DEMONSTRATIO MATHEMATICA
Vol. XXXI No 2 1998

Piotr Multarzyński, Wiesław Sasin

ON THE DIMENSION OF A TANGENT SPACE TO A STRUCTURED SPACE

In this paper we investigate the dimension of a tangent space (in the sense of Zarisky) at a point of a structured space. The paper is a continuation and generalization of [4] and [5]. The notion of a structured space is a generalization of the concept of the differential space introduced by Sikorski [7]. This generalization was originally considered by Mostov [3]. Some foundations of structured spaces with applications to relativistic physics are presented in [1] and [2].

In Section 1 we present some basic notions and definitions from structured space theory. In Section 2 we discuss the dimension of the tangent space in terms of germs of the smooth cross-sections.

1.Preliminaries

Let M be a topological space with the topology τ_M. The sheaf $\mathcal{C}$ of real functions on M is said to be a *differential structure* on M if, for any open set $U \in \tau_M$ and any functions $f_1, \ldots, f_n \in \mathcal{C}(U)$, $\omega \in C^\infty(\mathbb{R}^n)$, there is $\omega \circ (f_1, \ldots, f_n) \in \mathcal{C}(U)$, where $\mathcal{C}(U)$ denotes the set of all cross-sections of $\mathcal{C}$ over U. For a topological space M and a differential structure $\mathcal{C}$ on M, the pair $(M, \mathcal{C})$ is said to be a *structured space*.

For a point $p \in M$, let $\mathcal{C}_p$ be the set of all germs at p of sections of the sheaf $\mathcal{C}$. The set $\mathcal{C}_p$ with the natural operations of addition and multiplication is an algebra over R. By $\widehat{f}_p$ we will denote the germ of a section $f \in \mathcal{C}(U)$ at $p \in U$, $U \in \tau_M$.

A linear mapping $v : \mathcal{C}_p \to \mathbb{R}$ such that

$$(1) \qquad v(\mathbf{f} \cdot \mathbf{g}) = \mathbf{f}(p) \cdot v(\mathbf{g}) + \mathbf{g}(p) \cdot v(\mathbf{f}),$$

for any $\mathbf{f}, \mathbf{g} \in \mathcal{C}_p$, is said to be a *tangent vector* to the structured space $(M, \mathcal{C})$ at p. By T_pM we shall denote the linear space of all tangent vectors to $(M, \mathcal{C})$ at $p \in M$, called the *tangent space* to $(M, \mathcal{C})$ at $p \in M$.

A continuous mapping $F : M \to N$ is said to be a *smooth mapping* of a structured space $(M, \mathcal{C})$ into a structured space $(N, \mathcal{D})$ if,

$$\alpha \circ (F|F^{-1}(U)) \in \mathcal{C}(F^{-1}(U)),$$

for any $U \in \tau_N$ and any section $\alpha \in \mathcal{D}(U)$. In such a case we will write $F : (M, \mathcal{C}) \to (N, \mathcal{D})$.

If $F : (M, \mathcal{C}) \to (N, \mathcal{D})$ then, for any point $p \in M$ and a vector $v \in T_pM$, the mapping $F_{*p}v : \mathcal{D}_{F(p)} \to \mathbb{R}$ defined by

$$(F_{*p}v)(\mathbf{g}) = v(F^*\mathbf{g}), \tag{2}$$

is a tangent vector to $(N, \mathcal{D})$ at $F(p)$, where $F^*\mathbf{g}$ is the pull-back of the germ $\mathbf{g}$ by the mapping F.

Now we prove the following

LEMMA 1.1. *Let $(M, \mathcal{C})$ be a structured space and $p \in M$ be an arbitrary point. The following conditions are equivalent*:

(i) *vectors $v_1, \ldots, v_n \in T_pM$ are linearly independent*,

(ii) *the mapping $L : \mathcal{C}_p \to \mathbb{R}^n$ defined by $L(\mathbf{f}) = (v_1(\mathbf{f}), \ldots, v_n(\mathbf{f}))$, for $\mathbf{f} \in \mathcal{C}_p$, is a surjection*,

(iii) *there exist germs $\mathbf{f}_1, \ldots, \mathbf{f}_n \in \mathcal{C}_p$ such that $v_i(\mathbf{f}_j) = \delta_{ij}$ for $i, j = 1, \ldots, n$*,

(iv) *there exist germs $\mathbf{f}_1, \ldots, \mathbf{f}_n \in \mathcal{C}_p$ such that* $\det(v_i(\mathbf{f}_j)) \neq 0$.

Proof. (i)⇒(ii) If L is not a surjection then the image $ImL \subset \mathbb{R}^n$ is a linear subspace of $\mathbb{R}^n$. There exists a non-zero vector $(\lambda_1, \ldots, \lambda_n)$ which is normal to ImL, i.e. $\lambda_1 v_1(\mathbf{f}) + \cdots + \lambda_n v_n(\mathbf{f}) = 0$, for any $\mathbf{f} \in \mathcal{C}_p$ or, equivalently, $\lambda_1 v_1 + \ldots + \lambda_n v_n = 0$. In this case $v_1, \ldots, v_n$ are linearly dependent. This gives us a contradiction.

(ii)⇒(iii) Let $e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$ be the standard basis in $\mathbb{R}^n$. For e_j, $j = 1, \ldots, n$, there exists a germ $\mathbf{f}_j \in \mathcal{C}_p$ such that $L(\mathbf{f}_j) = e_j$. Now it is obvious that $v_i(f_j) = \delta_{ij}$, for $i, j = 1, \ldots, n$.

(iii)⇒(iv) This implication is evident.

(iv)⇒(i) Let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ be such numbers that $\lambda_1 v_1 + \ldots + \lambda_n v_n = 0$. Let us consider the linear system

$$\begin{aligned}
\lambda_1 v_1(\mathbf{f}_1) + \ldots + \lambda_n v_n(\mathbf{f}_1) &= 0 \\
\lambda_1 v_1(\mathbf{f}_2) + \ldots + \lambda_n v_n(\mathbf{f}_2) &= 0 \\
&\vdots \\
\lambda_1 v_1(\mathbf{f}_n) + \ldots + \lambda_n v_n(\mathbf{f}_n) &= 0.
\end{aligned}$$

Since the determinant $\det(v_i(\mathbf{f}_j)) \neq 0$ then the system has the unique solution $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$. This ends the proof. ∎

In the sequel, for any $n \in \mathbb{N}$, $\omega \in C^\infty(\mathbb{R}^n)$ and germs $\mathbf{f}_1, \ldots, \mathbf{f}_n \in \mathcal{C}_p$ let $\omega \circ (\mathbf{f}_1, \ldots, \mathbf{f}_n) \in \mathcal{C}_p$ be the germ with a representative $\omega \circ (f_1, \ldots, f_n)$, where $\widehat{f_i}_p = \mathbf{f}_i$, for $i = 1, \ldots, n$.

LEMMA 1.2. *Let $(M, \mathcal{C})$ be a structured space, $p \in M$ an arbitrary point and $v \in T_pM$ be a vector. Then for any $\mathbf{f}_1, \ldots, \mathbf{f}_n \in \mathcal{C}_p$ and $\omega \in C^\infty(\mathbb{R}^n)$*

$$v(\omega \circ (\mathbf{f}_1, \ldots, \mathbf{f}_n)) = \sum_{i=1}^{n} \omega'_{|i}(\mathbf{f}_1(p), \ldots, \mathbf{f}_n(p)) \cdot v(\mathbf{f}_i). \tag{3}$$

Proof. Let $f_i \in \mathcal{C}(U)$, for some open set $U \ni p$, be a section such that $\mathbf{f}_i = \widehat{f_i}_p$, for $i = 1, \ldots, n$. Consider the mapping $F = (f_1, \ldots, f_n)$. Of course, $F : (U, \mathcal{C}(U)) \to (\mathbb{R}^n, C^\infty(\mathbb{R}^n))$ is smooth. It is easy to see that the mapping $\iota_{*p} : T_pU \to T_pM$ is an isomorphism of linear spaces, where $\iota : (U, \mathcal{C}_U) \to (M, \mathcal{C})$ is an embedding. Let $\tilde{v} \in T_pU$ be a vector such that $\iota_{*p}\tilde{v} = v$. Of course, $F_{*p}\tilde{v} \in T_{F(p)}\mathbb{R}^n$. It is clear that, for any $\omega \in C^\infty(\mathbb{R}^n)$, we have the equality

$$(F_{*p}\tilde{v})(\omega) = \sum_{i=1}^{n} \tilde{v}(\widehat{f_i}_p) \cdot \omega'_{|i}(F(p)).$$

Hence we obtain

$$\tilde{v}(\omega \circ (\widehat{f_1}_p, \ldots, \widehat{f_n}_p)) = \sum_{i=1}^{n} \omega'_{|i}(F(p)) \cdot \tilde{v}(\widehat{f_i}p)$$

or equivalently

$$v(\omega \circ (\mathbf{f}_1, \ldots, \mathbf{f}_n)) = \sum_{i=1}^{n} \omega'_{|i}(\mathbf{f}_1(p), \ldots, \mathbf{f}_n(p)) \cdot v(\mathbf{f}_i). \blacksquare$$

2. Main results

Let $(M, \mathcal{C})$ be a structured space and $p \in M$ an arbitrary point.

DEFINITION 2.1. A germ $\mathbf{f} \in \mathcal{C}_p$ is said to be differentially dependent on germs $\mathbf{g}_1, \ldots, \mathbf{g}_n \in \mathcal{C}_p$ if there exists a function $\omega \in C^\infty(\mathbb{R}^n)$ such that

$$\mathbf{f} = \omega \circ (\mathbf{g}_1, \ldots, \mathbf{g}_n).$$

DEFINITION 2.2. A set $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\} \subset \mathcal{C}_p$ is said to be differentially independent if no germ $\mathbf{f}_i$, for $i \in \{1, \ldots, n\}$, depends differentially on the remaining ones. Any set $\mathcal{F} \subset \mathcal{C}_p$ is said to be differentially independent if every finite subset of $\mathcal{F}$ is differentially independent.

Now we prove

PROPOSITION 2.1. *Let $(M, \mathcal{C})$ be a structured space and $p \in M$. A subset $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\} \subset \mathcal{C}_p$ is differentially independent iff, for any $\omega \in C^\infty(\mathbb{R}^n)$ the following implication is true:*

$$\omega \circ (\mathbf{f}_1, \ldots, \mathbf{f}_n) = 0 \Longrightarrow \forall_{1 \le i \le n}\ \omega'_{|i}(\mathbf{f}_1(p), \ldots, \mathbf{f}_n(p)) = 0. \tag{$*$}$$

Proof. ($\Rightarrow$) Let $\{\mathbf{f}_1,\ldots,\mathbf{f}_n\} \subset \mathcal{C}_p$ be differentially independent, and let us assume that $\omega \circ (\mathbf{f}_1,\ldots,\mathbf{f}_n) = 0$, for some function $\omega \in C^\infty(\mathbb{R}^n)$. There exist an open neighbourhood $U \in \tau_M$ of p and sections $f_1,\ldots,f_n \in C^\infty(U)$ such that $\widehat{f}_{i_p} = \mathbf{f}_i$ for $i = 1,\ldots,n$ and $\omega\circ(f_1,\ldots,f_n) = 0$. Let us consider the differential space $(U, sc\{f_1,\ldots,f_n\}_U)$ finitely generated by $\{f_1,\ldots,f_n\}$ [6]. It is easy to see that the set $\{f_1,\ldots,f_n\}$ is differentially independent at p (see [4]). From Proposition 3 in [4] it follows that $\forall_{1\leq i\leq n}\ \omega'_{|i}(f_1(p),\ldots,f_n(p)) = 0$. Hence $\omega'_{|i}(\mathbf{f}_1(p),\ldots,\mathbf{f}_n(p)) = 0$, for every $i = 1,\ldots,n$.

($\Leftarrow$) Let $\{\mathbf{f}_1,\ldots,\mathbf{f}_n\} \subset \mathcal{C}_p$ be a subset satisfying $(*)$, for any $\omega \in C^\infty(\mathbb{R}^n)$. Suppose that the set $\{\mathbf{f}_1,\ldots,\mathbf{f}_n\}$ is differentially dependent and, without losing of generality, let us assume that there exists a function $\theta \in C^\infty(\mathbb{R}^{n-1})$ such that $\mathbf{f}_1 = \theta\circ(\mathbf{f}_2,\ldots,\mathbf{f}_n)$. Then $\mathbf{f}_1 - \theta\circ(\mathbf{f}_2,\ldots,\mathbf{f}_n) = 0$. Let $\omega \in C^\infty(\mathbb{R}^n)$ be a function given by the formula

$$\omega(x_1,\ldots,x_n) = x_1 - \theta(x_2,\ldots,x_n), \tag{4}$$

for $(x_1,\ldots,x_n) \in \mathbb{R}^n$. Of course, $\omega\circ(\mathbf{f}_1,\ldots,\mathbf{f}_n) = 0$ and $\omega'_{|1}(x_1,\ldots,x_n) = 1$, for any $(x_1,\ldots,x_n) \in \mathbb{R}^n$. Thus $\omega'_{|1}(\mathbf{f}_1(p),\ldots,\mathbf{f}_n(p)) = 1$ and the condition $(*)$ is not satisfied. ∎

COROLLARY 2.2. *If tangent vectors $v_1,\ldots,v_n \in T_pM$ are linearly independent then any set of germs $\mathbf{f}_1,\ldots,\mathbf{f}_n \in \mathcal{C}_p$, such that $v_i(\mathbf{f}_j) = \delta_{ij}$, for $i,j = 1,\ldots,n$, is differentially independent.*

Proof. We will show that the set $\{\mathbf{f}_1,\ldots,\mathbf{f}_n\}$ satisfies condition $(*)$. Let $\omega \in C^\infty(\mathbb{R}^n)$ be a function such that $\omega \circ (\mathbf{f}_1,\ldots,\mathbf{f}_n) = 0$. We have $v_j(\omega \circ (\mathbf{f}_1,\ldots,\mathbf{f}_n)) = 0$, for $j = 1,\ldots,n$. Hence and from Lemma 1.2 we have

$$\sum_{i=1}^{n} \omega'_{|i}(\mathbf{f}_1(p),\ldots,\mathbf{f}_n(p)) \cdot v_j(\mathbf{f}_i) = 0,$$

for $j = 1,\ldots,n$ Since the set $\{\mathbf{f}_1,\ldots,\mathbf{f}_n\}$ satisfies condition $(*)$, on the strength of Proposition 2.1, this set of germs is differentially independent.

DEFINITION 2.3. A set $B \subset \mathcal{C}_p$ is said to be a differential basis of $\mathcal{C}$ at p if B is differentially independent and, for any germ $\mathbf{f} \in \mathcal{C}_p$ there exist $n \in \mathbb{N}$, $\omega \in C^\infty(\mathbb{R}^n)$ and $\mathbf{g}_1,\ldots,\mathbf{g}_n \in B$ such that $\mathbf{f} = \omega \circ (\mathbf{g}_1,\ldots,\mathbf{g}_n)$.

LEMMA 2.1. *Let $(M,\mathcal{C})$ be a structured space and B be a differential basis of $\mathcal{C}$ at $p \in M$. For each function $u_0 : B \to \mathbb{R}$, there exists a unique tangent vector $u : \mathcal{C}_p \to \mathbb{R}$ such that $u|B = u_0$.*

Proof. Let $u : \mathcal{C}_p \to \mathbb{R}$ be given by

$$u(\mathbf{f}) = \sum_{i=1}^{n} \omega'_{|i}(\mathbf{f}_1(p),\ldots,\mathbf{f}_n(p)) \cdot u_0(\mathbf{f}_i), \tag{5}$$

for $\mathbf{f} \in \mathcal{C}_p$, where $n \in \mathbb{N}$, $\mathbf{f}_1, \ldots, \mathbf{f}_n \in B$, $\omega \in C^\infty(\mathbb{R}^n)$ are such that $\mathbf{f} = \omega \circ (\mathbf{f}_1, \ldots, \mathbf{f}_n)$. The correctness of definition (5) is a consequence of the condition $(*)$ in Proposition 2.1. Indeed, for a germ $\mathbf{f}$ composed in two different ways we can take the sum of the two sets of its components from the differential basis B, which allows us to write $\mathbf{f}$ in the forms $\mathbf{f} = \omega \circ (\mathbf{f}_1, \ldots, \mathbf{f}_n)$ and $\mathbf{f} = \tilde{\omega} \circ (\mathbf{f}_1, \ldots, \mathbf{f}_n)$, for some $\mathbf{f}_1, \ldots, \mathbf{f}_n \in B$, ω, $\tilde{\omega} \in C^\infty(\mathbb{R}^n)$, $n \in \mathbb{N}$. Then

$$(\omega - \tilde{\omega}) \circ (\mathbf{f}_1, \ldots, \mathbf{f}_n) = 0$$

and consequently, from $(*)$,

$$(\omega - \tilde{\omega})'_{|j}(\mathbf{f}_1(p), \ldots, \mathbf{f}_n(p)) = 0,$$

for $j = 1, \ldots, n$. Hence we obtain

$$\sum_{i=1}^{n} \omega'_{|i}(\mathbf{f}_1(p), \ldots, \mathbf{f}_n(p)) \cdot u_0(\mathbf{f}_i) = \sum_{i=1}^{n} \tilde{\omega}'_{|i}(\mathbf{f}_1(p), \ldots, \mathbf{f}_n(p)) \cdot u_0(\mathbf{f}_i),$$

which proves the correctness of definition (5). It can be easily shown that the function u given by (5) is a tangent vector and that u is the unique vector from T_pM such that $u|B = u_0$. ∎

COROLLARY 2.3. *Let $(M, \mathcal{C})$ be a structured space and B be a differential basis of $\mathcal{C}$ at $p \in M$. Then the mapping $\mu : T_pM \to \mathbb{R}^B$, given by*

(6) $$\mu(u) = u|B,$$

for $u \in T_pM$, is an isomorphism of vector spaces.

Proof. It is easy to see that μ is a linear mapping. From Lemma 2.1 follows that μ is a surjection. Let us also notice that μ is a monomorphism. Indeed, suppose that $u \in T_pM$ is a vector such that $\mu(u) = 0$. It means that $u|B = 0$. In view of Lemma 2.1 from the uniqueness of prolongation of $u|B$, it follows that $u = 0$. ∎

COROLLARY 2.4. *Let $(M, \mathcal{C})$ be a structured space and B be a differential basis of $\mathcal{C}$ at $p \in M$. Then*

(a) $\operatorname{Card} B < \aleph_0 \Rightarrow dim(T_pM) = \operatorname{Card} B$,

(b) $\operatorname{Card} B \geq \aleph_0 \Rightarrow dim(T_pM) > \operatorname{Card} B$,

(b*) $\operatorname{Card} B \geq \aleph_0 \Rightarrow dim(T_pM) = 2^{CardB}$, *if the generalized continuum hypothesis is assumed.*

Now, for any $p \in M$, we shall denote by a_p the linear subspace of $\mathcal{C}_p$ of all germs $\mathbf{f} \in \mathcal{C}_p$ for which there exist $\mathbf{f}_1, \ldots, \mathbf{f}_n \in \mathcal{C}_p$, $\omega \in C^\infty(\mathbb{R}^n)$, for some $n \in \mathbb{N}$, such that

(7) $$\mathbf{f} = \omega \circ (\mathbf{f}_1, \ldots, \mathbf{f}_n) \ \ \textit{and} \ \ \omega'_{|j}(\mathbf{f}_1(p), \ldots, \mathbf{f}_n(p)) = 0,$$

for $j = 1, \ldots, n$. Let $\mathcal{C}_p/a_p$ be the quotient linear space and $[\mathbf{f}]$ be the equivalence class of $\mathbf{f} \in \mathcal{C}_p$. Now, one can prove

LEMMA 2.2. *Let $(M,\mathcal{C})$ be a structured space, $p \in M$ an arbitrary point. Then*

(i) $[\theta \circ (\mathbf{f}_1, \ldots, \mathbf{f}_n)] = \sum_{i=1}^{n} \theta'_{|i}(\mathbf{f}_1(p), \ldots, \mathbf{f}_n(p))[\mathbf{f}_i]$,
for any $\mathbf{f}_1, \ldots, \mathbf{f}_n \in \mathcal{C}_p$, $\theta \in C^\infty(\mathbb{R}^n)$, $n \in \mathbb{N}$,

(ii) $[\mathbf{f} \cdot \mathbf{g}] = \mathbf{f}(p)[\mathbf{g}] + [\mathbf{f}] \cdot \mathbf{g}(\mathbf{p})$, *for any* $\mathbf{f}, \mathbf{g} \in \mathcal{C}_p$,

(iii) *if* $\mathbf{k} \in \mathcal{C}_p$ *is a germ of a constant function then* $[\mathbf{k}] = \mathbf{0}$.

Proof. (i) Let $\omega \in C^\infty(\mathbb{R}^n)$ be a function given by the formula

$$\omega(x_1, \ldots, x_n) = \theta(x_1, \ldots, x_n) - \sum_{i=1}^{n} \theta'_{|i}(\mathbf{f}_1(p), \ldots, \mathbf{f}_n(p)) \cdot x_i,$$

for $(x_1, \ldots, x_n) \in \mathbb{R}^n$. It is easy to see that

$$\omega \circ (\mathbf{f}_1, \ldots, \mathbf{f}_n) = \theta \circ (\mathbf{f}_1, \ldots, \mathbf{f}_n) - \sum_{i=1}^{n} \theta'_{|i}(\mathbf{f}_1(p), \ldots, \mathbf{f}_n(p)) \cdot \mathbf{f}_i,$$

and $\omega'_{|i}(\mathbf{f}_1(p), \ldots, \mathbf{f}_n(p)) = 0$, for $i = 1, \ldots, n$. Hence

$$\theta \circ (\mathbf{f}_1, \ldots, \mathbf{f}_n) - \sum_{i=1}^{n} \theta'_{|i}(\mathbf{f}_1(p), \ldots, \mathbf{f}_n(p)) \cdot \mathbf{f}_i \in a_p,$$

or equivalently

$$[\theta \circ (\mathbf{f}_1, \ldots, \mathbf{f}_n)] = \sum_{i=1}^{n} \theta'_{|i}(\mathbf{f}_1(p), \ldots, \mathbf{f}_n(p)) \cdot [\mathbf{f}_i],$$

(ii) follows from (1) if we take $\theta \in C^\infty(\mathbb{R}^2)$ given by $\theta(x_1, x_2) = x_1 \cdot x_2$, for $(x_1, x_2) \in \mathbb{R}^2$.

(iii) is obvious. ∎

Now, let $v \in T_pM$ be an arbitrary vector tangent to $(M,\mathcal{C})$ at $p \in M$. Note that $v|a_p = 0$. Thus v induces a linear functional $l_v \in (\mathcal{C}_p/a_p)^*$ defined by

$$l_v([\mathbf{f}]) = v(\mathbf{f}), \tag{8}$$

for any $\mathbf{f} \in \mathcal{C}_p$.

PROPOSITION 2.5. *The mapping $I : T_pM \to (\mathcal{C}_p/a_p)^*$ defined by*

$$I(v) = l_v, \tag{9}$$

for any $v \in T_pM$, is an isomorphism of linear spaces.

Proof. It is easy to see that I is a linear monomorphism. So it is enough to show that I is an epimorphism. For any $l \in (\mathcal{C}_p/a_p)^*$, let $v_l : \mathcal{C}_p \to \mathbb{R}$ be given by

(10) $$v_l(\mathbf{f}) = l([\mathbf{f}]),$$

for $\mathbf{f} \in \mathcal{C}_p$. It follows from condition (2) of Lemma 2.2 that v_l is a tangent vector to $(M,\mathcal{C})$ at p such that $I(v_l) = l$. ∎

Let us notice that a mapping $u : \mathcal{C}_p \to \mathbb{R}$ is a tangent vector iff it is linear and $u|a_p = 0$.

COROLLARY 2.6. *Let $(M,\mathcal{C})$ be a structured space and $p \in M$. Then for any $n \in \mathbb{N}$, $dim(T_pM) = n$ iff $dim(\mathcal{C}_p/a_p) = n$. In particular, $dim(T_pM) = 0$ iff $\mathcal{C}_p = a_p$.*

COROLLARY 2.7. *Let $(M,\mathcal{C})$ be a structured space and $p \in M$. If $\mathbf{f} \in \mathcal{C}_p$ satisfies $v(\mathbf{f}) = 0$ for each $v \in T_pM$, then $\mathbf{f} \in a_p$.*

Proof. For any linear functional $l \in (\mathcal{C}_p/a_p)^*$,

$$l([\mathbf{f}]) = 0.$$

Hence we get $[\mathbf{f}] = 0$ or equivalently $\mathbf{f} \in a_p$. ∎

DEFINITION 2.4. A set $\mathcal{F} \subset \mathcal{C}_p$ is said to be a linearized basis of the differential structure $\mathcal{C}$ at $p \in M$ (l-basis, for short) if any germ $\mathbf{f} \in \mathcal{C}_p$ can be uniquely expressed in the form

$$\mathbf{f} = \lambda_1\mathbf{f}_1 + \ldots + \lambda_n\mathbf{f}_n + \mathbf{g},$$

where $\mathbf{f}_1, \ldots, \mathbf{f}_n \in \mathcal{F}$, $\lambda_1, \ldots \lambda_n \in \mathbb{R} \setminus \{0\}$, $\mathbf{g} \in a_p$.

DEFINITION 2.5. The algebra $\mathcal{C}_p$ is generated by a subset $\mathcal{C}_0 \subset \mathcal{C}_p$, if for any germ $\mathbf{f} \in \mathcal{C}_p$ there exist $\mathbf{f}_1, \ldots, \mathbf{f}_n \in \mathcal{C}_0$, $\omega \in C^\infty(\mathbb{R}^n)$, $n \in \mathbb{N}$, such that

$$\mathbf{f} = \omega \circ (\mathbf{f}_1, \ldots, \mathbf{f}_n).$$

PROPOSITION 2.8. *Let $(M,\mathcal{C})$ be a structured space and $p \in M$ an arbitrary point. If the algebra $\mathcal{C}_p$ is generated by a subset $\mathcal{C}_0 \subset \mathcal{C}_p$, then there exists a linearized basis $\mathcal{F}$ of $\mathcal{C}$ at p such that $\mathcal{F} \subset \mathcal{C}_0$.*

Proof. Consider the quotient space $\mathcal{C}_p/a_p$. It is evident that the set $\{[\mathbf{f}] : \mathbf{f} \in \mathcal{C}_0\}$ generates the linear space $\mathcal{C}_p/a_p$. Let $B = \{[\mathbf{f}_s] : \mathbf{f}_s \in \mathcal{C}_0, s \in S\}$, where S is a set of indices, be a basis of $\mathcal{C}_p/a_p$. One can verify that the set $\mathcal{F} = \{\mathbf{f}_s : s \in S\}$ is clearly an l-basis of $\mathcal{C}$ at p. ∎

LEMMA 2.3. *Let $(M,\mathcal{C})$ be a structured space and let $\mathcal{F}$ be an l-basis of $\mathcal{C}$ at $p \in M$. For any function $u_0 : \mathcal{F} \to \mathbb{R}$ there exists exactly one tangent vector $u : \mathcal{C}_p \to \mathbb{R}$ at p such that $u|\mathcal{F} = u_0$.*

Proof. Let $u : \mathcal{C}_p \to \mathbb{R}$ be a mapping defined by

(11) $$u(\mathbf{f}) = \sum_{i=1}^{n} \lambda_i u_0(\mathbf{f}_i),$$

for $\mathbf{f} \in C_p \setminus a_p$, and $u(\mathbf{f}) = 0$ for $\mathbf{f} \in a_p$, where $\mathbf{f}_1, \ldots, \mathbf{f}_n \in \mathcal{F}$, $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ are elements such that $\mathbf{f} = \sum_{i=1}^{n} \lambda_i \mathbf{f}_i + \mathbf{g}$, where $\mathbf{g} \in a_p$. The mapping u is linear and $u|a_p = 0$. Therefore $u \in T_pM$ and $u|\mathcal{F} = u_0$. The uniqueness of u is clear. ∎

LEMMA 2.4. *All linearized bases of C at $p \in M$ are of the same cardinality. If C_0 generates C_p then, for any linearized basis $\mathcal{F}$ of C_p,* $\operatorname{Card} \mathcal{F} \leq \operatorname{Card} C_0$.

Proof. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two l-bases of C at p. Then the sets $[\mathcal{F}_1] = \{[\mathbf{f}] : \mathbf{f} \in \mathcal{F}_1\}$, $[\mathcal{F}_2] = \{[\mathbf{f}] : \mathbf{f} \in \mathcal{F}_2\}$ are bases of the linear space C_p/a_p, and $\operatorname{Card}[\mathcal{F}_i] = \operatorname{Card}[\mathcal{F}_i]$, for $i = 1, 2$. Of course, $\operatorname{Card}[\mathcal{F}_1] = \operatorname{Card}[\mathcal{F}_2]$. Hence $\operatorname{Card}[\mathcal{F}_1] = \operatorname{Card}[\mathcal{F}_2]$. The second assertion follows from Proposition 2.8. ∎

PROPOSITION 2.9. *Let (M, C) be a structured space and let $\mathcal{F} \subset C_p$ be an l-basis of C at $p \in M$. Then the mapping $\Phi : T_pM \to \mathbb{R}^{\mathcal{F}}$ defined by*

$$\Phi(u) = u|\mathcal{F}, \tag{12}$$

for $u \in T_pM$, is an isomorphism of linear spaces.

Proof. This proposition follows immediately from Lemma 2.3. ∎

COROLLARY 2.10. *Let (M, C) be a structured space and let $\mathcal{F}$ be a linearized basis of C at $p \in M$. Then*

(a) $\operatorname{Card} \mathcal{F} < \aleph_0 \Rightarrow \operatorname{Card} \mathcal{F} = dim(T_pM)$,

(b) $\operatorname{Card} \mathcal{F} \geq \aleph_0 \Rightarrow \operatorname{Card} \mathcal{F} < dim(T_pM)$,

(b*) $\operatorname{Card} \mathcal{F} \geq \aleph_0 \Rightarrow 2^{Card \mathcal{F}} = dim(T_pM)$, *if the generalized continuum hipothesis is assumed.*

Now, let us compare the notions of the differential basis and the linearized basis.

PROPOSITION 2.11. *Let (M, C) be a structured space and $p \in M$ be an arbitrary point. If a set $\mathcal{F} \subset C_p$ is a differential basis of C at p then $\mathcal{F}$ is a linearized basis of C at p.*

Proof. If the differential basis $\mathcal{F}$ is empty then, obviously, it is the linearized basis. In turn, if the differential basis is nonempty then also the set $C_p \setminus a_p$ is nonempty. In this case let $\mathbf{f} \in C_p \setminus a_p$ be an arbitrary element. There exist $n \in \mathbb{N}$, $\mathbf{f}_1, \ldots, \mathbf{f}_n \in \mathcal{F}$, $\omega \in C^{\infty}(\mathbb{R}^n)$ such that

$$\mathbf{f} = \omega \circ (\mathbf{f}_1, \ldots, \mathbf{f}_n). \tag{13}$$

Hence and from Lemma 2.2 obtain

$$[\mathbf{f}] = [\omega \circ (\mathbf{f}_1, \ldots, \mathbf{f}_n)] = \sum_{i=1}^{n} \omega'_{|i}(\mathbf{f}_1(p), \ldots, \mathbf{f}_n(p)) \cdot [\mathbf{f}_i]. \tag{14}$$

We will show that the set $\{[\mathbf{f}] : \mathbf{f} \in \mathcal{F}\}$ is a basis of the vector space C_p/a_p. It is enough to prove the linear independence of this set. Let

$$\lambda_1[\mathbf{f}_1] + \ldots + \lambda_n[\mathbf{f}_n] = 0,$$

for some $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and $\mathbf{f}_1, \ldots, \mathbf{f}_n \in \mathcal{F}$, $n \in \mathbb{N}$. Then

$$[\lambda_1\mathbf{f}_1 + \ldots + \lambda_n\mathbf{f}_n] = 0,$$

or equivalently

$$\lambda_1\mathbf{f}_1 + \ldots + \lambda_n\mathbf{f}_n \in a_p.$$

From the above formula one concludes that $\lambda_1, \ldots, \lambda_n = 0$. In fact, without losing of generality let us assume that $\lambda_1 \neq 0$. Then $\mathbf{f}_1$ can be presented in the form

$$\mathbf{f}_1 = -\frac{\lambda_2}{\lambda_1}\mathbf{f}_2 - \ldots - \frac{\lambda_n}{\lambda_1}\mathbf{f}_n + \mathbf{g},$$

for some $\mathbf{g} \in a_p$. Let $u_0 : \mathcal{F} \to \mathbb{R}$ be such that $u_0(\mathbf{f}_1) = 1$ and $u_0(\mathbf{h}) = 0$ for $\mathbf{h} \in B \setminus \{\mathbf{f}_1\}$. From Lemma 2.1 there exists exactly one $u \in T_pM$ such that $u|B = u_0$. Then

$$u(\mathbf{f}_1) = 1 \quad \text{and} \quad u(\mathbf{f}_1) = u\Big(-\frac{\lambda_2}{\lambda_1}\mathbf{f}_2 - \ldots - \frac{\lambda_n}{\lambda_1}\mathbf{f}_n + \mathbf{g}\Big) = 0,$$

which means a contradiction.

Now it is clear from (14) that $\mathbf{f}$ may be expressed in the form

$$\mathbf{f} = \sum_{i=1}^{n} \lambda_i\mathbf{f}_i + \mathbf{g},$$

where $\mathbf{g} \in a_p$, $\lambda_i = \omega'_{|i}(\mathbf{f}_1(p), \ldots \mathbf{f}_n(p))$ for $i = 1, \ldots, n$ and $\lambda_1, \ldots, \lambda_n$ are uniquely determined. ■

PROPOSITION 2.12. *Let (M, C) be a structured space and $p \in M$. A linearized basis $\mathcal{F}$ of C at p is a differential basis iff the algebra C_p is generated by $\mathcal{F}$.*

Proof. Let $\mathcal{F}$ be a linearized basis of C at p and assume that C_p is generated by $\mathcal{F}$. It remains to show that $\mathcal{F}$ is differentially independent. Let $\omega \in C^\infty(\mathbb{R}^n)$, $\mathbf{f}_1, \ldots, \mathbf{f}_n \in \mathcal{F}$ be arbitrary elements such that

$$\omega \circ (\mathbf{f}_1, \ldots, \mathbf{f}_n) = 0.$$

In view of Lemma 2.2 we have

$$[\omega \circ (\mathbf{f}_1, \ldots, \mathbf{f}_n)] = 0$$

or

$$\sum_{i=1}^{n} \omega'_{|i}(\mathbf{f}_1(p), \ldots, \mathbf{f}_n(p)) \cdot [\mathbf{f}_i] = 0.$$

Since $[\mathbf{f}_1], \ldots, [\mathbf{f}_1]$ are linearly independent we observe that $\omega'_{|i}(\mathbf{f}_1(p), \ldots$ $\ldots, \mathbf{f}_n(p)) = 0$, for $i = 1, \ldots, n$. We have proved that $\mathcal{F}$ satisfies the condition $(*)$ in Proposition 2.1. Thus $\mathcal{F}$ is differentially independent. Therefore $\mathcal{F}$ is a differential basis of C at p. ■

Let us notice that a linearized basis $\mathcal{F}$ does not need to generate $\mathcal{C}_p$. For example, if (M, τ) is a topological space and $\mathcal{C}$ is a non-trivial sheaf of all continuous functions, then $(M, \mathcal{C})$ is a structured space with the dimension equal zero at every point. In this case, for every $p \in M$, $\mathcal{C}_p = a_p$ and, consequently, the linearized basis $\mathcal{F}$ is empty. If the stalk $\mathcal{C}_p$ is non-trivial, then $\mathcal{C}_p$ is not generated by the empty linearized basis $\mathcal{F}$. One should notice that in this case the differential basis does not exist at the point $p \in M$. In general, a differential basis at a point does not always exist while a linearized basis does exist always, however it can be empty. Of course, if a differential basis is empty, then the linearized basis is also empty and the dimension at a given point is zero. Conversly, if at a given point the dimension is zero, then necessarily the linearized basis is empty and the differential basis is either also empty or it does not exist. Many examples of nonempty linearized bases which do not generate the stalk $\mathcal{C}_p$ one can obtain, for instance, by defining the Cartesian products of a non-trivial zero dimensional space with structured spaces of higher dimension.

References

[1] M. Heller, W. Sasin, *The structure of the b-boundary of space-time*, General Rel. and Gravitation 26 (1994), 797–811.

[2] M. Heller, W. Sasin, *Structured spaces and their application to relativistic physics*, J. Math. Phys. 36 (1995), 3644–3662.

[3] M. A. Mostov, *The differentiable space structures of Milnor classifying spaces, simplicial complex and geometric relations*, J. Diff. Geometry 14 (1979), 256–293.

[4] P. Multarzyński, W. Sasin, *On the dimension of differential spaces*, Demonstratio Math. 23 (1990), 405–415.

[5] P. Multarzyński, W. Sasin, *Algebraic charakterization of the dimension of differential spaces*, Supplemento ai Rendiconti del Circolo Matematico di Palermo, Ser. II, 22 (1989), 193–199.

[6] W. Sasin, Z. Z?ekanowski?, *On locally finitely generated differential spaces*, Demonstratio Math. 20 (1987), 477–487.

[7] R. Sikorski, *Abstract covariant derivative*, Colloq. Math. 18 (1967), 251–272.

INSTITUTE OF MATHEMATICS
WARSAW UNIVERSITY OF TECHNOLOGY
Pl. Politechniki 1
00-661 WARSZAWA, POLAND

Received November 25, 1996; revised version January 27, 1998.