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Nalini Shukla, Pratibha Shukla

A CONVOLUTION APPROACH TO CERTAIN SUBCLASSES OF UNIVALENT FUNCTIONS RELATED TO COMPLEX ORDER

1. Introduction

Let $\mathcal{A}$ denote the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.0}$$

analytic in $\mathcal{U} = \{z : |z| < 1\}$. If f and g are any two such functions of $\mathcal{A}$ with f given by (1.0) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then their convolution or their Hadamard product denoted by $f * g$, is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Let $\lambda \in N_0 = \{0, 1, 2, \ldots\}$. The λ-th order Ruscheweyh derivative of f, denoted by $D^\lambda f(z)$, is defined by (cf. [1])

$$D^\lambda f(z) = \frac{z}{\lambda!}\frac{d^\lambda}{dz^\lambda}(z^{\lambda-1} f(z)).$$

Ruscheweyh showed that

$$D^\lambda f(z) = z(1-z)^{-\lambda-1} * f(z), \quad z \in \mathcal{U}.$$

Notice that $D^0 f(z) = f(z)$, $D^1 f(z) = zf'(z)$, $D^2 f(z) = zf'(z) + \frac{1}{2}z^2 f''(z)$.

Goel and Sohi [8] studied the class $\mathcal{M}_\lambda(\alpha)$ of such functions of $\mathcal{A}$ which satisfy

$$\operatorname{Re}\left\{\frac{1}{z} D^{\lambda+1} f(z)\right\} > \alpha, \quad 0 \le \alpha < 1, \quad z \in \mathcal{U}. \tag{1.1}$$

They showed that the functions of $M_\lambda(\alpha)$ are univalent in $\mathcal{U}$. As usual for other classes of univalent functions, α may be called the order of the function of $\mathcal{M}_\lambda(\alpha)$. Recently, in [2], [3], [13], [16] have been introduced some classes of univalent and p-valent function of complex order. But no one has, so far, introduced a class of functions of complex order defined by convolution in this direction. Considering this natural problem, we now introduce a class $V^b_{\lambda,\mu}[A,B]$ as follows.

A function f of $\mathcal{A}$ belongs to $V^b_{\lambda,\mu}[A,B]$ if and only if there exists a function $w(z)$ analytic in $\mathcal{U}$, satisfying $w(0)=0$ and $|w(z)|<1$ for $z\in\mathcal{U}$ and such that

$$\frac{1}{b}\left\{\frac{1}{z}D^{\lambda+1}f(z)-1\right\}=-\mu+\mu\left\{\frac{1+Aw(z)}{1+Bw(z)}\right\},\qquad z\in\mathcal{U},\tag{1.2}$$

where $-1\le B<A\le 1$, $0<\mu\le 1$, $\lambda>-1$ and b is any non-zero complex number.

It is easy to see that, by (1.2), the condition $|w(z)|<1$ is equivalent to

$$\left|\frac{\frac{1}{z}D^{\lambda+1}f(z)-1}{\mu(A-B)b-B\{\frac{1}{z}D^{\lambda+1}f(z)-1\}}\right|<1,\qquad z\in\mathcal{U}.\tag{1.3}$$

By giving specific values to b,A,B,λ,μ in (1.3), we obtain the following subclasses:

(i) For $b=e^{-i\alpha}\cos\alpha$, $\alpha\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, we obtain the class [5] of functions f satisfying

$$\left|\frac{e^{i\alpha}\{\frac{1}{z}D^{\lambda+1}f(z)-1\}}{\mu(A-B)\cos\alpha-Be^{i\alpha}\{\frac{1}{z}D^{\lambda+1}f(z)-1\}}\right|<1,\qquad z\in\mathcal{U}.$$

(ii) For $\mu=1$, $b=1$ we obtain the class [12] of functions f satisfying

$$\left|\frac{\frac{1}{z}D^{\lambda+1}f(z)-1}{A-\frac{B}{z}D^{\lambda+1}f(z)}\right|<1,\qquad z\in\mathcal{U}.$$

(iii) For $\mu=1$, $A=1-2\alpha$, $B=-1$, $b=1$ we obtain the class [8] of functions f satisfying (1.1).

(iv) For $\mu=1$, $\lambda=0$ we obtain the class [6] of functions f satisfying

$$\left|\frac{f'(z)-1}{b(A-B)-B\{f'(z)-1\}}\right|<1,\qquad z\in\mathcal{U}.$$

(v) For $\mu=1$, $\lambda=0$, $b=e^{-i\alpha}\cos\alpha$, $\alpha\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, we obtain the class [19] of functions f satisfying

$$\left|\frac{e^{i\alpha}\{f'(z)-1\}}{Be^{i\alpha}f'(z)-(A\cos\alpha+B\sin\alpha)}\right|<1,\qquad z\in\mathcal{U}.$$

(vi) For $\mu=1$, $b=1$, $\lambda=0$ we obtain the class [7] of functions f satisfying

$$\left|\frac{f'(z)-1}{Bf'(z)-A}\right|<1,\qquad z\in\mathcal{U}.$$

(vii) For $\mu = 1$, $A = \delta$, $B = -\delta$, $b = 1$, $\lambda = 0$ we obtain the class [4], [17] of functions f satisfying

$$\left|\frac{f'(z)-1}{f'(z)+1}\right| < \delta, \quad z \in \mathcal{U}.$$

(viii) For $\mu = 1$, $A = (1-2\rho)\delta$, $B = -\delta$, $b = 1$ and $\lambda = 0$ we obtain the class [10] of functions f satisfying

$$\left|\frac{f'(z)-1}{f'(z)+1-2\rho}\right| < \delta, \quad z \in \mathcal{U},$$

where $0 \le \rho < 1$, $0 < \delta \le 1$.

Thus the study of the class $V^b_{\lambda,\mu}[A,B]$ provides a unified approach for the classes above-mentioned.

2. Preliminary lemmas

LEMMA 2.1 [9]. *If the function W is analytic for $|z| \le r < 1$, $W(0) = 0$ and $|W(z_0)|) = \max_{|z|=r} |W(z)|$, then $z_0 W'(z_0) = \xi W(z_0)$, where ξ is a real number such that $\xi \ge 1$.*

LEMMA 2.2 [11]. *Let $W(z) = \sum_{k=1}^{\infty} c_k z^k$ be analytic with $|W(z)| < 1$ in $\mathcal{U}$. If d is any complex number, then*

$$|c_2 - dc_1^2| \le \max\{1, |d|\}.$$

Equality may be attained with the functions $W(z) = z^2$ and $W(z) = z$.

LEMMA 2.3. *A function f belongs to $V^b_{\lambda,\mu}[A,B], -1 < B < A \le 1$, if and only if*

$$|G(z) - m| < M, \quad z \in \mathcal{U}, \tag{2.1}$$

where

$$G(z) = 1 + \frac{1}{b}\left\{\frac{1}{z}D^{\lambda+1}f(z) - 1\right\} \tag{2.2}$$

and

$$m = 1 - \frac{B\mu(A-B)}{1-B^2}, \quad M = \frac{\mu(A-B)}{1-B^2}.$$

Proof. Suppose that $f \in V^b_{\lambda,\mu}[A,B]$. Then, from (1.2), we get

$$G(z) = \frac{1 + \{B + \mu(A-B)\}W(z)}{1 + BW(z)}.$$

Therefore

$$G(z) - m = \frac{1 - m + \{B + \mu(A-B) - Bm\}W(z)}{1 + BW(z)}$$

$$= M\left\{\frac{BW(z)}{1+BW(z)}\right\} =: Mh(z),$$

where h satisfies $|h(z)| < 1$. Hence, (2.3) implies (2.1).

Conversely, suppose that (2.1) holds. Then we have $\left|\frac{G(z)}{M} - \frac{m}{M}\right| < 1$. Let $g(z) := \frac{G(z)}{M} - \frac{m}{M}$, then, by (2.3),

$$W(z) = \frac{g(z)-g(0)}{1-g(0)g(z)} = \frac{G(z)-1}{\mu(A-B)-B\{G(z)-1\}}, \tag{2.4}$$

with $W(0) = 0$ and $|W(z)| < 1$. Rearranging (2.4), we arrive at (1.2). Hence f belongs to $V^b_{\lambda,\mu}[A,B]$.

NOTE. The condition (2.1) can be written as

$$\left|\frac{(1-B)\{G(z)-1\}+\mu(A-B)}{\mu(A-B)} - \frac{1}{1+B}\right| < \frac{1}{1+B}, \qquad z \in \mathcal{U}.$$

Now, as $B \to -1$, the above condition reduces to

$$\mathrm{Re}\{G(z)\} > \frac{1}{2}\{2-\mu(1+A)\}, \qquad z \in \mathcal{U},$$

which is equivalent to (1.2), when $B = -1$. Thus, including the limiting case $B \to -1$, the results proved with the help of Lemma 2.3 will hold for $-1 \le B < A \le 1$.

3. Main results

We give some properties of the class $V^b_{\lambda,\mu}[A,B]$ in the following theorems, omitting however their proofs, because they run in parallel to that from [6] in the special case $b = e^{-i\alpha}\cos\alpha$.

THEOREM 3.1. *Let λ_0 be any integer greater than λ. Then* $V^b_{\lambda_0,\mu}[A,B] \subset V^b_{\lambda,\mu}[A,B]$.

THEOREM 3.2. *If f, given by* (1.0), *belongs to $V^b_{\lambda,\mu}[A,B]$ then*

$$|a_n| \le \mu(A-B)|b|[\delta(\lambda,n)]^{-1}, \qquad n = 2,3,\ldots,$$

where $\delta(\lambda,n) = \binom{\lambda+n}{\lambda+1}$. *The result is sharp.*

THEOREM 3.3. *If f, given by* (1.0), *belongs to $V^b_{\lambda,\mu}[A,B]$, then*

$$(1-B^2)\sum_{j=2}^{\infty}\{\delta(\lambda,j)\}^2|a_j|^2 \le \mu^2(A-B)^2|b|^2.$$

THEOREM 3.4. *Let f, given by* (1.0), *be analytic in $\mathcal{U}$. If* $\sum_{n=2}^{\infty}(1-B)\delta(\lambda,n)|a_n| \le \mu(A-B)|b|$ *for* $-1 < B < 0$, *then* $f \in V^b_{\lambda,\mu}[A,B]$.

The result is sharp. Although the converse need not be true.

THEOREM 3.5. *If $f \in V^b_{\lambda,\mu}[A,B]$, then*

$$R^- \le \operatorname{Re}\left\{\frac{1}{z}D^{\lambda+1}f(z)\right\} \le R^+,$$

where

$$R^{\mp} = [1 - B^2r^2 - \mu Br^2(A-B)\operatorname{Re}(b) \mp \mu(A-B)|b|r](1-B^2r^2)^{-1}.$$

The bounds are sharp.

THEOREM 3.6. *If f, given by (1.0), belongs to $V^b_{\lambda,\mu}[A,B]$ and β is any complex number, then*

$$|a_3 - \beta a_2^2| \le \mu(A-B)|b|\,[\delta(\lambda,3)]^{-1}\max\{1,|d|\},$$

where $d = [B\{\delta(\lambda,2)\}^2 + \mu(A-B)b\beta\delta(\lambda,3)]\{\delta(\lambda,2)\}^{-2}$.

The result is sharp.

THEOREM 3.7. *If $f \in V^b_{\lambda,\mu}[A,B]$, then F defined by*

$$F(z) = \frac{\gamma+1}{\gamma}\int_0^z t^{\gamma-1}f(t)\,dt, \qquad \gamma > -1,$$

also belongs to $V^b_{\lambda,\mu}[A,B]$.

THEOREM 3.8. *If $f,g \in V^b_{\lambda,\mu}[A,B]$ and $0 \le S \le 1$, then a function F given by $F(z) = Sf(z) + (1-S)g(z)$ also belongs to $V^b_{\lambda,\mu}[A,B]$.*

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Nalini Shukla
DEPARTMENT OF MATHEMATICS
JANTA COLLEGE
BAKEWAR - 206 124
ETAWAH (U.P.), INDIA;

Pratibha Shukla
DEPARTMENT OF MATHEMATICS
CHRIST CHURCH COLLEGE
KANPUR - 208 001, (U.P.), INDIA

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