DEMONSTRATIO MATHEMATICA
Vol. XXXI No 2 1998

Ömer Gök

CHARACTERIZATION OF THE DUAL CENTER OF BARRELLED SPACES

Introduction

It is our aim to prove that the center of the continuous dual X' of a barrelled locally convex Hausdorff space X consists precisely of those operators which are the adjoints of the operators belonging to the center of the barrelled locally convex Hausdorff space X. In [1], Y.A.Abramovich, E.L.Arenson and A.K.Kitover asked the following question. Let $C(K)$ be the set of all real or complex valued continuous functions defined on a compact Hausdorff space K , and let X be a Banach $C(K)$-module or a Banach lattice. Under what conditions on X we have that $Z^*(X) = Z(X')$? where $Z(X')$ is the dual center of X , $Z^*(X)$ is the adjoint operators belonging to the center of X. In [1], they asserted that a Banach lattice X has an order continuous norm if and only if $Z^*(X) = Z(X')$. Therefore, our approach to this question is new. For unexplained notion and terminology we refer to [2],[6] and [7].

We begin with stating some basic conventions, definitions and notation that will be used throughout our work.

We say that a locally convex space X is a locally convex $C(K)$-module if the following conditions are satisfied :

(i) $C(K) \times X \to X, (a,x) \to a.x$, is a bilinear mapping,
(ii) $1.x = x$ for all $x \in X$, $1 \in C(K)$,
(iii) $a.(b.x) = (ab).x$, for $a, b \in C(K)$, $x \in X$,
(iv) Bilinear mapping (i) is separately continuous.

Let X be a barrelled locally convex Hausdorff space and suppose that X is a locally convex $C(K)$-module. Then we call it as a barrelled locally convex $C(K)$-module. By X', we denote the continuous dual of a barrelled locally convex Hausdorff space X , by X'' we denote the second dual of X , i.e. , $(X', \beta(X', X))' = X''$. By $L(X)$ we denote the set of all continuous linear operators with the identity operator I. Then we introduce the following bilinear mappings

(A) $X \times X' \to C(K)', (x, x') \to (x.x')(a) = x'(a.x)$,
(B) $C(K)'' \times X' \to X', (a, x') \to (a.x')x = a(x.x')$.

On X', $C(K)'$ and $C(K)''$ we put $\sigma(X', X)$, $\sigma(C(K)', C(K)'')$ and $\sigma(C(K)'', C(K)')$-topologies , respectively. It is well-known that $C(K)$ is $\sigma(C(K)'', C(K)')$ dense in $C(K)''$, [2]. Then the bilinear mappings(A) and (B) are separately continuous with respect to respective topologies. Multiplication on $C(K)''$ defined by

$$C(K)'' \times C(K)'' \to C(K)'', (a, b) \to (a.b)c = b(a.c)$$

is known as Arens product [3],[4], and [5]. It is well-known that $C(K)$ is an AM-space with unit and the second dual $C(K)''$ of $C(K)$ is a Dedekind complete AM-space with unit [5]. Hence, $C(K)''$ is isomorphic to $C(S)$ with S hyperstonian [6]. We now introduce the following lemma which is familiar from [3].

LEMMA 1. *Let X be a barrelled locally convex $C(K)$-module. Then the following implications are true.*

(i) *The mapping $m : C(K) \to L(X)$ defined by $m(a)x = a.x$ is norm to strong operator continuous unital algebra homomorphism.*

(ii) *The mapping $m^* : C(K)'' \to L(X')$ defined by $m^*(a)x' = a.x'$ is $\sigma(C(K)'', C(K)')$ to w^*- operator continuous unital algebra homomorphism.*

(iii) *$m^*(a) = (m(a))^*$ for all $a \in C(K)$, where $(m(a))^*$ is the adjoint of $m(a)$.*

Proof. (ii) Let $\{a_\alpha\}$ be a net in $C(K)''$ which converges to a in $\sigma(C(K)''$, $C(K)')$, and let $x \in X$, $x' in X'$ be fixed but arbitrary elements. By the bilinear mapping (A) , we have $x.x' \in C(K)'$. Then $a_\alpha(x.x') \to a(x.x')$. By the bilinear mapping (B), $(a_\alpha.x')x \to (a.x')x$,i.e., $m^*(a_\alpha)x' \to m^*(a)x'$. Since X' is a locally convex $C(S)$-module , it follows that m^* is a unital algebra homomorphism.

DEFINITION 2. (i) Let X be a barrelled locally convex $C(K)$-module, and let $x \in X$ be fixed. Then $\Delta(x)$ is defined by

$$\Delta(x) = \{ax : \|a\| \leq 1, a \in C(K)\}$$

where the closure is taken with respect to the given topology in X.

(ii) Let X be a barrelled locally convex $C(K)$-module and let $Y \subset X$ be a subspace. We say that Y is an ideal in X if for all $x \in Y$, $\Delta(x) \in Y$.

(iii) Let X be a barrelled locally convex $C(K)$-module. Then the center

$Z(X)$ of X is defined by

$$Z(X) = \{T : X \to X | (\exists \lambda > 0)(\forall x \in X), (Tx \in \lambda \Delta(x))\}.$$

Similar definitions, by using bilinear mapping (B), can be done as follows.

$$\Delta(x') = \{a.x' : \|a\| \leq 1\}$$

where the closure is taken with respect to $\sigma(X', X)$ and $x' \in X'$. The center of X', $Z(X')$, is defined by

$$Z(X') = \{T : X' \to X' | (\exists \lambda > 0)(\forall x' \in X')(Tx' \in \lambda \Delta(x'))\}$$

By $Z^*(X)$ we denote the adjoints of the operators belonging to $Z(X)$. Similar concepts were introduced in [1].

PROPOSITION 3. *Let X be a barrelled locally convex $C(K)$-module . Then*

(i) $$\overline{m(C(K))} = Z(X)$$

where the closure is taken with respect to the strong operator topology.

(ii) $$\overline{m^*(C(K))} = \overline{m^*(C(K)'')} = Z(X')$$

where the closure is taken with respect to the w^*- operator topology.

Our principal result is the following theorem.

THEOREM 4. *Let X be a barrelled locally convex $C(K)$-module. Then*

$$Z(X') = Z^*(X).$$

Proof. Assume that $T' \in Z^*(X)$ for $T \in Z(X) = \overline{m(C(K))}$. There exists a net $\{a_\alpha\}$ in $C(K)$ such that $m(a_\alpha)x \to Tx\ forall x \in X$. Then $x'(a_\alpha.x) \to x'(Tx)$ or $m^*(a_\alpha)x'(x) \to T'x'(x)$. Since $m^*(a_\alpha)$ belongs to $\overline{m^*(C(K))}$, we have that $T' \in Z(X')$.

For the converse, let $T \in Z(X')$. Then there exists a net $\{a_\alpha\}$ in $C(K)$ such that $m^*(a_\alpha)x' \to Tx'$ in X'. Hence for all $x \in X$ $(a_\alpha.x')x \to Tx'(x)$. Since $m(a_\alpha)$ in $m(C(K))$, it follows that $T' \in Z(X)$. Therefore, $T \in Z^*(X)$.

References

[1] Y. A. Abramovich, E. L. Arenson and A.K . Kitover, *Banach $C(K)$-modules and operators preserving disjointness*, Pitman Res. Notes in Math. Ser., England, 1992.

[2] C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, Academic Press, London, 1985.

[3] Ö. Gök, *On a Bade type reflexive algebra*, Ph.D.Thesis, Metu, 1990.

[4] D. Hadwin and M. Orhon, *Reflexivity and approximate reflexivity for Boolean algebras of projections*, J. Funct. Anal. 87(2) (1989), 348–358.

[5] C. B. Huijsmans and B. de Pagter, *The order bidual of lattice ordered algebras*, J. Funct. Anal. 59 (1984), 41–64.
[6] H. H. Schaefer, *Banach Lattices and Positive Operators*, New York, Springer Verlag 1974.
[7] H. H. Schaefer, *Topological Vector Space*, Mc. Millan 1966.

YILDIZ TECHNICAL UNIVERSITY
FACULTY OF ARTS AND SCIENCES
DEPARTMENT OF MATHEMATICS
80270 ŞIŞLİ, ISTANBUL, TURKEY

Received January 2, 1997.