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ON SOLUTIONS OF SOME SYSTEM OF FUNCTIONAL EQUATIONS

Let us denote by $\mathbf{R}$ the set of real numbers and let $\mathbf{R}_0 = \mathbf{R} \setminus \{0\}$. In paper [7] (p. 79), the following system of functional equations

$$\begin{aligned}(1)\qquad & F(y_1x_1, y_1x_2 + y_2x_1^4 + 4x_1F(y_1,y_2)g(x_1,x_2) + \\ & +6x_1^2g(y_1,y_2)F(x_1,x_2) + 3F(y_1,y_2)F^2(x_1,x_2)) = \\ & y_1F(x_1,x_2) + x_1^2F(y_1,y_2)\,,\end{aligned}$$

$$\begin{aligned}(2)\qquad & g(y_1x_1, y_1x_2 + y_2x_1^4 + 4x_1F(y_1,y_2)g(x_1,x_2) + \\ & +6x_1^2g(y_1,y_2)F(x_1,x_2) + 3F(y_1,y_2)F^2(x_1,x_2)) = \\ & y_1g(x_1,x_2) + 3x_1F(y_1,y_2)F(x_1,x_2) + x_1^3g(y_1,y_2)\end{aligned}$$

in the class of functions $F : \mathbf{R}_0 \times \mathbf{R} \to \mathbf{R}$, $g : \mathbf{R}_0 \times \mathbf{R} \to \mathbf{R}$ was considered and two solutions $F(x_1,x_2) = g(x_1,x_2) = 0$ for any x_1, x_2, and $F(x_1,x_2) = g(x_1,x_2) = \frac{1}{3}(x_1 - x_1^2)$, where pointed out.

In the present paper we shall show that the system (1), (2) has also other solutions in certain classes of functions.

The system (1), (2) appeared when some subsemigroups of the group L_4^1 were determined. The definition of L_S^1 one can find in [2]. In papers [1]-[16] the authors dealt with the determination of subgroups and subsemigroups by means of functional equations.

First let us consider the system (1), (2) in the case of

$$(3)\qquad F(x_1,x_2) \not\equiv 0$$

for arbitrary x_1 and x_2. Then the equation (2) takes the form

$$(4)\qquad g(y_1x_1, y_1x_2 + y_2x_1^4) = y_1g(x_1,x_2) + x_1^3g(y_1,y_2)\,.$$

Hence, for $x_2 = y_2 = 0$ we get

$$(5)\qquad g(x_1y_1, 0) = y_1g(x_1,0) + x_1^3g(y_1,0)\,.$$

In view of Theorem 1 in [2], the general solution of (5) is the set of functions of the form

$$g(x_1,0)=\mathbf{p}(x_1^3-x_1)\,, \tag{6}$$

where $\mathbf{p}$ is any real number. If we set $x_1=y_1=1$ in (4), we obtain

$$g(1,x_2+y_2)=g(1,x_2)+g(1,y_2)\,. \tag{7}$$

Therefore, the function

$$\phi(x):=g(1,x) \tag{8}$$

is additive. If we set $x_1=1$ and $y_2=0$ in (4), then due to (6) and (8), we have

$$g(y_1,x_2y_1)=y_1\phi(x_2)+\mathbf{p}\left(y_1^3-y_1\right)\,. \tag{9}$$

Hence for $x_2=\frac{y_2}{y_1}$ we get

$$g(y_1,y_2)=y_1\phi\left(\frac{y_2}{y_1}\right)+\mathbf{p}\left(y_1^3-y_1\right)\,. \tag{10}$$

Putting (10) into (4), we can write

$$\begin{aligned}&x_1y_1\phi\left(\tfrac{y_1x_2+y_2x_1^4}{x_1y_1}\right)+\mathbf{p}\left[(x_1y_1)^3-x_1y_1\right]=\\&y_1x_1\phi\left(\tfrac{x_2}{x_1}\right)+\mathbf{p}\left(x_1^3-x_1\right)y_1+\\&+x_1^3y_1\phi\left(\tfrac{y_2}{y_1}\right)+x_1^3\mathbf{p}\left(y_1^3-y_1\right)\,.\end{aligned}$$

Therefore, due to the fact that ϕ is an additive function for rational x_1, y_1 one has the equality

$$y_1\phi(x_2)+x_1^4\phi(y_2)=y_1\phi(x_2)+x_1^3\phi(y_2)\,.$$

becoming $\phi(y_2)\left(x_1^4-x_1^3\right)=0$ for any $y_2\in\mathbf{R}$. Hence $\phi(y_2)=0$ and it follows from (10) that

$$g(y_1,y_2)=\mathbf{p}\left(y_1^3-y_1\right)\,. \tag{11}$$

We have shown that the general solution of (4) is the family of functions (11), where $\mathbf{p}\in\mathbf{R}$.

The system (1), (2) has then infinitely many solutions, namely the set of pairs of the functions

$$F(y_1,y_2)=0,\; g(y_1,y_2)=\mathbf{p}\left(y_1^3-y_1\right)\,,$$

where $\mathbf{p}\in\mathbf{R}$.

Let us consider now the system (1), (2) with F not depending on the second variable, i.e.,

$$F(x_1,x_2)=F(x_1,0)\,. \tag{12}$$

Then (1) reduces to the equation

$$F(x_1y_1,0)=y_1F(x_1,0)+x_1^2F(y_1,0)\,. \tag{13}$$

In virtue of Theorem 1 in [2], the general solution of (13) is

$$F(x_1, 0) = \mathbf{c}(x_1^2 - x_1) \,, \tag{14}$$

where $\mathbf{c} \in \mathbf{R}$. Hence, under the assumption (12),

$$F(x_1, x_2) = \mathbf{c}(x_1^2 - x_1) \,. \tag{15}$$

Substituting (15) into (2), we get

$$\begin{aligned} g\Big(x_1y_1, y_1x_2 + y_2x_1^4 + 4x_1g(x_1,x_2)\mathbf{c}(y_1^2 - y_1) +& \\ +6x_1^2g(y_1,y_2)\mathbf{c}(x_1^2 - x_1) + 3\mathbf{c}^3(x_1^2 - x_1)^2(y_1^2 - y_1)\Big) &= \\ = y_1g(x_1,x_2) + 3x_1\mathbf{c}^2(x_1^2 - x_1)(y_1^2 - y_1) + x_1^3g(y_1,y_2) &\,. \end{aligned} \tag{16}$$

If we put first $x_1 = 1$ and then $y_1 = 1$ in (16), we obtain

$$g\Big(y_1, y_1x_2 + y_2 + 4\mathbf{c}g(1,x_2)(y_1^2 - y_1)\Big) = y_1g(1,x_2) + g(y_1,y_2) \,, \tag{17}$$

$$g\Big(x_1, x_2 + y_2x_1^4 + 6x_1^2g(1,y_2)\mathbf{c}(x_1^2 - x_1)\Big) = g(x_1,x_2) + x_1^3g(1,y_2) \,. \tag{18}$$

If we set $y_1 = 1$ in (17), we have $g(1, x_2 + y_2) = g(1, x_2) + g(1, y_2)$. Therefore the function

$$\psi(x) := g(1, x) \tag{19}$$

is additive. Let us consider the cases

$$\mathbf{c} = 0 \,, \tag{20}$$

$$\mathbf{c} \neq 0 \,. \tag{21}$$

If $\mathbf{c} = 0$, then (17) for $y_2 = 0$ and (18) for $x_2 = 0$ become

$$g(y_1, y_1x_2) = y_1g(1,x_2) + g(y_1, 0) \tag{22}$$

and

$$g(x_1, y_2x_1^4) = g(x_1, 0) + x_1^3g(1, y_2) \,, \tag{23}$$

respectively. Setting $y_1 = x_1$ and $x_2 = y_2$ in (22), we get

$$g(x_1, x_1y_2) = x_1g(1, y_2) + g(x_1, 0) \tag{24}$$

and then, from (23) and (24)

$x_1^3g(1,y_2) + g(x_1,0) = g(x_1, x_1^4y_2) = g(x_1, x_1(x_1^3y_2)) = x_1g(1, x_1^3y_2) + g(x_1, 0)$,

implying $x_1^3g(1,y_2) = x_1g(1,x_1^3y_2)$. By using the additivity of $x \to g(1,x)$ for rational x_1, we get from the last equality $x_1^3g(1,y_2) = x_1^4g(1,y_2)$ and then

$$g(1, y_2) = 0 \tag{25}$$

for any $y_2 \in \mathbf{R}$. Hence, by (22) with $x_2 = \frac{y_2}{y_1}$, one has

$$g(y_1, y_2) = g(y_1, 0) \tag{26}$$

for any $y_2 \in \mathbf{R}$.

Consider now the case of $\mathbf{c} \neq 0$. Let us write (17), (18) in the forms

(27) $g\left(y_1, y_2 + y_1(x_2 + 4\mathbf{c}g(1,x_2)(y_1 - 1))\right) = y_1 g(1,x_2) + g(y_1,y_2)$,

(28) $g\left(y_1, y_2 + y_1^3(x_2 y_1 + 6\mathbf{c}g(1,x_2)(y_1 - 1))\right) = y_1^3 g(1,x_2) + g(y_1,y_2)$.

Fix x_2 arbitrarily. The following cases may occur:

(a) $g(1,x_2) = 0$,

(b) $g(1,x_2) \neq 0$.

In the case (b) we choose $y_1 \neq 0$ such that

(29) $$x_2 + 4\mathbf{c}g(1,x_2)(y_1 - 1) = 0$$

or

(30) $$x_2 y_1 + 6\mathbf{c}g(1,x_2)(y_1 - 1) = 0\,.$$

If $g(1,x_2) \neq \frac{x_2}{4\mathbf{c}}$, it suffices to set $y_1 = 1 - \frac{x_2}{4\mathbf{c}g(1,x_2)}$ in order to get (29). While if $g(1,x_2) = \frac{x_2}{4\mathbf{c}}$ the equality (30) is satisfied for $y_1 = \frac{3}{5}$. Then (27), (28) take the forms

(31) $$g(y_1,y_2) = y_1 g(1,x_2) + g(y_1,y_2)\,,$$

(32) $$g(y_1,y_2) = y_1^3 g(1,x_2) + g(y_1,y_2)\,.$$

Hence it follows $g(1,x_2) = 0$ for $y_1 \neq 0$. So, we have obtained a contradiction in the case (b). Therefore, $g(1,x_2) = 0$ Then (27), will take the form

(33) $$g(y_1, y_2 + y_1 x_2) = g(y_1,y_2)\,.$$

Since $y_1 \neq 0$ and x_2 is an arbitrary real number, we have, by (33) for $x_2 = -\frac{y_2}{y_1}$,

(34) $$g(y_1,y_2) = g(y_1,0)$$

for any $y_2 \in \mathbf{R}$. In the case of $\mathbf{c} = 0$ we have obtained (26). Therefore, the equality (34) is true for any $\mathbf{c}$. It follows that (16) will have the form

(35) $g(x_1y_1,0) = y_1 g(x_1,0) + 3x_1\mathbf{c}^2(x_1^2 - x_1)(y_1^2 - y_1) + x_1^3 g(y_1,0)$.

Observe that the left hand side of (35) is symmetric with respect to x_1, y_1, and so the right-hand side must be too. Then we have

$$y_1 g(x_1,0) + 3x_1\mathbf{c}^2(x_1^2 - x_1)(y_1^2 - y_1) + x_1^3 g(y_1,0) =$$
$$x_1 g(y_1,0) + 3y_1\mathbf{c}^2(y_1^2 - y_1)(x_1^2 - x_1) + y_1^3 g(x_1,0)$$

Hence for any fixed x_1 such that $x_1^3 - x_1 \neq 0$

$$g(y_1,0) = \frac{g(x_1,0)}{x_1^3 - x_1}(y_1^3 - y_1) + \frac{3\mathbf{c}^2(x_1^2 - x_1)}{x_1^3 - x_1}(y_1^2 - y_1)(y_1 - x_1)\,.$$

Therefore, there exist constants $\mathbf{b}$, $\mathbf{d}$, $\mathbf{e}$ such that

(36) $$g(y_1,0) = \mathbf{b}y_1^3 + \mathbf{d}y_1^2 + \mathbf{e}y_1\,.$$

Putting (36) into (35), we get

$$\begin{aligned}&\mathbf{b}x_1^3y_1^3+\mathbf{d}x_1^2y_1^2+\mathbf{e}x_1y_1=\\&=\mathbf{b}y_1x_1^3+\mathbf{d}y_1x_1^2+\mathbf{e}y_1x_1+3x_1\mathbf{c}^2(x_1^2-x_1)(y_1^2-y_1)+\\&+\mathbf{b}y_1^3x_1^3+\mathbf{d}x_1^3y_1^3+\mathbf{e}x_1^3y_1\,.\end{aligned}$$

Hence $\mathbf{b}=+3\mathbf{c}^2-\mathbf{e}$, $\mathbf{d}=-3\mathbf{c}^2$, and (36) becomes

$$g(y_1,0)=(3\mathbf{c}^2-\mathbf{e})y_1^3-3\mathbf{c}^2y_1^2+\mathbf{e}y_1\,. \tag{37}$$

So, we have proved the following lemma.

LEMMA 1. *The general solution of the equation* (35) *is the family of functions of the form* (37), *where* $\mathbf{e}$ *is an arbitrary real number.*

By Lemma 1 and the equality (34) we get the result al follows.

LEMMA 2. *The general solution of the equation* (16) *is the family*

$$g(y_1,y_2)=(3\mathbf{c}^2-\mathbf{e})y_1^3-3\mathbf{c}^2y_1^2+\mathbf{e}y_1\,, \tag{38}$$

where $\mathbf{e}$ *is any real number.*

THEOREM 1. *The general solution of the system* (1), (2) *in the set of fonctions satysfying* (12) *is the set of pairs of functions of the form* (15) *and* (38), *where* $\mathbf{c},\mathbf{e}$ *are arbitrary real numbers.*

So we have shown that the system (1), (2) possesses infinitely many solutions. Unfortunately, the general solution of this system is still not known.

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