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A. K. Gaur, Mursaleen

DIFFERENCE SEQUENCE SPACES DEFINED BY A SEQUENCE OF MODULI

Abstract. The idea of difference sequences was introduced by H. Kizmaz [1]. In this paper we define difference sequence spaces by a sequence of moduli and establish some inclusion relations.

1. Introduction

Let l_∞, c, and c_0 be the sequence spaces of bounded, convergent and null sequences $x = (x_k)$ respectively. Recently, Kizmaz [1] has defined the following sequence spaces

$$l_\infty(\triangle) := \{x = (x_k) : \triangle x \in l_\infty\},$$
$$c(\triangle) := \{x = (x_k) : \triangle x \in c\},$$
$$c_0(\triangle) := \{x = (x_k) : \triangle x \in c_0\},$$

where $\triangle x = (\triangle x_k)_{k=1}^\infty = (x_k - x_{k+1})_{k=1}^\infty$.

DEFINITION 1.1. A function $f : [0,\infty) \to [0,\infty)$ is called a *modulus* if

1° $f(t) = 0$ if and only if $t = 0$,
2° $f(t+u) \le f(t) + f(u)$ for all $t, u \ge 0$,
3° f is increasing, and
4° f is continuous from the right of 0.

DEFINITION 1.2. Let X be a sequence space. Then we define the sequence space for a modulus f as follows [4], [5]

$$X(f) := \{x = (x_k) : (f(|x_k|)) \in X\}.$$

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Recently, Kolk [2, 3] gave an extension of $X(f)$ by considering a sequence of moduli $F = (f_k)$, i.e.

$$X(F) := \{x = (x_k) : (f_k(|x_k|)) \in X\}.$$

Define the following sequence spaces for a sequence of modulus $F = (f_k)$,

$$l_\infty(F, \triangle) := \{x = (x_k) : \triangle x \in l_\infty(F)\},$$
$$c_0(F, \triangle) := \{x = (x_k) : \triangle x \in c_0(F)\}$$

for a sequence of moduli $F = (f_k)$. We determine a necessary and a sufficient condition for the inclusions between $X(\triangle)$ and $Y(F, \triangle)$, where $X, Y \neq l_\infty$ or $X, Y \neq c_0$. We will use the following lemmas by Kolk [2].

LEMMA 1.1. *The condition* $\sup_k f_k(t) < \infty$, $t > 0$ *holds if and only if there is a point* $t_0 > 0$ *such that* $\sup_k f_k(t_0) < \infty$.

LEMMA 1.2. *The condition* $\inf_k f_k(t) > 0$ *holds if and only if there exists a point* $t_0 > 0$ *such that* $\inf_k f_k(t_0) > 0$.

2. Main results

THEOREM 2.1. *Let* $F = (f_k)$ *be a sequence of moduli. Then the following statements are equivalent*:

(1) $l_\infty(\triangle) \subseteq l_\infty(F, \triangle)$;
(2) $c_0(\triangle) \subseteq l_\infty(F, \triangle)$;
(3) $\sup_k f_k(t) < \infty$ $(t > 0)$.

Proof. (1) implies (2) is obvious.

(2) implies (3): Let $c_0(\triangle) \subset l_\infty(F, \triangle)$. Suppose that (3) is not true. Then, by Lemma 1.1, $\sup_k f_k(t) = \infty$ for all $t > 0$, and, therefore, there is an index sequence (k_i) such that

$$f_{k_i}(1 + \frac{1}{2} + \cdots + \frac{1}{i-1}) > i \quad \text{for } i = 1, 2, \ldots \tag{2.1}$$

Define $x = (x_k)$ as follows

$$x_k := \begin{cases} 1 + \frac{1}{2} + \cdots + \frac{1}{i-1}, & \text{if } k = k_i,\ i = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

Then $x \in c_0(\triangle)$ but by (2.1), $x \neq l_\infty(F, \triangle)$ which contradicts (2). Hence (3) must hold.

(3) implies (1): Let (3) be satisfied and $x \in l_\infty(\triangle)$. If we suppose that $x \notin l_\infty(F, \triangle)$, then

$$\sup_k f_k(|\triangle x_k|) = \infty \quad \text{for} \quad \triangle x \in l_\infty.$$

Let $t = |\triangle x|$. Then $\sup_k f_k(t) = \infty$ which contradicts (3). Hence $l_\infty(\triangle) \subseteq l_\infty(F, \triangle)$.

THEOREM 2.2. *The following statements are equivalent for a sequence of moduli* $F = (f_k)$:

(1) $c_0(F, \triangle) \subseteq c_0(\triangle)$;
(2) $c_0(F, \triangle) \subseteq l_\infty(\triangle)$;
(3) $\inf_k f_k(t) > 0, \ (t > 0)$.

Proof. (1) implies (2) is obvious.

(2) implies (3): Let $c_0(F, \triangle) \subseteq l_\infty(\triangle)$. Suppose that (3) does not hold. Then, by Lemma 1.2,

$$\inf_k f_k(t) = 0 \ (t > 0). \tag{2.2}$$

For an index sequence (k_i) with

$$f_{k_i}(i^2) < \frac{1}{i} \quad \text{for } i = 1, 2, \ldots$$

define the sequence $x = (x_k)$ by

$$x_k := \begin{cases} i^2, & \text{if } k = k_i \text{ for } i = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

By (2.2), $x \in c_0(F, \triangle)$ but $x \notin l_\infty(\triangle)$ which contradicts (2). Hence (3) must hold.

(3) implies (1): Let (3) hold and $x \in c_0(F, \triangle)$, i.e.

$$\lim_k f_k(|\triangle x_k|) = 0.$$

Suppose that $x \notin c_0(\triangle)$. Then for some number $\epsilon_0 > 0$ and index k_0 we have $|\triangle x_k| \geq \epsilon_0$ for $k \geq k_0$. Therefore $f_k(\epsilon_0) \leq f_k(|\triangle x_k|$ for $k \geq k_0$ and consequently $\lim_k f_k(\epsilon_0) = 0$ which contradicts (3). Hence $c_0(F, \triangle) \subseteq c_0(\triangle)$.

THEOREM 2.3. *The inclusion* $l_\infty(F, \triangle) \subseteq c_0(\triangle)$ *holds if and only if*

$$\lim_k f_k(t) = \infty \quad \textit{for } t > 0. \tag{2.3}$$

Proof. Let $l_\infty(F, \triangle) \subseteq c_0(\triangle)$ such that (2.3) does not holds. Then there is a number $t_0 > 0$ and an index sequence (k_i) such that

$$f_{k_i}(t_0) \leq M < \infty. \tag{2.4}$$

Define the sequence $x = (x_k)$ by

$$x_k := \begin{cases} -t_0 i, & \text{if } k = k_i \text{ for } i = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

Thus $x \in l_\infty(F, \triangle)$, by (2.4). But $x \notin c_o(\triangle)$. so that (2.3) must hold for $l_\infty(F, \triangle) \subseteq c_0(\triangle)$. Conversly, let (2.3) hold. If $x \in l_\infty(F, \triangle)$, then

$f_k(|\triangle x_k|) \leq M < \infty$ for $k = 1, 2, \ldots$ Suppose that $x \notin c_0(\triangle)$. Then for some number $\epsilon_0 > 0$ and index $k \leq k_0$ we have $|\triangle x_k| \geq \epsilon_0$ for $k \geq k_0$. Therefore $f_k(\epsilon_0) \leq f_k(|\triangle x_k|) \leq M$ for $k \geq k_0$ which contradicts (2.3). Hence $x \in c_0(\triangle)$.

THEOREM 2.4. *The inclusion* $l_\infty(\triangle) \subseteq c_0(F, \triangle)$ *holds, if*

$$\lim_k f_k(t) = 0 \quad \text{for } t > 0. \tag{2.5}$$

Proof. Let $l_\infty(\triangle) \subseteq c_0(F, \triangle)$. Suppose that (2.5) does not hold.
Then for some $t_0 > 0$

$$\lim_k f_k(t_0) = l \neq 0. \tag{2.6}$$

Define $x = (x_k)$ by $x_k = -t_0 k$ for $k = 1, 2, \ldots$ Then $x \notin c_0(F, \triangle)$ by (2.6). Hence, (2.5) must hold.

Conversely, suppose that (2.5) holds and $x \in l_\infty$. Then $|\triangle x_k| \leq M < \infty$ for $k = 1, 2, \ldots$ Therefore $f_k(|\triangle x_k|) \leq f_k(M)$ for $k = 1, 2, \ldots$ and $\lim_k f_k(|\triangle x_k|) \leq \lim_k f_k(M) = 0$, by (2.5). Hence $x \in c_0(F, \triangle)$.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCES
DQUESNE UNIVERSITY
440 COLLEGE HALL
PITTSBURGH, PA 15282 U.S.A.
e-mail: gaur@mathcs.duq.edu

DEPARTMENT OF MATHEMATICS
ALIGARH MUSLIM UNIVERSITY
ALIGARH 202002, INDIA

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