

$$\|f\|_X = \sup_{x \in I} \|f(x)\|, \tag{2.1}$$

where

$$\|f(x)\| = \sum_{i=1}^{n} |f_i(x)|, \quad f = (f_1, \ldots, f_n) \in X.$$

When $I \subset R$ is an unbounded interval, we denote by $X = C_b(I; R^n)$ the Banach space of the functions $f : I \to R^n$ continuous, bounded on I with respect to the norm (2.1).

We write the system (1.1) in the form of operational equation in X as follows

$$f = Tf, \tag{2.2}$$

where $f = (f_1, \ldots, f_n)$, $Tf = ((Tf)_1, \ldots, (Tf)_n)$ with
(2.3)

$$(Tf)_i(x) = \sum_{j=1}^{n} \sum_{k=1}^{m} a_{ijk}[x, f_j(S_{ijk}(x))] + g_i(x), \quad i = 1, 2, \ldots, n, x \in I.$$

We admit the following hypotheses:
(H_1) $S_{ijk} : I \to I$ are continuous,
(H_2) $g \in X$,
(H_3) $a_{ijk} : I \times R \to R$ are continuous and satisfy the condition: there exists $\widetilde{\alpha}_{ijk} : I \to R$ bounded and nonnegative such that

$$|a_{ijk}(x, y) - a_{ijk}(x, \widetilde{y})| \le \widetilde{\alpha}_{ijk}(x)|y - \widetilde{y}|, \quad y, \widetilde{y} \in R, \ x \in I, \tag{2.4}$$

Denote

$$\alpha := \sum_{i,j=1}^{n} \sum_{k=1}^{m} (\sup_{x \in I} \widetilde{\alpha}_{ijk}(x)) < 1. \tag{2.5}$$

THEOREM 1. *Under hypotheses* (H_1)—(H_3), *there exists a unique function* $f \in X$ *such that* $f = Tf$. *Moreover*, f *is stable with respect to* g *in* X.

Proof. It is evident that $Tf \in X$, for every $f \in X$. Considering f, $\widetilde{f} \in X$, we easily verify, by (H_3) and (2.5), that

$$\|Tf - T\widetilde{f}\|_X \le \alpha \|f - \widetilde{f}\|_X. \tag{2.6}$$

Then, using Banach fixed point theorem, we have the existence of unique $f \in X$ such that $f = Tf$.

Consider f and $\widetilde{f}$ from X being two solutions of (2.2) corresponding to g and $\widetilde{g}$ from X, respectively. By the analogous evaluation, we have

$$\|f - \widetilde{f}\|_X \leq \frac{1}{1-\alpha}\|g - \widetilde{g}\|_X. \tag{2.7}$$

Hence, f is stable with respect to g.

Remark 1. Theorem 1 gives a consecutive approximate algorithm

$$f^{(\nu)} = Tf^{(\nu-1)}, \quad \nu = 1, 2, \ldots, \quad f^{(0)} \in X \text{ given.} \tag{2.8}$$

Then the sequence $\{f^{(\nu)}\}$ converges in X to the solution f of (2.2) and we have an evaluation of the error

$$\|f^{(\nu)} - f\|_X \leq \frac{\|Tf^{(0)} - f^{(0)}\|_X}{1-\alpha}\alpha^\nu, \quad \nu = 1, 2, \ldots \tag{2.9}$$

Consider now the case of $a_{ijk}(x, y)$ of form (1.2) and denote

$$\beta := \sum_{i,j=1}^{n} \sum_{k=1}^{m} |\widetilde{a}_{ijk}| < 1. \tag{2.10}$$

THEOREM 2. *Suppose that* (H_1), (H_2) *hold. Then there exists a unique function* $f = (f_1, \ldots, f_n) \in X$ *being the solution of the following system*

$$f_i(x) = \sum_{j=1}^{n} \sum_{k=1}^{m} \widetilde{a}_{ijk} f_j(S_{ijk}(x)) + g_i(x), \quad i = 1, 2, \ldots, n, \quad x \in I. \tag{2.11}$$

Moreover, the solution of (2.11) is stable with respect to $g = (g_1, \ldots, g_n)$ *in* X.

Proof. We apply Theorem 1 for $a_{ijk}(x, y) = \widetilde{a}_{ijk} y$. Then $\widetilde{\alpha}_{ijk} = |\widetilde{a}_{ijk}|$ in (2.4) and $\alpha = \beta < 1$, by (2.5), (2.10).

Remark 2. Let $S_{ijk}(x)$ be the binomials of first degree

$$S_{ijk}(x) = b_{ijk}x + c_{ijk} \tag{2.12}$$

and $I = [-b, b]$. Suppose that the real numbers b_{ijk}, c_{ijk} satisfy the condition

$$|b_{ijk}| < 1, \quad i, j = 1, \ldots, n, \quad k = 1, \ldots, m, \tag{2.13}$$

$$\max_{\substack{1 \leq i,j \leq n \\ 1 \leq k \leq m}} \frac{|c_{ijk}|}{1 - |b_{ijk}|} \leq b. \tag{2.14}$$

Then (H_1) holds.

THEOREM 3. *Suppose that* $I = [-b, b]$, *the real numbers* $\widetilde{a}_{ijk}$, b_{ijk}, c_{ijk} *satisfy* (2.10), (2.13), (2.14) *and* $S_{ij}(x)$ *are of the form* (2.12). *Then, for each*

$g \in X$, there exists a unique $f \in X$ being the solution of the system (2.11). *Moreover, this solution is stable with respect to $g = (g_1, \ldots, g_n)$ in X.*

Remark 3.

(i) The result in [1] is a special case of Theorem 3 with $n = m = 2$.

(ii) Theorem 3 is true for $I = R$ and in this case the terms b_{ijk}, c_{ijk} need not satisfy the conditions (2.13), (2.14).

3. Maclaurin expansion of the solution

From here, we consider $I = [-b, b]$ and the numbers $\widetilde{a}_{ijk}$, b_{ijk}, c_{ijk} as in Theorem 3.

Suppose that $g \in C^1(I; R^n)$ and $f \in C^1(I; R^n)$ is the unique solution of the corresponding system (2.11). Differentiating two members of (2.11), we obtain

$$(3.1) \quad f_i'(x) = \sum_{j=1}^{n} \sum_{k=1}^{m} \widetilde{a}_{ijk} b_{ijk} f_j'(S_{ijk}(x)) + g_i'(x), \quad i = 1, 2, \ldots, n, \ x \in I.$$

Let $f_i'(-b)$ and $f_i'(b)$ mean the forward derivative at $-b$ and the backward derivative at b of f_i, respectively. Put

$$(3.2) \qquad a_{ijk}^{(1)} = \widetilde{a}_{ijk} b_{ijk}.$$

From (2.10), (2.13), we have

$$(3.3) \qquad \beta^{(1)} := \sum_{i,j=1}^{n} \sum_{k=1}^{m} |a_{ijk}^{(1)}| \leq \beta < 1.$$

By Theorem 3, there exists a unique function

$$F^{[1]} = (F_1^{[1]}, \ldots, F_n^{[1]}) \in C(I, R^n)$$

being the solution of the system

$$(3.4) \quad F_i^{[1]}(x) = \sum_{j=1}^{n} \sum_{k=1}^{m} a_{ijk}^{(1)} F_j^{[1]}(S_{ijk}(x)) + g_i'(x), \quad i = 1, 2, \ldots, n, \ x \in I.$$

Moreover, from the uniqueness, this solution is also the deriverative $f' = (f_1', \ldots, f_n')$ of f.

Similarly, we consider the case of $f \in C^r(I; R^n)$ being the solution of the system (2.11) corresponding to $g \in C^r(I; R^n)$. Differentiating r times two members of (2.11), we have

(3.5)

$$f_i^{(r)}(x) = \sum_{j=1}^{n} \sum_{k=1}^{m} \widetilde{a}_{ijk} b_{ijk}^{r} f_j^{(r)}(S_{ijk}(x)) + g_i^{(r)}(x), \quad i = 1, 2, \ldots, n, \ x \in I.$$

From (2.10), (2.13), we denote

$$\beta^{(r)} = \sum_{i,j=1}^{n} \sum_{k=1}^{m} |\widetilde{a}_{ijk} b_{ijk}^{r}| \leq \beta < 1. \tag{3.6}$$

Therefore, the following system

$$F_i^{[r]}(x) = \sum_{j=1}^{n} \sum_{k=1}^{m} \widetilde{a}_{ijk} b_{ijk}^{r} F_j^{[r]}(S_{ijk}(x)) + g_i^{(r)}(x), \quad i = 1, 2, \ldots, n, \ x \in I, \tag{3.7}$$

has a unique solution

$$F^{[r]} = (F_1^{[r]}, \ldots, F_n^{[r]}) \in C(I; R^n), \tag{3.8}$$

equal to the derivative $f^{(r)} = (f_1^{(r)}, \ldots, f_n^{(r)})$ of the solution f.

Therefore, we have the following theorem.

THEOREM 4. *Let $g \in C^r(I; R^n)$. Then there exist $f \in C^r(I; R^n)$ and $F^{[r]} \in C(I; R^n)$ being the unique solutions of the systems* (2.11) *and* (3.7), *respectively. Moreover, $F^{[r]}$ is the r-order derivative of f.*

Remark 4. In the case of $I = R$, we suppose additionally that the real numbers $\widetilde{a}_{ijk}$, b_{ijk}, c_{ijk} satisfy the condition

$$\max_{0 \leq s \leq r} \sum_{i,j=1}^{n} \sum_{k=1}^{m} |\widetilde{a}_{ijk} b_{ijk}^{s}| < 1. \tag{3.9}$$

Then, if

$$g \in C_b^r(I; R^n) = \{g \in C_b(I; R^n) / g^{(1)}, g^{(2)}, \ldots, g^{(r)} \in C_b(I; R^n)\}, \tag{3.10}$$

the conclusion of Theorem 4 is still true, where the functional spaces $C(I; R^n)$ and $C^r(I; R^n)$ appearing in Theorem 4 are replaced by $C_b(I; R^n)$ and $C_b^r(I; R^n)$, respectively. The proof of this result is the same as that of Theorem 4.

Now we return to the same case of $I = [-b, b]$. Suppose that $f \in C^p(I; R^n)$ is the unique solution of (2.11) corresponding to $g \in C^p(I; R^n)$. For each $r = 1, 2, \ldots, p$, we have $F^{[r]}$ as in Theorem 4. Then, from Maclaurin formula we have

$$f_i(x) = \sum_{r=0}^{p-1} \frac{f_i^{(r)}(0)}{r!} x^r + \frac{1}{(p-1)!} \int_0^x (x-t)^{p-1} f_i^{(p)}(t) dt, \quad i = 1, 2, \ldots, n. \tag{3.11}$$

On the other hand, we have

$$F^{[r]} = f^{(r)}, \quad r = 1, 2, \ldots, p. \tag{3.12}$$

Put $F^{[0]} = f$. From (3.11), (3.12) we have
(3.13)

$$f_i(x) = \sum_{r=0}^{p-1} \frac{F_i^{[r]}(0)}{r!} x^r + \frac{1}{(p-1)!} \int_0^x (x-t)^{p-1} F_i^{[p]}(t)dt, \qquad i = 1,2,\ldots,n.$$

Inversely, suppose that a function $\widetilde{f} = (\widetilde{f}_1, \ldots, \widetilde{f}_n) \in C(I;R^n)$ is given by the formula
(3.14)

$$\widetilde{f}_i(x) = \sum_{r=0}^{p-1} \frac{F_i^{[r]}(0)}{r!} x^r + \frac{1}{(p-1)!} \int_0^x (x-t)^{p-1} F_i^{[p]}(t)dt, \qquad i = 1,2,\ldots,n,.$$

Then, from (3.12), (3.14) we have

$$\text{(3.15)} \quad \widetilde{f}_i(x) = \sum_{r=0}^{p-1} \frac{f_i^{(r)}(0)}{r!} x^r + \frac{1}{(p-1)!} \int_0^x (x-t)^{p-1} f_i^{(p)}(t)dt = f_i(x),$$
$$i = 1,2,\ldots,n, \ x \in I.$$

Therefore $\widetilde{f}$ is a solution of (2.11).

Then, we have the following theorem.

THEOREM 5. *Let* $g \in C^p(I;R^n)$. *Then, the solution* $f \in C^p(I;R^n)$ *of* (2.11) *is represented by* (3.13), *where* $F^{[r]} \in C(I;R^n)$ *is the unique solution of* (3.7). *Inversely, every function* $f \in C^p(I;R^n)$ *represented by* (3.13) *is a solution of* (2.11).

Remark 5. We consider the case $I = R$ and the real numbers $\widetilde{a}_{ijk}$, b_{ijk}, c_{ijk} satisfying the condition (3.9). If $g \in C_b^p(I;R^n)$, the conclusion of Theorem 5 is still true, where the functional spaces $C(I;R^n)$ and $C^p(I;R^n)$ appearing in Theorem 5 are replaced by $C_b(I;R^n)$ and $C_b^p(I;R^n)$, respectively.

Returning to the case of $I = [-b,b]$ we have the following corollary.

COROLLARY 1. *If* $g_i, \ldots, g_n$ *are polynomials of degree not greater than* $r-1$, *the solution* f *of* (2.11) *is also a sequence of such polynomials.*

Proof. We have

$$\text{(3.16)} \qquad g_i^{(r)}(x) = 0, i = 1,2,\ldots,n, x \in [-b,b].$$

Then $F^{[r]} = 0$ is the unique solution of the system (3.7). Applying (3.13) with $p = r$, we have

$$\text{(3.17)} \qquad f_i(x) = \sum_{s=0}^{r-1} \frac{F_i^{[s]}(0)}{s!} x^s.$$

THEOREM 6. *Suppose that $f \in C^p(I; R^n)$ is the solution of the system (2.11) corresponding to $g \in C^p(I; R^n)$ and that $\widetilde{f}$ is the sequence of polynomials of degree not greater than $p-1$ and satisfies the system (2.11) corresponding to $\widetilde{g} = (\widetilde{g}_1, \ldots, \widetilde{g}_n)$, where*

$$\widetilde{g}_i(x) = \sum_{r=0}^{p-1} \frac{g_i^{(r)}(0)}{r!} x^r, i = 1, 2, \ldots, n. \tag{3.18}$$

Then, we have

$$\|f - \widetilde{f}\|_X \le \frac{1}{1-\beta} \frac{b^p}{p!} \|g^{(p)}\|_X. \tag{3.19}$$

Proof. We have Maclaurin expansion of $g_i(x)$ in the form

$$g_i(x) = \widetilde{g}_i(x) + \frac{1}{(p-1)!} \int_0^x (x-t)^{p-1} g_i^{(p)}(t)dt. \tag{3.20}$$

Applying the estimate (2.7) with $\alpha = \beta$, we have

$$\|f - \widetilde{f}\|_X \le \frac{1}{1-\beta} \|g - \widetilde{g}\|_X. \tag{3.21}$$

From (3.20) we have

$$\begin{aligned} \|g - \widetilde{g}\|_X &= \sup_{|x| \le b} \sum_{i=1}^n |g_i(x) - \widetilde{g}_i(x)| \\ &= \frac{1}{(p-1)!} \sup_{|x|\le b} \sum_{i=1}^n |\int_0^x (x-t)^{p-1} g_i^{(p)}(t)dt|, \end{aligned} \tag{3.22}$$

where

$$\begin{aligned} \sum_{i=1}^n |\int_0^x (x-t)^{p-1} g_i^{(p)}(t)dt| &\le \int_0^x (x-t)^{p-1} \|g^{(p)}(t)\| dt \\ &\le \|g^{(p)}\|_X \frac{x^p}{p} \le \|g^{(p)}\|_X \frac{b^p}{p} \end{aligned} \tag{3.23}$$

for $0 \le x \le b$. Similarly, for $-b \le x \le 0$ the inequality (3.23) is still true. Therefore, from (3.22), (3.23) we have

$$\|g - \widetilde{g}\|_X \le \|g^{(p)}\|_X \cdot \frac{b^p}{p!} \tag{3.24}$$

and, by (3.21), we get (3.19).

COROLLARY 2. *If for $g \in C^\infty(I; R^n)$ there exists $d > 0$ such that*

$$\|g^{(p)}\|_X \le d^p, \quad p = 0, 1, 2, \ldots \tag{3.25}$$

and f is a solution of the system (2.11) *corresponding to g and $\widetilde{f}^{[p]}$ being the sequence of polynomials of degree not greater than $p-1$ satisfies the system* (2.11) *corresponding to $\widetilde{g}$ as in Theorem 6, then*

$$\lim_{p\to\infty} \|f - \widetilde{f}^{[p]}\|_X = 0.$$

Moreover, we have the estimates

$$\|f - \widetilde{f}^{[p]}\|_X \leq \frac{1}{1-\beta} \cdot \frac{(bd)^p}{p!}, \quad p = 1, 2, \dots \tag{3.26}$$

Proof. The estimates (3.19) and (3.25) imply (3.26).

COROLLARY 3. *Let $g \in C(I; R^n)$ and f be the solution of the system (2.11) corresponding to g. Then, there exists a sequence of polynomials of degree non greater than $p-1$: $\widetilde{f}^{[p]} = (\widetilde{f}_1^{[p]}, \dots, \widetilde{f}_n^{[p]})$ such that*

$$\lim_{p\to\infty} \|\widetilde{f}^{[p]} - f\|_X = 0. \tag{3.27}$$

Proof. According to Weierstrass theorem, each function g_i is approximated by a sequence of polynomials converging uniformly to $p_i^{[p]}$ when the degree $p-1 \to +\infty$. Therefore, $p^{[p]} = (p_1^{[p]}, \dots, p_n^{[p]})$ converges in $C(I; R^n)$ to g when $p \to +\infty$. Let $\widetilde{f}^{[p]}$ be a polynomial solution of (2.11) corresponding to $g = p^{[p]}$. According to the estimate (3.21) we have

$$\|\widetilde{f}^{[p]} - f\|_X \leq \frac{1}{1-\beta}\|p^{[p]} - g\|_X \to 0. \tag{3.28}$$

as $p \to +\infty$.

4. Numerical results

In this part, we consider the algorithm (2.8) with $I = [-b, b]$. More concretely, for $x \in [-b, b]$ and $1 \leq i \leq n$, we put

$$f_i^{(\nu)}(x) = \sum_{j=1}^{n} \sum_{k=1}^{m} a_{ijk}[x, f_j^{(\nu-1)}(S_{ijk}(x))] + g_i(x), \tag{4.1}$$

$$f_i^{(0)}(x) = 0, \tag{4.2}$$

Basting on (4.1), we calculate the values $f_i^{(\nu)}(x_\mu)$ at some discrete points

$$x_\mu = -b + \mu\Delta x, \quad \Delta x = 2b/N, \quad \mu = 0, 1, \dots, N. \tag{4.3}$$

Afterwards, we interpolate the values $f_i^{(\nu)}(x_\mu)$ by spline functions of first degree on $[-b, b]$, basing on the knot points $x_0, x_1, \dots, x_N$,

$$\widetilde{f}_i^{(\nu)}(x) = \sum_{\mu=0}^{N} \widetilde{f}_i^{(\nu)}(x_\mu) W_\mu(x), \tag{4.4}$$

where the functions $W_0(x), W_1(x), \ldots, W_N(x)$ are defined as follows

$$(4.5)\qquad W_\mu(x) = \begin{cases} (x - x_{\mu-1})/\Delta x & \text{for } x_{\mu-1} \le x \le x_\mu, \\ (x_{\mu+1} - x)/\Delta x & \text{for } x_\mu \le x \le x_{\mu+1}, \\ 0 & \text{for } x \notin [x_{\mu-1}, x_{\mu+1}],\ 1 \le \mu \le N-1, \end{cases}$$

$$(4.6)\qquad W_0(x) = \begin{cases} (x_1 - x)/\Delta x, & -b \le x \le x_1, \\ 0, & x_1 \le x \le b, \end{cases}$$

$$(4.7)\qquad W_N(x) = \begin{cases} (x - x_{N-1})/\Delta x, & x_{N-1} \le x \le b, \\ 0, & -b \le x \le x_{N-1}. \end{cases}$$

Put

$$(4.8)\qquad \tilde{f}_{i\mu}^{(\nu)} = \tilde{f}_i^{(\nu)}(x_\mu).$$

From there, we define $\tilde{f}_{i\mu}^{(\nu)}$ by recurrence according to repeat steps $\nu = 1, 2, \ldots$.

$$(4.9)\qquad \tilde{f}_{i\mu}^{(\nu)} = \sum_{j=1}^{n} \sum_{k=1}^{m} a_{ijk}[x_\mu, \sum_{\eta=0}^{N} \tilde{f}_{j\eta}^{(\nu-1)} W_\eta(S_{ijk}(x_\mu))] + g_i(x_\mu),$$

$$0 \le \mu \le N,\ 1 \le i \le n,\ \nu \ge 1,$$

$$(4.10)\qquad \tilde{f}_{i\mu}^{(0)} \equiv 0.$$

Numerical application is effectuated on the two following examples

EXAMPLE 1. Case of nonlinearity. Consider the following system with $n = 2$, $m = 1$, $b = 1$, $x \in [-1, 1]$.

$$(4.11)\qquad \begin{cases} f_1(x) = \dfrac{1}{100} \sin f_1\left(\dfrac{x}{2} + \dfrac{1}{3}\right) + \dfrac{1}{200}\left| f_2\left(\dfrac{x}{3} + \dfrac{1}{2}\right)\right| + g_1(x), \\ f_2(x) = \dfrac{1}{200} f_1\left(\dfrac{x}{3} + \dfrac{1}{2}\right) + \dfrac{1}{100} \cos f_2\left(\dfrac{x}{3} + \dfrac{1}{2}\right) + g_2(x), \end{cases}$$

where

$$(4.12)\qquad \begin{cases} g_1(x) = x - \dfrac{1}{200}\left(\dfrac{x}{3} + \dfrac{1}{2}\right)^2 - \dfrac{1}{100} \sin\left(\dfrac{x}{2} + \dfrac{1}{3}\right), \\ g_2(x) = x^2 - \dfrac{1}{200}\left(\dfrac{x}{3} + \dfrac{1}{2}\right) - \dfrac{1}{100} \cos\left(\dfrac{x}{3} + \dfrac{1}{2}\right)^2. \end{cases}$$

The exact solution of the system (4.11), (4.12) is

$$(4.13)\qquad f_1^{ex}(x) = x, f_2^{ex}(x) = x^2.$$

We calculate with algorithm (4.9), (4.10) until ν^{th} repeat step satisfying

$$(4.14)\qquad \max_{\substack{0 \le \mu \le N \\ 1 \le i \le 2}} |\tilde{f}_{i\mu}^{(\nu)} - \tilde{f}_{i\mu}^{(\nu-1)}| < 10^{-8}.$$

Afterwards, let N increase respectively with $N = 5, 10, 15, 20, 100$. The result given by the tables 1, 2 as follows indicate the calculating values $\widetilde{f}_{i\mu}^{(\nu)}$ comparing to the exact values $f_1^{ex}(x_\mu)$ at knot-points x_μ, $0 \le \mu \le N = 5$. The tables 3 and 4 give the result of variation of error when N increase gradually.

μ	$\widetilde{f}_{1\mu}^{(\nu)}$	$f_1^{ex}(x)(x_\mu)$	$E_{1\mu}$
0	−1.000	−1.000	0.000061
1	−0.600	−0.600	0.000149
2	−0.200	−0.200	0.000193
3	0.200	0.200	0.000060
4	0.600	0.600	0.000149
5	1.000	1.000	0.000194

Table 1: $N = 5$, $E_{1\mu} = |\widetilde{f}_{1\mu}^{(\nu)} - f_1^{ex}(x_\mu)|$

μ	$\widetilde{f}_{2\mu}^{(\nu)}$	$f_2^{ex}(x_\mu)$	$E_{2\mu}$
0	1.000	1.000	0.000004
1	0.360	0.360	0.000071
2	0.040	0.040	0.000239
3	0.040	0.040	0.000372
4	0.359	0.360	0.000525
5	1.000	1.000	0.000251

Table 2: $N = 5$, $E_{2\mu} = |\widetilde{f}_{2\mu}^{(\nu)} - f_2^{ex}(x\mu)|$

Table 3. **Table 4.**

N	$e_1 = \max\limits_{0\le\mu\le N} E_{1\mu}$	$e_2 \max\limits_{0\le\mu\le N} E_{2\mu}$
5	0.000194	0.000525
10	0.000049	0.000429
15	0.000020	0.000415
20	0.000010	0.000406
100	0.000002	0.000397

EXAMPLE 2. Case of linearity : $a_{ijk}(x, y) = \widetilde{a}_{ijk}y$. Consider the following system with $n = m = 2$, $b = 1$, $-1 \le x \le 1$,

(4.15)
$$\begin{cases} f_1(x) = \dfrac{1}{100} f_1\left(\dfrac{x}{2} + \dfrac{1}{3}\right) + \dfrac{1}{200} f_2\left(\dfrac{x}{2} + \dfrac{1}{4}\right) + \dfrac{1}{200} f_2\left(\dfrac{x}{3} + \dfrac{1}{2}\right) + g_1(x), \\ f_2(x) = \dfrac{1}{200} f_1(\dfrac{x}{3} + \dfrac{1}{2}) + \dfrac{1}{200} f_1\left(\dfrac{x}{2} + \dfrac{1}{3}\right) + \dfrac{1}{100} f_2\left(\dfrac{x}{3} + \dfrac{1}{4}\right) + g_2(x), \end{cases}$$

where

$$(4.16)\qquad \begin{cases} g_1(x) = \dfrac{97}{100}x - \dfrac{31}{1200}, \\ g_2(x) = \dfrac{7171}{1200}x - \dfrac{23}{1200}. \end{cases}$$

The numbers $\widetilde{a}_{ijk}$, b_{ijk}, c_{ijk}, with i, j, k equal to 1 or 2 satisfy (2.10), (2.13), (2.14). The exact solution of (4.15), (4.16) is

$$f_1^{ex}(x) = x, \qquad f_2^{ex}(x) = 6x.$$

We calculate by algorithm (4.9), (4.10) with repeat steps $\nu = 1, 2, 3, \ldots$ such that

$$(4.17)\qquad \max_{\substack{1\le i\le 2,\\ 0\le \mu\le N}} |\widetilde{f}_{i\mu}^{(\nu)} - \widetilde{f}_{i\mu}^{(\nu-1)}| < 10^{-8}.$$

The result given by the tables 5 and 6 as follows indicate the calculating values $\widetilde{f}_{i\mu}^{(\nu)}$ comparing to the exact values $f_i^{ex}(x_\mu)$ at knot-points $\mu_0, \mu_1, \ldots, \mu_5$ for $N = 5$.

μ	$\widetilde{f}_{1\mu}^{(\nu)}$	$f_1^{ex}(x_\mu)$	$E_{1\mu}$
0	−1.000	−1.000	0.000000
1	−0.600	−0.600	0.000000
2	−0.200	−0.200	0.000000
3	0.200	0.200	0.000000
4	0.600	0.600	0.000000
5	1.000	1.000	0.000000

Table 5: $N = 5$, $E_{1\mu} = |\widetilde{f}_{1\mu}^{(\nu)} - f_1^{ex}(x_\mu)|$, $\max_{0\le\mu\le 5} E_{1\mu} = 3.035150.10^{-11}$.

μ	$\widetilde{f}_{2\mu}^{(\nu)}$	$f_2^{ex}(x_\mu)$	$E_{2\mu}$
0	−6.000	−6.000	0.000000
1	−3.600	−3.600	0.000000
2	−1.200	−1.200	0.000000
3	1.200	1.200	0.000000
4	3.600	3.600	0.000000
5	6.000	6.000	0.000000

Table 6: $N = 5$, $E_{2\mu} = |\widetilde{f}_{2\mu}^{(\nu)} - f_2^{ex}(x_\mu)|$, $\max_{0\le\mu\le 5} E_{2\mu} = 3.035150.10^{-11}$.

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