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ON SOME GENERALIZATION OF UNIFORMLY LIPSCHITZIAN MAPPINGS AND ITS FIXED POINTS

1. Introduction

Let (E,d) be a metric space. A mapping $T : E \to E$ is called *uniformly k-Lipschitzian* if there exists a constant $k > 0$ such that

$$d(T^n x, T^n y) \leq k \cdot d(x, y)$$

for any points x, y in E and any positive integer n.

The first fixed point theorem for uniformly Lipschitzian mappings in Banach spaces was given by Goebel and Kirk [3] who states a relationships between the existence fixed point for these mappings and the Clarkson modulus of convexity. The existence of a fixed point of uniformly k-Lipschitzian mappings have been investigated by many authors, cf. [1]. Recently, Tan and Xu [8] presented new fixed point theoren for uniformly k-Lipschitzian mappings in uniformly convex Banach spaces.

A more general approach is proposed by Lifschitz [5] , who defines the following coefficient in metric space (E,d):

> $\kappa(E)$ is the supremum of all positive numbers b such that there exists $a > 1$ such that for every x, y in E and $r > 0$ with $d(x, y) > 0$ there exists z in E satisfying $\bar{B}(x, br) \cap \bar{B}(y, ar) \subset \bar{B}(z, r)$.

It is clear that $\kappa(E) \geq 1$ and $\kappa(E) > 1$ for strictly convex spaces. Lifshitz proved the following Theorem:

THEOREM 1. *Let (E,d) be a complete metric space and $T : E \to E$ a uniformly k-Lipschitzian mapping with constant $k < \kappa(E)$. If there exists $x \in E$ such that the sequence $\{T^n x\}$ is bounded, then T has a fixed point in E.*

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In a Banach space E we denote by $\kappa_0(E)$ the infimum of the numbers $\kappa(M)$ when M is a nonempty closed convex bounded subsed of E. The direct computation of $\kappa_0(E)$ is a difficult problem, because we can consider many different convex subsets of E with arbitrary shape. The value of $\kappa_0(E)$ is only known when E is a Hilbert space H (cf. [5,1]), $\kappa_0(H) = \sqrt{2}$, or for some classes of Banach spaces which are isomorphic to l^2 (cf. [2]): for James's spaces $E_\lambda = (l^2, \|x\|_\lambda = max\{\|x\|_{l^2}, \lambda \cdot \|x\|_\infty\}), \lambda \geq 1$,

$$\kappa_0(E_\lambda) = \sqrt{1 + \lambda^{-2} - 2 \cdot \lambda^{-2} \cdot \sqrt{\lambda^2 - 1}} \quad \text{if } 1 \leq \lambda \leq \frac{1}{2} \cdot \sqrt{5},$$

$$\kappa_0(E_\lambda) = 1 \quad \text{if } \lambda > \frac{1}{2} \cdot \sqrt{5}.$$

Recently, Dominguez Benavides [2] using Bynum's normal structure coefficient $N(E)$ extended Lifshitz's Theorem.

We recall,

$$N(E) = \left\{ \frac{\operatorname{diam}(M)}{\inf_{y \in M}(\sup_{x \in M} \|x - y\|)} : M \subset E \text{ convex closed bounded with } \operatorname{diam}(M) > 0 \right\}.$$

A different form of this coefficient was given by Lim [6]:

$$N(E) = \left\{ \frac{\operatorname{diam}_a\{x_n\}}{r_a\{x_n\}} : \{x_n\} \text{ is a bounded sequence which is not norm convergent} \right\},$$

where

$$\operatorname{diam}_a\{x_n\} = \lim_{k \to \infty} (\sup\{\|x_n - x_m\| : n,\ m \geq k\}),$$

$$r_a\{x_n\} = \inf\{ \overline{\lim_{n \to \infty}} \|x_n - y\| : y \in \overline{conv}\{x_n\}\}.$$

It is known that [1]:

1) for a Hilbert space H, $N(H) = \sqrt{2}$;
2) for James's space E_λ, $N(E_\lambda) = \frac{1}{\lambda} \cdot \sqrt{2}$ if $1 \leq \lambda \leq \sqrt{2}$;
3) $N(l^p) = N(L^p) = \min\{2^{1/p}, 2^{1-1/p}\}$, $1 < p < +\infty$;
4) $\kappa_0(E) \leq N(E)$.

Dominguez Benavides proved the following Theorem:

THEOREM 2. *Let E be a Banach space, M a nonempty closed convex bounded subset of E and $T : M \to M$ uniformly k-Lipschitzian mapping. If*

$$k < \frac{1}{2} \cdot \left(1 + \sqrt{1 + 4 \cdot N(E) \cdot (\kappa_0(E) - 1)}\right),$$

then T has a fixed point in M.

2. Main result

In the present paper we extend Theorem 2 to more general class of mappings (not necessarily continuous) $T : M \to M$ whose n-th iterate $(n = 1, 2, \ldots)$ satisfying the following condition:

$$(*) \quad \|T^n x - T^n y\|$$
$$\leq A\cdot\|x-y\|+B\cdot\{\|x-T^n x\|+\|y-T^n y\|\}+C\cdot\{\|x-T^n y\|+\|y-T^n x\|\}$$

for all x, y in M, where the nonnegative constants A, B, C satisfy $B+C < 1$.

The mappings T satisfying (*) are called *generalized uniformly Lipschitzian mappings.*

Recently, Singh and Jung [7] have given the following generalization of Lifshitz's Theorem:

THEOREM 3. *Let (E, d) be a complete metric space. If $T : E \to E$ is a generalized uniformly Lipschitzian mapping with $\frac{A+3(B+C)}{1-(B+C)} < \kappa(E)$, and for some $x \in E$ the sequence $\{T^n x\}$ is bounded, then T has a fixed point in M.*

We follow an idea of [2] and prove the following fixed point theorem for generalized uniformly Lipschitzian mappings:

THEOREM 4. *Let E be a Banach space, M a nonempty closed convex bounded subsed of E and $T : M \to M$ be a mapping satisfying for all x, y in $M, n = 1, 2, \ldots$ the condition $(*)$. If*

$$\frac{A+3(B+C)}{1-(B+C)} < \frac{1}{2}\cdot\left(1+\sqrt{1+4\cdot N(E)\cdot(\kappa_0(E)-1)}\right),$$

then T has a fixed point in M.

Proof. If $\frac{1}{2}\cdot(1+\sqrt{1+4\cdot N(E)\cdot(\kappa_0(E)-1)}) = 1$, then $\kappa_0(E) = 1$. In this case the existence of fixed points follows from Hardy-Rogers's Theorem [4]. We prove our result if

$$\frac{1}{2}(1+\sqrt{1+4\cdot N(E)\cdot(\kappa_0(E)-1)}) > 1.$$

Denote $N = N(E)$ and $\kappa_0 = \kappa_0(E)$. We can assume $k = \frac{A+3(B+C)}{1-(B+C)} > 1$ and observe that the condition $\kappa_0 \leq N$ implies

$$\frac{1}{2}\cdot(1+\sqrt{1+4\cdot N\cdot(\kappa_0-1)}) \leq N,$$

and hence $k < N$. Next note that the condition

$$k < \frac{1}{2}\cdot(1+\sqrt{1+4\cdot N\cdot(\kappa_0-1)})$$

is equivalent to

$$\frac{k}{N} < \frac{\kappa_0-1}{k-1}.$$

Choose $b < \kappa_0$ such that

$$\frac{k}{N} < \frac{b-1}{k-1}.$$

Let $a > 1$ be the corresponding number to b in definition of $\kappa(M)$. We can assume

$$\frac{k}{N} < \frac{\frac{b}{a}-1}{k-1}.$$

Choose $\varepsilon > 0$ such that $\frac{1}{a} \cdot (1+2\varepsilon) = \alpha < 1$.

For every $x \in M$ define

$$R(x) = \inf\{r > 0 : \text{ there exists } y \in M \text{ satisfy } \overline{\lim}_{n\to\infty}\|x - T^n y\| \leq r\}.$$

We shall prove that $R(x) = 0$ for some $x \in M$. Let x be an arbitrary point in M. If $R(x) > 0$ choose $y \in M$ such that

$$\overline{\lim_{n\to\infty}}\|x - T^n y\| < R(x) \cdot (1+\epsilon).$$

There are two cases:

Case I.

$$\sup\{\|x - T^n x\| : n \geq 1\} \leq \frac{N \cdot R(x) \cdot (1+\varepsilon)}{k \cdot a}.$$

In this case, since

$$\|T^n x - T^m x\| \leq \frac{A+B+C}{1-(B+C)}\|x - T^{n-m}x\| + \frac{2(B+C)}{1-(B+C)}\|x - T^n x\|$$

if $n > m$ we know that

$$\operatorname{diam}_a\{T^n x\} \leq \frac{N \cdot R(x) \cdot (1+\varepsilon)}{a}.$$

Since $diam_a\{T^n x\} \geq r_a\{T^n x\} \cdot N$, we obtain

$$\frac{N \cdot R(x) \cdot (1+\varepsilon)}{a} \geq N \cdot r_a\{T^n x\},$$

which implies

$$r_a\{T^n x\} \leq \frac{R(x) \cdot (1+\varepsilon)}{a}.$$

Then there exists $z \in M$ such that

$$\overline{\lim_{n\to\infty}}\|T^n x - z\| < \frac{R(x) \cdot (1+2\varepsilon)}{a} = \alpha \cdot R(x).$$

Thus $R(z) < \alpha \cdot R(x)$. Furthermore $\|z - x\| \leq \|z - T^n x\| + \|T^n x - x\|$ which implies

$$\|z-x\| \le \overline{\lim_{n\to\infty}}\|z-T^nx\| + \overline{\lim_{n\to\infty}}\|T^nx-x\|$$
$$\le \alpha\cdot R(x) + \frac{N\cdot\alpha\cdot R(x)}{k} = \alpha\cdot(1+\frac{N}{k})\cdot R(x).$$

CASE II.

$$\sup\{\|x-T^nx\| : n\ge 1\} > \frac{N\cdot R(x)\cdot(1+\varepsilon)}{k\cdot a}.$$

In this case there exists $i\in \mathbf{N}$ such that

$$\|x-T^ix\| > \frac{N\cdot R(x)\cdot(1+\varepsilon)}{k\cdot a}.$$

Choose $j\in \mathbf{N}$ such that $\|x-T^ny\| < R(x)\cdot(1+\varepsilon)$ for $n\ge j$. Hence for $n\ge j$, we have

$$\|T^{i+n}y-T^ix\| \le \frac{A+B+C}{1-(B+C)}\|x-T^ny\| + \frac{2(B+C)}{1-(B+C)}\|x-T^{i+n}y\|$$
$$\le k\cdot R(x)\cdot(1+\varepsilon).$$

Choose $\lambda\in(0,1)$ such that $\frac{k}{N}<\lambda<\frac{\frac{b}{a}-1}{k-1}$. Then for $n\ge i+j$, we get

$$\|T^ny-\lambda\cdot T^ix+(1-\lambda)\cdot x\| \le$$
$$\le \lambda\cdot\|T^ny-T^ix\| + (1-\lambda)\cdot\|T^ny-x\| \le$$
$$\le \lambda\cdot k\cdot R(x)\cdot(1+\varepsilon) + (1-\lambda)\cdot R(x)\cdot(1+\varepsilon) \le$$
$$\le [\lambda\cdot(k-1)+1]\cdot R(x)\cdot(1+\varepsilon) <$$
$$< \frac{b}{a}\cdot R(x)\cdot(1+\varepsilon).$$

Furthermore

$$\|x-\lambda\cdot T^ix+(1-\lambda)\cdot x\| = \lambda\cdot\|T^ix-x\| \ge$$
$$\ge \lambda\cdot\frac{N\cdot R(x)\cdot(1+\varepsilon)}{k\cdot a} \ge R(x)\cdot\frac{1+\varepsilon}{a}.$$

By the definition of b there exists $z\in M$ such that

$$\|T^ny-z\| \le \frac{R(x)\cdot(1+\varepsilon)}{a} \le \alpha\cdot R(x)$$

for n $\ge i+j$. Thus $R(z)\le\alpha\cdot R(x)$, and

$$\|z-x\| \le \|z-T^ny\| + \|T^ny-x\| \le R(x)\cdot(1+\varepsilon+\alpha).$$

Define $f(x)=z$, z choosen as in the case I or II. By induction, take x_0 arbitrary in M and put $x_n=f(x_{n-1}), n=1,2,\dots$. Then

$$R(x_n)\le\alpha\cdot R(x_{n-1})\le\dots\le\alpha^n\cdot R(x_0).$$

We shall prove that $\{x_n\}$ is a Cauchy sequence. Indeed, if S= max $\{1+\varepsilon+\alpha, \alpha\cdot(1+\frac{N}{k}\}$, we have

$$\|x_{n+1}-x_n\| \leq S\cdot R(x_{n-1}) \leq \ldots \leq S\cdot\alpha^{n-1}\cdot R(x_0).$$

Thus $\{x_n\}$ converges to some x in M. It is clear that $R(x)=0$. We shall prove that x is a fixed point of T. Indeed, for any $\gamma>0$ there exists $y\in M$ such that $\|x-T^n y\|<\gamma$ if $n\geq P=P(\gamma)$. Thus for $n\geq P$, we have

$$(**)\qquad \|T^n x - x\| \leq \|T^n x - T^{2n}y\| + \|T^{2n}y - x\| \leq$$
$$\leq \frac{A+B+C}{1-(B+C)}\|x-T^n y\| + \frac{2(B+C)}{1-(B+C)}\|x-T^{2n}y\| + \|T^{2n}y-x\| < k\cdot\gamma+\gamma$$

and by $(**)$

$$\|T^{n+1}x - Tx\| \leq \frac{A+B+C}{1-(B+C)}\|x-T^n x\| + \frac{2(B+C)}{1-(B+C)}\|x-T^{n+1}x\|$$
$$< (k\cdot\gamma+\gamma)\cdot k.$$

Hence

$$\|Tx-x\| \leq \|Tx-T^{n+1}x\| + \|T^{n+1}x-x\| < (k\cdot\gamma+\gamma)\cdot k + k\cdot\gamma+\gamma =$$
$$(k+1)^2\cdot\gamma \to 0$$

as $\gamma\downarrow 0$. This completes the proof.

3. Final remarks

1. Since $\kappa_0(E)\leq N(E)$ it is easy to prove that

$$\kappa_0(E) \leq \frac{1}{2}\cdot\left(1+\sqrt{1+4\cdot N(E)\cdot(\kappa_0(E)-1)}\right) \leq N(E)$$

and the first equality only holds if $\kappa_0(E)=1$ or $\kappa_0(E)=N(E)$.

Note that for James's spaces $E_\lambda, \lambda>1$,

$$1<\kappa_0(E_\lambda)<N(E_\lambda).$$

Hence, for these Banach spaces Theorem 4 is strictly more general that the Singh-Jung's Theorem [7].

2. Since $\frac{1}{2}\cdot\left(1+\sqrt{1+4\cdot N(E_\lambda)\cdot(\kappa_0(E_\lambda)-1)}\right)$ converges to $\sqrt{2}$ as $\lambda\to 1$ it is clear that for λ close to 1 the constant which appears in Theorem 4 is strictly bigger than the constant $\sqrt{N(E_\lambda)}$ which appears in Casini-Maluta's Theorem, cf. [1].

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