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RELAXATION OF THE DIFFERENTIAL INCLUSIONS OF THE STURM–LIOUVILLE TYPE

Abstract. Let $T = (0;\pi)$, W be a linear normed space of Sobolev type and consider an operator $\mathcal{P} : \mathcal{W} \longrightarrow \mathcal{H}$, and a multifunction $\mathcal{F} : \operatorname{dom}\mathcal{F} \subset T \times \mathbf{R}^k \longrightarrow \operatorname{cl}\left(\mathbf{R}^k\right)$. The paper deals with a problem of the connection of the topological properties of the solution set $\mathcal{R}_{\mathcal{F}}$ for the boundary value problem for a differential inclusion:

$$\textit{Find } w \in W \textit{ such that } \mathcal{P}w \in \mathcal{F}(t,w),$$

and the solution set $\mathcal{R}_{\operatorname{clco}\mathcal{F}}$ of the relaxed problem

$$\textit{Find } w \in W \textit{ such that } \mathcal{P}w \in \operatorname{clco}\mathcal{F}(t,w),$$

where W is a Sobolev space $W^{2,2}([0;\pi]) \cap W_0^{1,2}([0;\pi])$ and $\mathcal{P}w = -\frac{d^2w}{dt^2}$ with Dirichlet boundary data $w(0) = 0 = w(\pi)$. The density of the solution set $\mathcal{R}_{\mathcal{F}}$ in the solution set $\mathcal{R}_{\operatorname{clco}\mathcal{F}}$ is proved.

1. Introduction

Differential inclusions of the form $\mathcal{P}u(t) \in \mathcal{F}(t,u(t))$ where $\mathcal{P}$ is a differential operator are immediate generalization of the differential equations. The theory of properties of ordinary differential inclusions of the first order has been thriving since the early seventies and a lot is known on the existence of solutions and on their properties both in the framework of the Euclidean space $\mathbf{R}^n$ as well as in the framework of the Banach space X.

In general differential inclusions with ordinary differential operators of the higher order are much less examined although a remarkable amount of interest in this field has been observed lately [8], [3].

The present paper deals with linking the topological properties of the solution set to the inclusion

$$-\frac{d^2x(t)}{dt^2} \in \mathcal{F}(t,x(t)) \tag{1}$$

with boundary conditions

$$x(0) = 0 = x(\pi), \tag{2}$$

and the solution set to the inclusion boundary value problem

$$-\frac{d^2x(t)}{dt^2} \in \operatorname{clco}\mathcal{F}(t,x(t)) \tag{3}$$

with boundary conditions

$$x(0) = 0 = x(\pi). \tag{4}$$

About the multifunctions $\mathcal{F}(t,\cdot)$ we assume that they are Lipschitz with compact but not necessarily convex values in the Euclidean space $\mathbf{R}^k$. We prove that the set of solutions $\mathcal{R}_{\mathcal{F}}$ is dense in the set of solutions $\mathcal{R}_{\operatorname{clco}\mathcal{F}}$. In section 2 we prove a version of the Filippov lemma taking into account the theory of the Sturm–Liouville equation

$$-\frac{d^2x}{dt^2} - m(t)x = \lambda x, \qquad t \in T \tag{5}$$

and the classical Filippov lemma [6]. The main result on density in the strong L^2 topology is formulated and proved in the section 3 while the section 1 contains preliminary facts and definitions which are needed in this paper.

2. Preliminaries

Let $T = (0;\pi)$, $\mathcal{L}$ be the σ–algebra of Lebesgue measurable subsets of T. The spaces of functions integrable with p–th power on T for $1 \le p \le \infty$, equipped with the usual norms $|x|_p$, we shall denote by L^p and let $W^{m,p}$ and $W_0^{m,p}$ be Sobolev spaces endowed with the usual norms $|x|_{m,p}$.

Let us fix a function $w \in L^2$ and consider the Sturm–Liouville operator L_w defined for $x \in W^{2,2}$ by

$$(L_w x)(t) := -\frac{d^2x(t)}{dt^2} - w(t)x(t), \qquad t \in T. \tag{6}$$

This operator satisfies

$$\langle L_w x, z\rangle = \int_0^\pi (x'(t)z'(t) - w(t)x(t)z(t)) \tag{7}$$

for all $x \in W^{2,1}$ and $z \in W_0^{1,\infty}$. Denote the bilinear form on the right hand side of (7) by $a_w(x,z)$ and observe that it is can be extended uniquely to a bounded bilinear form on $W_0^{1,2} \times W_0^{1,2}$. It is known that for any real $w \in L^2$ such a form defines unbounded Sturm–Liouville operator in the space $\mathcal{H} := L_X^2$ with countable real point spectrum $\lambda_0 < \lambda_1 < \ldots$, $\lim_{k\to\infty} \lambda_k = +\infty$. Moreover, we may choose the positive eigenfunction x_0 corresponding to the eigenvalue λ_0 (*c.f.* [10], [4], [5]).

The simplest Sturm–Liouville operator is the operator $L_0 x = -\frac{d^2x}{dt^2}$ (for $w = 0$) with the spectrum $\lambda_0 = 1$, $\lambda_0 = 4$, $\lambda_0 = 9, \ldots$, and

$x_0 = \sqrt{\frac{2}{k\pi}}[\sin t, \ldots, \sin t]$.

The equation

$$L_0 x = u \tag{8}$$

with the boundary conditions (4)

$$x(0) = 0 = x(\pi) \tag{9}$$

has a solution $x = Gu \in W^{2,1} \cap W_0^{1,2}$ for any $u \in L^1$. This solution is expressed by the formula

$$Gu(t) = \int_0^\pi \mathcal{G}_0(t,s)u(s)ds \tag{10}$$

where

$$\mathcal{G}_0(t,s) = \begin{cases} \frac{1}{\pi}t(\pi - s) & \text{for } t \le s, \\ \frac{1}{\pi}s(\pi - t) & \text{for } t > s, \end{cases}$$

is the corresponding Green function.

The operator $G : L^2 \longrightarrow W^{2,2}$ is linear, and bounded, and positive, *i.e.* for any function $u \le 0$ we have $Gu \le 0$ where $u = (u_1, \ldots, u_k) \le 0$ means that all $u_i \le 0$, $i = 1, \ldots, k$. In particular, it implies that for any $u \in L^2$ the following estimate

$$|Gu(t)| \le G(|u\sqrt{k}|)(t) \quad a.e. \text{ in } \bar{T} \tag{11}$$

holds.

Let us consider the differential inclusion

$$-\frac{d^2x}{dt^2} \in \mathcal{F}(t,x), \tag{12}$$

where the multifunction $\mathcal{F}(t,x)$ satisfies the following hypotheses:

($H1$) the sets $\mathcal{F}(t,x)$ are compact subsets of $\mathbf{R}^k$ for any $t \in \bar{T}$ and $x \in \mathbf{R}^k$, and the multifunctions $t \mapsto \mathcal{F}(t,x)$ are measurable for any $x \in \mathbf{R}^k$;

($H2$) there exists $m \in L^2$ such that for any $x, y \in \mathbf{R}^k$ we have

$$\mathrm{d}_H(\mathcal{F}(t,x), \mathcal{F}(t,y)) \le m(t)|x - y|,$$

where $\mathrm{d}_H(K,L)$ stands for Hausdorff distance between sets K and $L \subset \mathbf{R}^k$;

($H3$) $\quad \|\mathcal{F}(t,0)\| := \sup\{|x| : x \in \mathcal{F}(t,0)\} \le a(t) \quad a.e.$ and $a \in L^2$.

R e m a r k 1. Let us notice that for any measurable selection $v(t) \in \mathcal{F}(t, Gu(t))$ the hypothesis (H2) implies

$$\mathrm{dist}\,(v(t), \mathcal{F}(t,0)) \le m(t)|G(u)(t)|$$

for *a.e.* $t \in T = [0;\pi]$. Then from $(H3)$ follows an estimate

$$|v(t)| \le a(t) + m(t)|G(|u|\sqrt{k})(t)|$$

and further $v \in L^2$. ∎

On the positive measurable function m we impose the hypothesis $(H4)$:

$$\gamma := \sqrt{k}\left[\frac{1}{3\pi}\int_0^\pi m^2(t)t^2(\pi - t)^2 dt\right]^{\frac{1}{2}} < 1 \tag{13}$$

or

$$|m|_2 < \frac{4}{\pi}\sqrt{\frac{3}{k\pi}} \quad \text{or} \quad |m|_\infty < 3\sqrt{10}/\pi^2\sqrt{k}. \tag{14}$$

Remark 2. The hypothesis $(H4)$ is a sufficient condition for

$$1 \notin \sigma(mG\sqrt{k})$$

i.e. for the invertability of the operator $I - mG\sqrt{k}$ in a Banach algebra $\mathcal{BH}$ of all linear bounded operators $S : \mathcal{H} \longrightarrow \mathcal{H}$. ∎

Let us consider the boundary value problem to the inclusion (12) with boundary conditions (4). By a solution of the problem (12), (4) we mean any function $x \in W^{2,2}$ such that

$$(L_0x)(t) \in \mathcal{F}(t, x(t)) \quad a.e. \text{ in} \quad T = (0;\pi). \tag{15}$$

In the present paper we deal with properties of the solution sets $\mathcal{R}_\mathcal{F}$ and $\mathcal{R}_{\text{clco}\,\mathcal{F}}$ of the problem (12), (4) and the problem (3), (4) respectively.

We prove that $\mathcal{R}_\mathcal{F}$ is dense in the strong L^2 topology in the solution set $\mathcal{R}_{\text{clco}\,\mathcal{F}}$.

Recall the notion of decomposability. The set $\mathcal{K} \subset L^2$ is called decomposable, if

$$\chi_A u + (1 - \chi_A)v \in \mathcal{K} \tag{16}$$

for any u, $v \in \mathcal{K}$ and $A \in \mathcal{L}$. The family of all non-empty, closed, and decomposable subsets of L^2 let us denote by $\text{dec}(L^2)$. Let us observe that to the multifunction $\mathcal{F}$ there corresponds the map $\mathcal{K}_\mathcal{F} : L^2 \longrightarrow \text{dec}(L^2)$ given by

$$\mathcal{K}_\mathcal{F}(x) = \{f \in L^2 : f(t) \in \mathcal{F}(t, Gx(t)) \ a.e.\}$$

3. A version of the Filippov lemma

Let us assume that the problem (1) and the operator L_m satisfy $(H1)$, ..., $(H4)$. The fundamental lemma in this paper is as follows:

LEMMA 1. *Assume* $(H1), \ldots, (H4)$. *Let* f_0 *be an arbitrary element in* L^2 *and* $p \in L^2$ *be such that* $\operatorname{dist}(f_0, \mathcal{F}(t, Gf_0(t)) \le p(t)$ *a.e.. Then there exist a solution* $x \in W_0^{1,2}([0;\pi]) \cap W^{2,2}([0;\pi])$ *of* (1) *such that*

$$|x(t) - Gf_0(t)| \le G\sqrt{k}(I - mG\sqrt{k})^{-1} p(t) \quad a.e. \quad in\ T, \tag{17}$$

$$\left| -\frac{d^2x}{dt^2}(t) - f_0(t) \right| \le (I - mG\sqrt{k})^{-1} p(t) \quad a.e. \quad in\ T. \tag{18}$$

Proof. Take into account the Filippov iterations technique. Observe that for any $p \in L^2$

$$|mG\sqrt{k}p(t)|_2 \le \gamma |p|_2 \quad \text{where by } (H4) \quad \gamma < 1.$$

So the series

$$\beta(t) = \sum_{\ell=0}^{\infty} [(mG\sqrt{k})^{\ell} p](t) \in L^2 \tag{19}$$

is convergent in L^2 as well as pointwisely.

We shall construct $x = Gf$, where $f \in L_X^2 \cap \operatorname{Fix} \mathcal{K}_{\mathcal{F}}$.

Since

$$\operatorname{dist}(f_0(t), \mathcal{F}(t, Gf_0(t))) \le p(t) \quad a.e.$$

there exists an L^2-selection f_1 of $\mathcal{F}(t, Gf_0(t))$ such that

$$|f_1(t) - f_0(t)| \le p(t) \quad a.e.$$

Then

$$|Gf_1(t) - Gf_0(t)| \le G\sqrt{k}p(t) \quad a.e.$$

and therefore, by $(H2)$,

$$\operatorname{dist}(f_1(t), \mathcal{F}(t, Gf_1(t))) \le mG\sqrt{k}p(t) \quad a.e.$$

Next, there exists an L^2-selection f_2 of $\mathcal{F}(t, Gf_1(t))$ such that

$$|f_1(t) - f_2(t)| \le mG\sqrt{k}p(t) \quad a.e.$$

and hence

$$\operatorname{dist}(f_2(t), \mathcal{F}(t, Gf_2(t))) \le (mG\sqrt{k})^2 p(t) \quad a.e.$$

Using the induction argument, we can find a sequence $\{f_n\}$ of L^2-selections such that

$$|f_{n+1}(t) - f_n(t)| \le (mG\sqrt{k})^n p(t) \quad a.e., \tag{20}$$

and hence

$$\operatorname{dist}(f_{n+1}(t), \mathcal{F}(t, Gf_{n+1}(t)) \le (mG\sqrt{k})^{n+1} p(t) \quad a.e.$$

Take

$$f = f_0 + \sum_{n=0}^{\infty}(f_{n+1} - f_n)$$

and observe that $f \in L^2_X \cap \operatorname{Fix} \mathcal{K}_{\mathcal{F}}$. The inequalities (17) and (18) are now a straightforward conclusion of (19) and (20). ∎

Denote

$$\mathcal{I}f(t) := \int_0^t f(s)\,ds. \tag{21}$$

LEMMA 2. *For every L^2-selection $h \in \operatorname{clco} \mathcal{F}(t, Gh(t))$ and for any $\varepsilon > 0$ there is $f \in \mathcal{F}(t, Gh(t))$ such that*

$$|\mathcal{I}f(t) - \mathcal{I}h(t)| < \varepsilon/2\pi, \tag{22}$$

$$|Gf(t) - Gh(t)| < \varepsilon. \tag{23}$$

Proof. Observe that for every $t \in T$ we have

$$\mathcal{I}h(t) \in \int_0^t \operatorname{clco} \mathcal{F}(\tau, Gh(\tau))d\tau = \int_0^t \mathcal{F}(\tau, Gh(\tau))d\tau.$$

Since $\|F(t, Gh(t))\| \in L^2$ (Remark 1), then the existence of a required $f(t) \in F(t, Gh(t))$ satisfying (22) is the first part of the proof of the classical Filippov–Ważewski Relaxation Theorem (see Aubin–Cellina cf. [1]).

The inequality (23) follows from the previous one and the representation $Gu(t) = \frac{t}{\pi}(\mathcal{I}^2 u)(\pi) - (\mathcal{I}^2 u)(t)$. ∎

4. Main result

Let us consider the problem of density of the solution set $\mathcal{R}_{\mathcal{F}} \subset W^{2,2} \cap W_0^{1,2}$ to the differential inclusion

$$-\frac{d^2 x}{dt^2} \in \mathcal{F}(t, x) \quad \text{with} \quad x(0) = 0 = x(\pi) \tag{24}$$

in the solution set $\mathcal{R}_{\operatorname{clco} \mathcal{F}} \subset W^{2,2} \cap W_0^{1,2}$ to the relaxed inclusion

$$-\frac{d^2 x}{dt^2} \in \operatorname{clco} \mathcal{F}(t, x) \quad \text{with} \quad x(0) = 0 = x(\pi). \tag{25}$$

Let us impose the conditions (H1) (H2) and (H3) on the right hand side $\mathcal{F}(t, x)$ and let us assume that operator L_m, where $m(t)$ is "Lipschitz constant" of the multifunction $\mathcal{F}(t, \cdot)$ satisfies $(H4)$. The solution set $\mathcal{R}_{\mathcal{F}}$ is the set of all x such that (24) is fulfiled almost everywhere in T with (4) on the boundary of T. Let us denote

$$\mathcal{K}_{\mathcal{F}}(u) = \left\{v \in L^2 : v(t) \in \mathcal{F}(t, G(u)(t)) \text{ } a.e. \text{ in } T\right\} \tag{26}$$

and $\operatorname{Fix} \mathcal{K}_{\mathcal{F}} = \{u : u \in \mathcal{K}_{\mathcal{F}}(u)\}$. It is clear that for $x = Gu$

$$x \in \mathcal{R}_{\mathcal{F}} \quad \text{iff} \quad u \in \operatorname{Fix} \mathcal{K}_{\mathcal{F}}.$$

The main result in this paper is the following:

THEOREM 1. *Let us assume that for the multifunction $\mathcal{F}(t,x)$ the hypotheses $(H1)$, $(H2)$, $(H3)$ and $(H4)$ hold. Then*

$$\mathcal{R}_{\mathcal{F}} \text{ is dense in } \mathcal{R}_{\text{clco}\,\mathcal{F}}$$

with respect to the strong topology L^2.

Proof. Our proof is based on the version of Filippov lemma formulated in Lemma 2.

We shall prove that $\mathcal{K} : L^2 \longrightarrow \operatorname{dec}(L^2)$ defined as in (26) is nonempty, closed valued multifunction.

To see that the sets $\mathcal{K}_{\mathcal{F}}(u) \neq \emptyset$, let v be a measurable selection of multifunction $t \mapsto \mathcal{F}(t, G(u)(t))$. The existence of v follows from the Kuratowski and Ryll–Nardzewski Theorem while its square integrability from Remark 1. The closedness of $\mathcal{K}_{\mathcal{F}}(u)$ follows from the fact that any L^2 convergent sequence of L^2-selections of $\mathcal{F}(t, Gu(t))$ contains, by Remark 1 a pointwisely convergent subsequence.

Take any $h \in \operatorname{clco} \mathcal{F}(t, Gh)$ and arbitrary $\varepsilon > 0$. By Lemma 2 there exists $f \in \mathcal{F}(t, Gh)$ such that, $|\mathcal{I}f(t) - \mathcal{I}h(t)| < \varepsilon/2\pi$ and $|Gf(t) - Gh(t)| < \varepsilon$ for all $f \in T$.

Observe that $\operatorname{dist}(f(t), \mathcal{F}(t, Gf(t))) \leq d_H \mathcal{F}(t, Gh(t) \mathcal{F}(t, Gf(t)) \leq m(t)|Gh(t) - Gf(t)| \leq m(t)\varepsilon$.

By the above version of Filippov Lemma 1 with $p(t) = \varepsilon m(t)$ there exists $\bar{f} \in \mathcal{F}(t, G\bar{f}(t))$ such that $|G\bar{f}(t) - Gf(t)| \leq \varepsilon G\sqrt{k}(I - mG\sqrt{k})^{-1} m(t)$ and therefore

$$|G\bar{f} - Gf|_2 \leq \sqrt{k} \|(I - mG\sqrt{k})^{-1}\|_{L^2 \to L^2} 2\pi^{\frac{3}{2}} |m|_1 \varepsilon.$$

But this in turn implies that

$$|G\bar{f} - Gf|_2 \leq \frac{2\pi^{\frac{3}{2}}\sqrt{k}}{1-\gamma} |m|_1 \varepsilon.$$

So, we have

$$|G\bar{f} - Gh|_2 \leq |G\bar{f} - Gf|_2 + |Gf - Gh|_2 \leq \alpha\varepsilon$$

where $\alpha = \frac{2\pi^{\frac{3}{2}}\sqrt{k}}{1-\gamma}|m|_1 + 1$. ■

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