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SOME VARIATIONAL FORMULAS IN THE CLASS $\widetilde{K}_n(E)$ AND THEIR APPLICATIONS

Introduction

If for each function $F(z)$ from a class V we succeed in isolating a family $F(z,\varepsilon)$ of functions uniformly differentiable with respect to ε within the domain D as $\varepsilon = 0$, then the expansion of the type

$$F(z,\varepsilon) = F(z) + \varepsilon Q(z) + o(|\varepsilon|, D)$$

is called a variational formula in the class V (written for F). The development of a variational formula is usually a difficult and separate task. The variational method is one of the fundamental tools in the geometric theory of functions of complex variable. This method allows to solve a series of extremal problems particularly in the theory of univalent functions. Applying the variational-geometric method, Lavrent'yev achieved significant results in applied problems [5]. The method of solwing extremal problems in the class of univalent functions proposed by an American mathematician Schiffer leads to differential equations for extremal function [6]. G.M. Goluzin [7] suggested his own variational method which led him to the same differential equations as that of Schiffer. Goluzin's method often leads to a final solution of various extremal problems of geometrical theory of analytical functions. Using the automorphism of a unit circle, French mathematican F. Marti arrived at variational formula in the class of univalent functions and solwed a few significant extremal problems [1].

In this work, we show the variational formulas for a class of analytical functions. Some extremal problems can be solved applying these variational formulas.

1. Let E be the unit circle, i.e. $|z| < 1$. Following [2] denote by $K_n(E)$ a class of analytic on E functions $F(z)$ such that the n-devided difference

$$[F(z); z_0, \dots, z_n] = \sum_{m=0}^{n} \frac{F(z_m)}{(z_m - z_0)\dots(z_m - z_{m-1})(z_m - z_{m+1})\dots(z_m - z_n)} \neq 0$$

for any pairwise different points $z_0, \dots, z_n \in E$.

When $n = 1$ we have a class $K_1(E)$ of univalent functions in E whose important role as means for realisation of conformal mapping is well known. Note, that if $F(z) \in K_n(E)$, then $F^{(n)}(z) \neq 0$ for any $z \in E$ (see [2]). Let us note the following property of the class $K_n(E)$, $n \geq 1$. In order that $F(z) \in K_n(E)$, for $n \geq 1$, it is necessary and sufficient, that functions

$$u \equiv 1, \quad u_1 = z, \dots, \quad u_{n-1} = z^{n-1}, \quad u_n = F(z)$$

form the Chebyshev system in E or that the equation

$$a_0 + a_1 z + \dots + a_{n-1} z^{n-1} + F(z) = 0$$

have no more than n roots in E for any $a_0, a_1, \dots, a_{n-1}$.

We will call the analytical function $F(z)$ in E to be n-normed in E, if its expansion in power series of z is of the form

$$F(z) = z^n + \sum_{k=2}^{\infty} a_{k,n} z^{n+k-1}.$$

The numbers $a_{k,n}$ are called the k-coefficient of function $F(z)$. Let us introduce a normalization operator $N_n[F(z)]$ for the class $K_n(E)$ according to the formula

$$N_n[F(z)] = \frac{n!}{F^{(n)}(0)} \left(F(z) - \sum_{m=0}^{n-1} \frac{1}{m!} F^{(m)}(0) z^m \right).$$

Such an operator transforms any function $F(z)$ from the class $K_n(E)$ into an n-normed function which due to the elementary properties of the n-divided difference, belongs to the class $K_n(E)$ as well (see [2]).

Let us denote by $\widetilde{K}_n(E)$ the set of all n-normed functions from the class $K_n(E)$. When $n = 1$, the class $\widetilde{K}_1(E)$ consists of univalent and normed in E functions.

Note that the class $\widetilde{K}_n(E), n \geq 1$ is compact with respect to uniform convergence inside E.

The properties of functions from the class $\widetilde{K}_n(E)$, for $n \geq 1$, have been studied in [2]–[4] and in other works of the author of this paper.

2. Let Λ be a set of all univalent mappings ω from E into itself having form

$$\omega = \omega(z) = \frac{z+\zeta}{1+\bar{\zeta}z}, \ \zeta \in E.$$

It is shown in [3], that if $F(z) \in K_n(E)$, then

$$(1-\bar{\omega}(0)\omega(z))^{1-n}F(\omega(z)) \in K_n(E), \quad \forall \omega(z) \in \Lambda.$$

Let us introduce an operator $\Omega_n\left[F(z)\right]$ by the formula

$$\Omega_n^{\omega}[F(z)] = N_n[(1-\bar{\omega}(0)\omega(z))^{1-n}F(\omega(z))]. \tag{1}$$

By virtue of the condition

$$\frac{\partial^n(1-\bar{\omega}(0)\omega(z))^{1-n}F(\omega(z))}{\partial z^n} \neq 0, \quad \forall z \in E, \text{ (see [3])}$$

we conclude that the operator (1) is for any $\omega \in E$ defined on $K_n(E)$. When $n = 1$ we have an operator

$$\Omega_1^{\omega}[F(z)] = \frac{F(\omega(z)) - F(\zeta)}{(1-|\zeta|^2)F'(\zeta)} \tag{2}$$

frequently used in the theory of univalent in E functions. From the elementary properties of the devided difference it easily follows.

LEMMA 1. *If $F(z) \in \widetilde{K}_n(E)$, then the function*

$$F(z,\zeta) = \Omega_n^{\omega}[F(z)] \in \widetilde{K}_n(E), \quad \forall \zeta \in E.$$

Let us mention that the operator (2) was applied by F. Marti [1] to deduce one of his variational formulas in the class $\widetilde{K}_1(E)$ of univalent and normed in E functions.

THEOREM 1 (F. Marti). *If the function*

$$F(z) = z + \sum_{k=2}^{\infty} a_{k,1}z^k \in \widetilde{K}_1(E),$$

then the function $F(z;\zeta) \in \widetilde{K}_1(E)$ for any $\zeta \in E$, and a formula

$$F(z;\zeta) = F(z) + (F'(z) - 1 - 2a_{2,1}F(z))\zeta - z^2F'(z)\bar{\zeta} + o(|\zeta|)$$

holds for sufficiently small ζ.

3. Let us define a variational formula in the class $\widetilde{K}_n(E)$ similar to that of Marti. The following theorem holds:

THEOREM 2. *If the function*

$$F(z) = z^n + \sum_{k=2}^{\infty} a_{k,n} z^{n+k-1} \in \widetilde{K}_n(E),$$

then the function $F(z;\zeta) \in \widetilde{K}_n(E)$ for any $\zeta \in E$, and a formula

$$(3) \quad F(z,\zeta) = F(z) + (F'(z) - (n+1)a_{2,n}F(z) - nz^{n-1})\zeta \\ -(z^2F'(z) - (n-1)zF(z))\bar{\zeta} + o(|\zeta|)$$

holds for sufficiently small ζ.

Proof. The fact that the function $F(z;\zeta) \in \widetilde{K}_n(E)$ for any fixed $\zeta \in E$ follows from Lemma 1. Let us expand $F(z;\zeta)$ into power series of z

$$(4) \qquad F(z;\zeta) = z^n + \sum_{k=2}^{\infty} a_{k,n}(\zeta) z^{n+k-1},$$

where (see [4])

$$(5) \quad a_{k,n}(\zeta) = \\ \sum_{m=0}^{k-1} (-1)^m \frac{(k-1)!}{m!(k-1-m)!} (1-|\zeta|^2)^{k-1-m} \bar{\zeta}^m \frac{n! F^{(n+k-1-m)}(\zeta)}{(n+k-1-m)! F^{(n)}(\zeta)}.$$

Obviously

$$(6) \qquad F(z;0) \equiv F(z), \quad a_{k,n}(0) = a_{k,n}.$$

Let $\zeta = xe^{i\gamma}, -1 < x < 1, 0 \le \gamma < 2\pi$. The function $F\Big(z; xe^{i\gamma}\Big)$ is analytic for any fixed γ and z at the point $x = 0$. Therefore, taking into account (6), we may write

$$(7) \qquad F(z; xe^{i\gamma}) = F(z) + Q(z)x + o(|x|)$$

where

$$Q(z) = \frac{\partial F(z; xe^{i\gamma})}{\partial x}\bigg|_{x=0}.$$

Furthermore, the k-coefficient $a_{k,n}(xe^{i\gamma})$ is an analytic function for any fixed γ at the point $x = 0$. From (5), one can easily to obtain an equality

$$(8)\qquad \left.\frac{\partial a_{k,n}(xe^{i\gamma})}{\partial x}\right|_{x=0} =$$
$$(n+k)a_{k+1,n}e^{i\gamma} - (n+1)a_{k,n}a_{2,n}e^{i\gamma} + (k-1)a_{k-1,n}e^{-i\gamma}.$$

Let us calculate $Q(z)$. Using (4) and (8) we obtain

$$\begin{aligned} Q(z) &= \sum_{k=2}^{\infty} \left.\frac{\partial a_{k,n}(xe^{i\gamma})}{\partial x}\right|_{x=0} z^{n+k-1} \\ &= z^n e^{i\gamma} \sum_{k=2}^{\infty}(n+k)a_{k+1,n}z^{k-1} - e^{-i\gamma}z^n \sum_{k=2}^{\infty}(k-1)a_{k-1,n}z^{k-1} \\ &\quad - z^n(n+1)a_{2,n}e^{i\gamma}\sum_{k=2}^{\infty} a_{k,n}z^{k-1} \\ &= e^{i\gamma}F'(z) - e^{i\gamma}nz^{n-1} - e^{-i\gamma}(z^2F'(z) - (n-1)zF(z)) \\ &\quad - (n+1)a_{2,n}e^{i\gamma}F(z). \end{aligned}$$

Multiplying $Q(z)$ by x and puting it into (7) we see that the function $F(z;\zeta)$ has an expression of the form (3) for all sufficiently small values of ζ. If $n = 1$ we obtain the Marti variational formula again.

Along with the variation of the function $F(z) \in \widetilde{K}_n(E)$ we show the variation of its k-coefficient. The following theorem holds:

THEOREM 3. *Let the function* $F(z) = z^n + \sum_{k=2}^{\infty} a_{k,n}z^{n+k-1} \in \widetilde{K}_n(E)$. *Then the k-coefficient* $a_{k,n}(\zeta)$ *of the function* $F(z;\zeta)$ *satisfies the formula*

$$(9)\quad a_{k,n}(\zeta) = a_{k,n}+((n+k)a_{k+1,n}-(n+1)a_{k,n}a_{2,n})\zeta-(k-1)a_{k-1,n}\bar{\zeta}+o(|\zeta|)$$

for all sufficiently small value of ζ.

Proof. Let $\zeta = xe^{i\gamma}, -1 < x < 1, 0 \le \gamma < 2\pi$. From (5) we see that the function $a_{k,n}(xe^{i\gamma})$ is analytic at $x = 0$ for any fixed γ. Therefore

$$a_{k,n}(xe^{i\gamma}) = a_{k,n} + \left.\frac{\partial a_{k,n}(xe^{i\gamma})}{\partial x}\right|_{x=0} x + o(|x|)$$

for any $|x| < p$, where p is sufficiently small. Using (8), we obtain (9).

We will show another simple variational formula in the class $\widetilde{K}_n(E)$.

THEOREM 4. *If* $F(z) \in \widetilde{K}_n(E)$, *then the function* $e^{-in\gamma}F(e^{i\gamma}z) \in \widetilde{K}_n(E)$ *for any real* γ *and we have the formula*

$$(10)\qquad e^{-in\gamma}F(e^{i\gamma}z) = F(z) + i(zF'(z) - nF(z))\gamma + o(|\gamma|).$$

P r o o f. The fact that the function $e^{-in\gamma}F(e^{i\gamma}z)$, for any real γ belongs to the class $\widetilde{K}_n(E)$ does not raise any doubts. Formula (10) can be easily obtained from the expansion of this function into a power series of z.

4. Let us solve a few extremal problems applying formulas (3), (9) and (10)

THEOREM 5. *Let functions*

$$F_m(z) = z^n + \sum_{k=2}^{\infty} a_{k,n}^{(m)} z^{n+k-1}, \quad m = 1,2$$

belong to the class $\widetilde{K}_n(E), n \geq 1$ and at any point $z_0 \neq 0$ their values are such that

$$|F_1(z_0)| = \max_{F(z)\in\widetilde{K}_n(E)} |F(z_0)|, \quad |F_2(z_0)| = \min_{F(z)\in\widetilde{K}_n(E)} |F(z_0)|.$$

Then

$$(11)\quad (1-|z_0|^2)F_m'(z_0)+((n-1)\bar{z}_0-(n+1)a_{2,n}^{(m)})F_m(z_0) = nz_0^{n-1}, \quad m = 1,2.$$

P r o o f. We will prove the theorem for the case when $m = 1$. The case $m = 2$ is treated in a similar fashion. The existence of the function $F_1(z)$ satisfying the condition of the theorem is provided by the compactness of the class $\widetilde{K}_n(E), n \geq 1$. According to Theorem 4, the function $e^{-in\gamma}F(e^{i\gamma}z) \in \widetilde{K}_n(E)$, $n \geq 1$ for any real γ. Using the variational formula (10) we obtain

$$|F_1(z_0) + i(z_0F_1'(z_0) - nF_1(z_0))\gamma + o(|\gamma|)|^2 \leq |F_1(z_0)|^2.$$

Hence for any real γ we have

$$Re\,\{i\bar{F}_1(z_0)(z_0F_0'(z_0) - nF_1(z_0))\gamma\} + Re\,\{o(|\gamma|)\} \leq 0.$$

But

$$Re\{i\bar{F}_1(z_0)(z_0F_1'(z_0) - nF_1(z_0))\} = 0,$$

hence

$$(12)\qquad Im\{z_0\bar{F}_1(z_0)F_1'(z_0)\} = 0.$$

Since $F_1(z) \in \widetilde{K}_n(E)$, $n \geq 1$, then $F(z;\zeta) \in \widetilde{K}_n(E)$, $n \geq 1$ for any $\zeta \in E$ and therefore, taking into account the property of the function $F_1(z)$, we have

$$|F_1(z_0;\zeta)| \leq F_1(z_0), \quad \forall\zeta \in E.$$

Hence, using the variational formula (3) we obtain the inequality

$$|F_1(z_0) + (F_1'(z_0) - nz_0^{n-1} - (n+1)a_{2,n}^{(1)}F_1(z_0))\zeta - (z_0^2 F_1'(z_0) - (n-1)z_0F_1(z_0))\bar{\zeta} + o(|\zeta|)|^2 \le |F_1(z_0)|^2$$

which holds for sufficiently small values of ζ. The last inequality may be replaced by the inequality

$$(13) \quad Re\{\zeta[\bar{F}_1(z_0)(F_1'(z_0) - nz_0^{n-1} - (n+1)a_{2,n}^{(1)}F_1(z_0)) - (F_1(z_0)(\bar{z}_0^2\bar{F}_1'(z_0) - (n-1)\bar{z}_0\bar{F}_1(z_0)))]\} + Re\{o(|\zeta|)\} \le 0,$$

which holds for any ζ under the condition that $|\zeta| < p$, where p is sufficiently small. From (13) we conclude that

$$(14) \quad \bar{F}_1(z_0)(F_1'(z_0) - nz_0^{n-1} - (n+1)a_{2,n}^{(1)}F_1(z_0)) - (F_1(z_0)(\bar{z}_0^2\bar{F}_1(z_0) - (n-1)\bar{z}_0\bar{F}_1(z_0))) = 0.$$

Using (12) we can write down the relation

$$\bar{z}_0^2\bar{F}_1'(z_0)F_1(z_0) = |z_0|^2 F_1'(z_0)\bar{F}_1(z_0),$$

which transforms equality (14) into (11).

5. Let us consider extremal problems related to the coefficients of functions from $\widetilde{K}_n(E)$, $n \ge 1$.

THEOREM 6. *Let the function*

$$F(z) = z^n + \cdots + a_{k-1,n}z^{n+k-2} + \hat{a}_{k,n}z^{n+k-1} + a_{k+1,n}z^{n+k} + \cdots$$

belongs to the class $\widetilde{K}_n(E), n \ge 1$ *and its k-coefficient* $\hat{a}_{k,n}, k \ge 2$ *has a property*

$$|\hat{a}_{k,n}| = \max_{F(z)\in\widetilde{K}_n(E)} \frac{|F^{(n+k-1)}(0)|}{(n+k-1)!}.$$

Then the equality

$$(15) \quad \bar{\hat{a}}_{k,n}((n+k)a_{k+1,n} - (n+1)\hat{a}_{k,n}a_{2,n}) - (k-1)\hat{a}_{k,n}\bar{a}_{k-1,n} = 0$$

holds.

Proof. The existence of the function $F(z) \in \widetilde{K}_n(E), n \ge 1$ whose k-coefficient of expansion assumes the maximum in absolute value, is ensured by the compactness of the class $\widetilde{K}_n(E)$, $n \ge 1$. Since $F(z) \in \widetilde{K}_n(E), n \ge 1$ then the function $F(z;\zeta) = z^n + \cdots + a_{k,n}(\zeta) + \cdots$ belongs to $\widetilde{K}_n(E), n \ge 1$ as well for any $\zeta \in E$. Applying the property of the coefficient $\hat{a}_{k,n}$ of the function $F(z)$ we have

$$|a_{k,n}(\zeta)| \leq |\hat{a}_{k,n}|, \quad \forall\zeta \in E.$$

Hence, using the variational formula (9), we obtain the inequality

$$|\hat{a}_{k,n} + ((n+k)a_{k+1,n} - (n+1)\hat{a}_{k,n}a_{2,n})\zeta - (k-1)a_{k-1,n}\overline{\zeta} + o(|\zeta|))| \\ \leq |\hat{a}_{k,n}|,$$

which holds for any sufficiently small ζ. The latter inequality can be replaced by

$$(16) \qquad Re\{[\hat{a}_{k,n}((n+1)a_{k+1,n} - (n+1)\hat{a}_{k,n}a_{2,n}) \\ -(k-1)\hat{a}_{k,n}\bar{a}_{k-1,n}]\zeta\} + Re\{o(|\zeta|)\} \leq 0,$$

which holds for any ζ under the condition $|\zeta| < p$ where p is sufficiently small. Since the argument of complex number ζ can be chosen arbitrarily, then (15) easily follows from (16).

Two following theorems can be proved in a similar way

THEOREM 7. *Let the functions*

$$F_m(z) = z^n + \cdots + a^{(m)}_{k-1,n}z^{n+k-2} + \hat{a}^{(m)}_{k,n}z^{n+k-1} + a^{(m)}_{k+1,n}z^{n+k} + \cdots$$

$m = 1,2$ *belong to the class* $\widetilde{K}_n(E), n \geq 1$ *and their k-coefficient* $\hat{a}^{(m)}_{k,n}, m = 1,2$ *have the properties*

$$Re\hat{a}^{(1)}_{k,n} = \max_{F(z)\in\widetilde{K}_n(E)} Re\frac{F^{(n+k-1)}(0)}{(n+k-1)!},$$
$$Re\hat{a}^{(2)}_{k,n} = \min_{F(z)\in\widetilde{K}_n(E)} Re\frac{F^{(n+k-1)}(0)}{(n+k-1)!}.$$

Then the following equalities

$$\hat{a}^{(m)}_{k,n} = \frac{(n+k)a^{(m)}_{k+1,n} - (k-1)\bar{a}^{(m)}_{k-1,n}}{(n+1)a^{(m)}_{2,n}}, \quad m = 1,2$$

hold.

We denote by $\widetilde{K}^r_n(E)$ the class of functions from $\widetilde{K}_n(E)$ whose all expansion coefficients are real numbers

THEOREM 8. *Let the function*

$$F(z) = z^n + \cdots + a_{k-1,n}z^{n+k-2} + \hat{a}_{k,n}z^{n+k-1} + a_{k+1,n}z^{n+k} + \cdots$$

belongs to the class $\widetilde{K}_n^r(E), n \geq 1$ *and its k-coefficient* $\hat{a}_{k,n}$ *possess the property*

$$|\hat{a}_{k,n}| = \max_{F(z)\in\widetilde{K}_n^r(E)} \frac{|F^{(n+k-1)}(0)|}{(n+k-1)!}.$$

Then the equality

$$\hat{a}_{k,n} = \frac{(n+k)a_{k+1,n} - (k-1)a_{k-1,n}}{(n+1)a_{2,n}}$$

holds.

COROLLARY 1. *Let the function*

$$F(z) = z^n + \hat{a}_{2,n}z^{n+1} + a_{3,n}z^{n+2} + \cdots$$

belongs to the class $\widetilde{K}_n(E)$, $n \geq 1$ *and its second coefficient* $\hat{a}_{2,n}$ *has the property*

$$|\hat{a}_{2,n}| = \max_{F(z)\in\widetilde{K}_n(E)} \frac{|F^{(n+1)}(0)|}{(n+1)!}.$$

Then

$$a_{3,n} = \frac{(n+1)\hat{a}_{2,n}^2 + 1}{n+2}.$$

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