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Andrzej Borzymowski, Mustafa Shaieb

GOURSAT–TYPE PROBLEMS CONTAINING THE NORMAL DERIVATIVES OF THE UNKNOWN FUNCTIONS

Abstract. The paper concerns both linear and nonlinear Goursat–type problems for the partial differential equation of the form $\square \frac{\partial^2 u}{\partial x \partial y} = F$ with the boundary conditions containing the normal derivatives of u. The linear problem is reduced to a functional equation and hence the solution is found in series form. The existence and uniqueness of a solution to the nonlinear problem is proved by way of the Banach fixed point theorem.

Introduction

Several papers were devoted to the boundary value problems with Neumann–type boundary conditions for second–order hyperbolic partial differential equations whose leading parts correspond to the second canonical form $\square u := \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}$ (cf. [7], [9]–[12] and references). As far as we know, analogous problem for the equations with the leading parts corresponding to the first canonical form $Lu := \frac{\partial^2 u}{\partial x \partial y}$ have not been taken up *) except paper [4] of the first author, where the local existence of a nonlinear Neumann problem for a system of high order integro–differential equations with the leading parts $L^p u, p \geq 1$, was proved.

In this paper we deal with two Goursat problems for the equation $Lu = F$ whose boundary conditions contain the normal derivatives of u (for the Goursat problem with the boundary conditions not containing the normal derivatives see [1]–[3], [6], [8] and references). In Sections 1, 2 we examine a linear problem, reduce it to a functional equation and hence find its solution in series form. In Section 3 we consider a nonlinear problem and prove the local existence and uniqueness of its solution (under the assumptions

*) It is easily seen that, in general, the problems for the equation $Lu = F$ cannot be obtained from those for $\square u = F$ by the linear map transforming $\square u$ into Lu.

different from those in [4]) by using the results obtained in the preceding sections and applying the Banach fixed point theorem.

To the best of our knowledge the said problems have not been considered so far.

1. Let Y be a Banach space with norm $\|.\|$, Ω the rectangle $[0,A]\times[0,B]$, where $0 < A, B < \infty$ and Γ and $\widetilde{\Gamma}$ two curves of equations $y = f(x)$ and $x = h(y)$, where $f : [0,A] \to [0,B]$ and $h : [0,B] \to [0,A]$, respectively. Denote by $\mathbf{n}$ and $\widetilde{\mathbf{n}}$ the normal unit vectors to Γ and $\widetilde{\Gamma}$, respectively, and introduce the class $\mathbf{K}$ of all functions $u : \Omega \to Y$ possesing continuous derivatives $D_x^r D_y^s u$, where $D_x = \frac{\partial}{\partial x}$, $D_y = \frac{\partial}{\partial y}$ and $r, s \in \{0,1\}$.

We examine the following Goursat problem (G_L): find a solution of the equation

$$Lu = F \tag{1.1}$$

(where $F : \Omega \to Y$ is a given function) in Ω, that is a function $u \in \mathbf{K}$ satisfying (1.1) at each point of Ω, fulfilling the boundary conditions

$$\frac{d}{d\mathbf{n}}u[x,f(x)] = M(x), \qquad \frac{d}{d\widetilde{\mathbf{n}}}u[h(y),y] = N(y), \tag{1.2}$$

where $(x,y) \in \Omega$, and $M : [0,A] \to Y$ and $N : [0,B] \to Y$ are given functions.

We make the following assumptions.

I. The functions f and h are of class $\mathbb{C}^1$, strictly increase and satisfy the conditions

$$f(0) = h(0) = 0, \quad \min(\overset{*}{f}, \overset{*}{h}) > 0, \tag{1.3}$$

$$\overset{*}{f}, \overset{*}{h} < 1, \tag{1.4}$$

where $\overset{*}{f} = f'(0)$; $\overset{*}{h} = h'(0)$.

Moreover, the curves Γ and $\widetilde{\Gamma}$ do not intersect one another in $\Omega\backslash O$, where $O(0,0)$.

II. The functions M and M are continuous and satisfy the conditions

$$\|M(x)\| \leq m_1 x^{1+\theta_1}, \|N(y) \leq m_1 y^{1+\theta_1}, (x,y) \in \Omega, \tag{1.5}$$

where m_1 and θ_1 are positive constans.

III. The function F is continuous and satisfies the inequality

$$\|F(x,y)\| \leq m_2(x^{\theta_2} + y^{\theta_2}, (x,y) \in \Omega, \tag{1.6}$$

where m_2 and θ_2 are positive constans.

Let us observe that we have the following lemmas whose validity follows from the above Assumptions I–III.

LEMMA 1.1. *If u is of the form*

$$u(x,y) = R(x,y) + \varphi(x) + \psi(y), \tag{1.7}$$

where

$$R(x,y) = \int_0^x\int_0^y F(\xi,\eta)\,d\eta\,d\xi, \tag{1.8}$$

$\varphi : [0,A] \to Y$ and $\psi : [0,B] \to Y$ being functions of class $\mathbb{C}^1$, then u is a solution to equation (1.1) in Ω. Conversely, for a given solution u to equation (1.1) in Ω there are functions $\varphi : [0,A] \to Y$ and $\psi : [0,B] \to Y$ of class $\mathbb{C}^1$ such that equality (1.7) holds good.

LEMMA 1.2. *For every number $\varepsilon_0 \in (0,1)$ there is a sufficiently small number $\delta \in (0, \min(A,B))$ such that the inequalities*

$$\begin{cases} (1-\varepsilon_0)\overset{*}{f} < f'(x) < (1+\varepsilon_0)\overset{*}{f}, \\ (1-\varepsilon_0)\overset{*}{h} < h'(y) < (1+\varepsilon_0)\overset{*}{h} \end{cases} \tag{1.9}$$

hold good for $(x,y) \in [0,\delta] \times [0,\delta]$.

LEMMA 1.3. *The following inequalities*

$$\left\| \frac{d}{d\mathbf{n}} R(x,y) \right\|_{y=f(x)} \leq \text{const } x^{1+\theta_2}, \tag{1.10}$$

$$\left\| \frac{d}{d\widetilde{\mathbf{n}}} \right\|_{x=h(y)} \leq \text{const } y^{1+\theta_2} \tag{1.11}$$

are valid.

Now, let us introduce the function (cf. [1], p.104; [8], p.103)

$$g(x) = h \circ f(x), \quad x \in [0,A]. \tag{1.12}$$

LEMMA 1.4 (cf. [1], p. 104). *The relation*

$$g^n \rightarrow 0 \quad \text{on } [0,A] \tag{1.13}$$

holds good, when n tends to infinity, with $\rightarrow$ denoting the uniform convergence.

2. Let us assume that the direction cosines of the vectors $\mathbf{n}$ and $\widetilde{\mathbf{n}}$ are given by

$$
(2.1)\qquad \begin{cases} \cos(x,\mathbf{n}) = -\dfrac{f'(x)}{e(x)}; & \cos(y,\mathbf{n}) = \dfrac{1}{e(x)} \\ \cos(x,\widetilde{\mathbf{n}}) = -\dfrac{1}{\widetilde{e}(y)}; & \cos(y,\widetilde{\mathbf{n}}) = \dfrac{h'(y)}{\widetilde{e}(y)} \end{cases}
$$

where

$$
(2.2)\qquad e(x) = (1+f'^2(x))^{\frac{1}{2}}, \quad \widetilde{e}(y) = (1+h'^2(y))^{\frac{1}{2}}.
$$

Imposing on function u (cf. (1.7)) the boundary conditions (1.2), one gets the following system of functional equations

$$
(2.3)\qquad \begin{cases} \varphi'(x) - (f'(x))^{-1}\psi' \circ f(x) = V(x), \\ \psi'(y) - (h'(y))^{-1}\varphi' \circ h(y) = W(y) \end{cases}
$$

for $(x,y) \in \Omega$, where φ', ψ' are the unknowns, sought in the class $\mathbb{C}^0$, and v, W are given by

$$
(2.4)\qquad \begin{cases} V(x) = -\dfrac{e(x)}{f'(x)}\left\{M(x) - \left[\dfrac{d}{d\mathbf{n}}R(x,y)\right]_{y=f(x)}\right\}, \\ W(y) = \dfrac{\widetilde{e}(y)}{h'(y)}\left\{N(y) - \left[\dfrac{d}{d\widetilde{\mathbf{n}}}R(x,y)\right]_{x=h(y)}\right\}. \end{cases}
$$

As a result, problem (G_L) is reduced to system (2.3). It is evident that (2.3) is equivalent to

$$
(2.5)\qquad \psi'(y) = (h'(y))^{-1}[\varphi' \circ h(y) + W(y)], \quad y \in [0,B];
$$

$$
(2.6)\qquad \varphi'(x) - b(x)\varphi' \circ g(x) = G(x), \quad x \in [0,A],
$$

where

$$
(2.7)\qquad b(x) = (f'(x)h' \circ f(x))^{-1},
$$

$$
(2.8)\qquad G(x) = V(x) + (f'(x))^{-1}W \circ f(x).
$$

Of course, it is sufficient to solve equation (2.6) and then substitute its solution φ' to (2.5) and find ψ'.

Equation (2.6) is a functional equation of the type studied for $Y = R$ in [8], Chapt.II.

PROPOSITION 2.1. *Equation* (2.6) *has a solution given by the formula*

$$
(2.9)\qquad \varphi'(x) = S(x),
$$

where

$$
(2.10)\qquad S(x) = \sum_{n=0}^{\infty} a_n(x),
$$

$$a_n(x) = B_n(x) G \circ g^n(x) \tag{2.11}$$

with

$$B_n(x) = \prod_{m=0}^{n-1} b \circ g^m(x) \tag{2.12}$$

(as usual, we set $\prod_{m=r}^{s} u_m := 1$ when $s < r$). It is the unique solution of (2.6) in the class $\mathbf{K}_1$ of all continuous functions $\varphi' : [0, A] \to Y$ such that

$$\|\varphi'(x)\| \le C_\varphi x^{1+\theta}, \tag{2.13}$$

where c_φ is a positive constans (which may be different for different functions φ) and $\theta = \min(\theta_1, \theta_2)$. The function φ' given by (2.9)–(2.12) belongs to the class $\mathbf{K}_1$.

Proof. We begin with the proof of the uniform convergence of series (2.10). Let ε_0 be a number such that

$$0 < \varepsilon_0 < 1 - q^\omega, \tag{2.14}$$

where (cf. (1.4))

$$q = \overset{*}{f}\overset{*}{h} \in (0,1), \tag{2.15}$$

$$\omega = \frac{\theta}{2(4+\theta)}. \tag{2.16}$$

It follows from Lemma 1.4 that there is a number $n_0 \in \mathbb{N}$ (where $\mathbb{N}$ denotes the set of all positive integers) such that for all $n_0 \le v \in \mathbb{N}$ and $x \in [0, A]$ the relations $g^v(x) \in [0, \delta]$, $f \circ g^v(x) \in [0, \delta]$ hold good. In the sequel we shall assume that $n > n_0$. By (2.7) and (2.12) we have

$$\begin{aligned} B_n(x) &= \prod_{m=0}^{n_0-1} b \circ g^m(x) \prod_{m=n_0}^{n-1} b \circ g^m(x) \\ &\le \text{const} \prod_{m=n_0}^{n-1} \left(f' \circ g^m(x) h' \circ f \circ g^m(x)\right)^{-1}, \end{aligned}$$

whence and from Lemma 1.2 (with ε_0 satisfying (2.14)) we obtain the estimate

$$B_n(x) \le \text{const } \left[(1-\varepsilon_0)^2 q\right]^{-n}. \tag{2.17}$$

Let us observe that (cf. (2.4) and (2.8))

$$\begin{aligned} \|G \circ g^n(x)\| \le \text{const } (&\|M \circ g^n(x)\| + \|N \circ f \circ g^n(x)\| \\ &+ \|\frac{d}{d\mathbf{n}} R[g^n(x), f \circ g^n(x)]\| + \|\frac{d}{d\tilde{\mathbf{n}}} R[g^{n+1}(x), g^n(x)]\|). \end{aligned} \tag{2.18}$$

In virtue of (1.5), (1.12) and Lemma 1.2, we have the following relations (cf. (2.15))

$$\|M \circ g^n(x)\| \le \text{const } \left[(1+\varepsilon_0)^2 q\right]^{n(1+\theta_1)} x^{1+\theta_1} \le \text{const } \left[(1-\varepsilon_0)^{-2} q\right]^{n(1+\theta_1)} x^{1+\theta_1} \tag{2.19}$$

and in similar way we get

$$\|N \circ f \circ g^n(x)\| \le \text{const } [(1-\varepsilon^{-2} q]^{n(1+\theta_1)} x^{1+\theta_1}. \tag{2.20}$$

Furthermore, Lemma 1.3 yields

$$\max\left(\left\|\frac{d}{d\mathbf{n}} R[g^n(x), f \circ g^n(x)]\right\|, \left\|\frac{d}{d\tilde{\mathbf{n}}} R[g^{n+1}(x), g^n(x)]\right\|\right) \le \text{const } [(1-\varepsilon_0)^{-2} q]^{n(1+\theta_2)} x^{1+\theta_2}. \tag{2.21}$$

On joining (2.18)–(2.21), we obtain the inequality

$$\|G \circ g^n(x)\| \le \text{const } \left[(1-\varepsilon_0)^{-2} q\right]^{n(1+\theta)} x^{1+\theta} \tag{2.22}$$

which, together with (2.17), implies (cf. (2.11))

$$\|a_n(x)\| \le \text{const } q_1^n x^{1+\theta} \le \text{const } q_1^n, e \in [0, A] \tag{2.23}$$

where

$$q_1 = (1-\varepsilon_0)^{-2(2+\theta)} q^\theta \tag{2.24}$$

(with q given by (2.15)). It follows from (2.14)–(2.16) that $q_1 \in (0,1)$ and hence (cf. (2.23)) the series (2.10) is uniformly convergent, as required. It is verified by direct calculation that the function φ' given by (2.9) is a solution to (2.6).

In order to prove the uniqueness of the solution in K_1, let us observe that if a function $\varphi : [0, A] \to Y$ is a solution to equation (2.6) in the interval $[0, A]$, then, for every $r \in \mathbb{N}$ and every $x \in [0, A]$, the following equality

$$\varphi'(x) = \sum_{n=0}^{r} \Big(\prod_{m=0}^{n-1} b \circ g^m(x)\Big) G \circ g^n(x) + \varrho_r(x) \tag{2.25}$$

holds good, where

$$\varrho_r(x) = \Big(\prod_{m=0}^{r} b \circ g^m(x)\Big) \varphi' \circ g^{r+1}(x). \tag{2.26}$$

For $r > n_0$, we have the following sequence of inequalities (cf. the derivation of (2.22))

$$\|\varrho_r(x)\| \le \text{const} \prod_{m=n_0+1}^{r} [(1-\varepsilon_0)^2 q]^{-r} [(1-\varepsilon_0)^{-2} q]^{r(1+\theta/2)} [g^{r+1}(x)]^{\theta/2}$$
$$\le \text{const}\ q_2^r x^{1+\theta/2} [g^{r+1}(x)]^{\theta/2},$$

where (cf. (2.14)–(2.16))

$$0 < q_2 = (1-\varepsilon_0)^{-(4+\theta)} q^{\theta/2} = [(1-\varepsilon_0)^{-2(4+\theta)} q^{\theta}]^{1/2} < 1.$$

As a consequence,

$$\|\varrho_r(x)\| \le \text{const}\ [g^{r+1}(x)]^{\theta/2},$$

whence and from Lemma 1.4 it follows that

$$\varrho_r(x) \to 0 \quad \text{when } r \to \infty. \tag{2.27}$$

Relations (2.25) and (2.27) imply that $\varphi'(x)$ is of the form (2.9) which ends the proof of the uniqueness.

Finally, it follows from (2.10) and (2.23) that the function φ' given by (2.9) is continuous and satisfies inequality (2.13), i.e., belongs to the class K_1.

Thus, the proof of Proposition 2.1 is completed

COROLLARY 2.1. *It follows from Proposition 2.1 that the functions* $\varphi'(x)$ (cf. (2.9)) *and*

$$\psi'(y) = (h'(y))^{-1}[S \circ h(y) + W(y)], \quad y \in [0, B], \tag{2.28}$$

satisfy system (2.3) *in* Ω *if, and for* $\varphi' \in \mathbf{K}_1$ *) *only if, the following equalities*

$$\varphi(x) = \int_0^x S(\xi)\,d\xi + C_1, \tag{2.29}$$

$$\psi(y) = \int_0^y [h'(\eta]^{-1}[S \circ h(y) + W(\eta)]\,d\eta + C_2 \tag{2.30}$$

hold good for $(x, y) \in \Omega$, *where* C_1, C_2 *are arbitrary constans.*

As a result we get

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*) It is easily proved that, if $\varphi' \in \mathbf{K}_1$, then (2.5) implies $\|\psi'(y)\| \le \widetilde{C}_\varphi y^{1+\vartheta}$, where $\widetilde{C}_\varphi$ is a positive constant depending on C_φ (cf. (2.13)).

THEOREM 2.1. *If Assumptions* I–III *are satisfied, then problem* (G_L) *has a solution given by formula* (1.7), *where* $\varphi(x)$ *and* $\psi(x)$ *are defined by* (2.29) *and* (2.30), *respectively. It is the only solution in the class of all solutions of equation* (1.1) *such that the functions* φ *and* ψ *in formula* (1.7) *satisfy the conditions* $\varphi(0) = a$; $\psi(0) = b$; $\varphi' \in \mathbf{K}_1$, *where* a *and* b *are given numbers.*

3. We shall now pose a nonlinear counterpart of the problem studied in Sections 1 and 2. Let us consider the following nonlinear partial differential equation

$$Lu = F\left[(x, y, u(x, y), \frac{\partial u}{\partial x}(x, y), \frac{\partial u}{\partial y}(x, y)\right] \tag{3.1}$$

(where $F : \Omega \times Y^3 \to Y$ is a given function), and mean its solution in Ω as a function $u \in \mathbf{K}$ satisfying (3.1) at each point $(x, y) \in \Omega$.

We examine the following problem (G_L): find a solution of equation (3.1) in Ω satisfying the boundary conditions (cf. (1.2))

$$\begin{cases} \dfrac{d}{d\mathbf{n}} u[x, f(x)] = M(x, u[x, f(x)], \dfrac{\partial u}{\partial x}[x, f(x)], \dfrac{\partial u}{\partial y}[x, f(x)]), \\ \dfrac{d}{d\bar{\mathbf{n}}}[h(y), y] = N(y, u[h(y), y], \dfrac{\partial u}{\partial x}[h(y), y], \dfrac{\partial u}{\partial y}[h(y), y]), \end{cases} \tag{3.2}$$

where $(x, y) \in \Omega$ and $M : [0, A] \times Y^3 \to Y$, $N : [0, B] \times Y^3 \to Y$ are given functions.

We retain Assumption **I** and we make the following ones.

II′. The functions M and N are continuous and satisfy the conditions

$$M(0, (0)) = N(0, (0)) = 0, \tag{3.3}$$

$$\begin{cases} \|M(x, \xi, \eta, \zeta) - M(\bar{x}, \bar{\xi}, \bar{\eta}, \bar{\zeta})\| \\ \leq K_1(\bar{x}^{2\theta_*} + \omega_1 + \omega_2 + \omega_3)(\bar{x} - x + \|\bar{\xi} - \xi\| + \|\bar{\eta} - \eta\| + \|\bar{\zeta} - \zeta\|), \\ \|N(y, \xi, \eta, \zeta) - N(\bar{y}, \bar{\xi}, \bar{\eta}, \bar{\zeta}) \\ \leq K_1(\bar{y}^{2\theta_*} + \omega_1 + \omega_2 + \omega_3)(\bar{y} - y + \|\bar{\xi} - \xi\| + \|\bar{\eta} - \eta\| + \|\bar{\zeta} - \zeta\|), \end{cases} \tag{3.4}$$

where $(0) = (0, 0, 0)$, $0 \leq x \leq \bar{x} \leq A$, $0 \leq y \leq \bar{y} \leq B$, K_1 is a positive constant, $\theta_* \in (\frac{1}{2}, 1)$ and

$$\omega_1 = \max(\|\xi\|, \|\bar{\xi}\|), \quad \omega_2 = \max(\|\eta\|, \|\bar{\eta}\|), \quad \omega_3 = \max(\|\zeta\|, \|\bar{\zeta}\|). \tag{3.5}$$

III′. The function F is continuous and satisfies the conditions

$$F(0, 0, (0)) = 0, \tag{3.6}$$

$$\|F(x, y, \xi, \eta, \zeta) - F(\bar{x}, \bar{y}, \bar{\xi}, \bar{\eta}, \bar{\zeta})\| \tag{3.7}$$

$$\leq K_2(\overline{x}^{2\widehat{\theta}} + \overline{y}^{2\widehat{\theta}} + \omega_1 + \omega_2 + \omega_3)(\overline{x} - x + \overline{y} - y + \|\overline{\xi} - \xi\| + \|\overline{\eta} + \eta\|),$$

where K_2 is a positive constant and $\widehat{\theta} \in (\frac{1}{2}, 1)$.

Remark 3.1. It follows from (3.3) and (3.4) that the following inequalities hold good

$$\tag{3.8} \begin{cases} \|M(x,\xi,\eta,\zeta)\| \leq \widetilde{K}_1(x + \|\xi\| + \|\eta\| + \|\zeta\|)^2, \\ \|N(y,\xi,\eta,\zeta)\| \leq \widehat{K}_1(y + \|\xi\| + \|\eta\| + \|\zeta\|)^2, \end{cases}$$

with $\widetilde{K}_1 = \max(1, A^{2\theta_*-1})$, $\widehat{K}_1 = \max(1, B^{2\theta_*-1})$ and

$$\tag{3.9} \|F(x,y,\xi,\eta,\zeta)\| \leq \widetilde{K}_2(x + y + \|\xi\| + \|\zeta\| + \|\zeta\|)^2$$

where $\widetilde{K}_2 = \max(1, \widehat{A}^{2\widehat{\theta}-1})$ with $\widehat{A} = \max(A, B)$.

4. We shall attempt to solve the problem (G_N). It follows from an argument analogous to that in Section 2 that the said problem is equivalent to the following system of integro-differential equations

$$\tag{4.1} u(x,y) = R_u(x,y) + \varphi(x) + \psi(y),$$

$$\tag{4.2} \begin{cases} \varphi'(x) - (f'(x))^{-1}\psi' \circ f(x) = V_u(x), \\ \psi'(y) - (h'(y))^{-1}\varphi' \circ f(x) = W_u(y), \end{cases}$$

where $(x,y) \in \Omega$,

$$\tag{4.3} R_u(x,y) = \int_0^x\int_0^y F\left[t,\tau,u(t,\tau),\frac{\partial u}{\partial x}(t,\tau),\frac{\partial u}{\partial y}(t,\tau)\right] d\tau\, dt$$

and $V_u(x)$ and $W_(y)$ are given by formulas (2.4), with the replacement of R by R_u, $M(x)$ by

$$M_u(x) := M(x, u[x,f(x)], \frac{\partial u}{\partial x}[x,f(x)], \frac{\partial u}{\partial y}[x,f(x)])$$

and $N(y)$ by

$$N_u(y) := N(y, u[h,(y),y], \frac{\partial u}{\partial x}[h(y),y], \frac{\partial u}{\partial y}[h(y),y]).$$

Let Λ be the Banach space of all functions $u : \Omega \to Y$ of class C^1, and let the distance in Λ be defined in the ordinary way

$$\tag{4.4} d(u,\overline{u}) = \sup_{\Omega} \|u(x,y) - \overline{u}(x,y)\| + \sup_{\Omega} \left\|\frac{\partial u}{\partial x}(x,y) - \frac{\partial \overline{u}}{\partial x}(x,y)\right\| + \sup_{\Omega} \left\|\frac{\partial u}{\partial y}(x,y) - \frac{\partial \overline{u}}{\partial y}(x,y)\right\|.$$

We consider the set Z of all points $u \in \Lambda$ such that

$$(4.5) \quad u(0,0) = 0,$$

$$(4.6) \quad \max\left(\left\|\frac{\partial u}{\partial x}(x,y)\right\|, \left\|\frac{\partial u}{\partial y}(x,y)\right\|\right) \le \varrho(x+y)^{1+\vartheta}, \quad (x,y) \in \Omega,$$

where $\frac{1}{2} < \varrho < 1$ is a parameter, and $\vartheta = \min(\theta_*, \widehat{\theta})$ (cf. (3.4), (3.7)). It is clear that relations (4.5), (4.6) imply

$$(4.7) \qquad \|u(x,y)\| \le \varrho(x+y)^{2+\vartheta}, \quad (x,y) \in \Omega, \quad u \in \mathbf{Z}.$$

Evidently, $\mathbf{Z}$ is a closed subset of Λ and hence it can be treated as a complete metric space with the distance function (4.4).

We shall express $\varphi'(x)$ and $\psi'(y)$ in terms of the function u by applying Proposition 2.1 to the equation *)

$$(4.8) \qquad \varphi'(x) - b(x)\varphi' \circ g(x) = G_u(x), \quad u \in \mathbf{Z}$$

and then substituting its solution to

$$(4.9) \qquad \psi'(y) = (h'(y))^{-1}[\varphi' \circ h(y) + W_u(y)].$$

To this end it is enough to show that the function $G_u \circ g^n(x)$ satisfies an inequality analogous to (2.22). By using (3.9), (4.3), (4.6) and (4.7), we have

$$(4.10) \quad \max\left(\left\|\frac{d}{d\mathbf{n}}R_u[g^n(x), f \circ g^n(x)]\right\|, \left\|\frac{d}{d\widetilde{\mathbf{n}}}R[g^{n+1}(x), g^n(x)]\right\|\right) \le \text{const}\ (1+\varrho)[(1-\varepsilon_0)^{-2}q]^{n(1+\vartheta)} A^{1-\vartheta}x^{1+\vartheta},$$

$$(4.11) \quad \begin{cases} \|M_u \circ g^n(x)\| \le \text{const}\ (1+\varrho)[(1-\varepsilon_0)^{-2}q]^{n(1+\vartheta)} A^{1-\vartheta}x^{1+\vartheta}, \\ \|N_u \circ f \circ g^n(x)\| \le \text{const}\ (1+\varrho)[(1-\varepsilon_0)^{-2}q]^{n(1+\vartheta)} A^{1-\vartheta}x^{1+\vartheta}. \end{cases}$$

As a result, we obtain

$$(4.12) \qquad \|G_u \circ g^n(x)\| \le \text{const}\ (1+\varrho)[(1-\varepsilon_0)^{-2}q]^{n(1+\vartheta)} A^{1-\vartheta}x^{1+\vartheta},$$

i.e., we have got for $\|G_u \circ g^n(x)\|$ an inequality analogous to (2.22) (by assumption (2.14), $(1-\varepsilon_0)^{-2}q < 1$).

On using Proposition 2.1, we can assert that, for every $u \in \mathbf{Z}$, equation (4.8) has a solution

$$(4.13) \qquad \varphi'(x) = S_u(x),$$

*) $G_u(x)$ is given by formula (2.8) with V and W replaced by V_u and W_u, respectively.

where

$$S_u(x) = \sum_{n=0}^{\infty} a_n^{(u)}(x), \tag{4.14}$$

$$a_n^{(n)}(x) = B_n(x) G_u \circ g^n(x) \tag{4.15}$$

(with $B_n(x)$ given by (2.12)), the series in (4.14) being uniformly convergent.

It is the unique solution of (4.8) in the class $\mathbf{K}_1$ to which the function $\varphi_u'(x)$ given by (4.13) belongs itself, satisfying the inequality (cf. (4.12))

$$\|\varphi_u'(x)\| \leq \text{const } (1+\varrho) A^{1-\vartheta} x^{1+\vartheta}. \tag{4.16}$$

As a result, the corresponding function $\psi_u'(y)$ is given by (4.9) with $\varphi'(x) = \varphi_u'(x)$, and satisfies the inequality

$$\|\psi_u'(y)\| \leq \text{const } (1+\varrho) A^{1-\vartheta} y^{1+\vartheta}. \tag{4.17}$$

Evidently, const in (4.16), (4.17) is independent of the choice of $u \in \mathbf{Z}$.

It is a consequence of (4.5), (4.9) and (4.13) that the function u is a solution of problem (G_N) vanishing at $(0,0)$ if *)

$$u(x,y) = R_u(x,y) + \varphi_u(x) + \psi_u(y), \tag{4.18}$$

where

$$\begin{cases} \varphi_u(x) = \int_0^x S_u(t)\,dt + C_0, \\ \psi_u(y) = \int_0^y (h'(t))^{-1} [S_u \circ h(t) + W_u(t)]\,dt - C_0 \end{cases} \tag{4.19}$$

for $(x,y) \in \Omega$, C_0 being an arbitrary constant. It follows directly from (4.16), (4.17) and (4.19) that the function u satisfying (4.18) fulfils the inequalities

$$\|u(x,y)\| \leq \text{const } (1+\varrho) \widehat{A}^{1-\vartheta} (x+y)^{2+\vartheta}, \tag{4.20}$$

$$\max\left(\left\| \frac{\partial u}{\partial x}(x,y) \right\|, \left\| \frac{\partial u}{\partial y}(x,y) \right\| \right) \leq \text{const } (1+\varrho) \widehat{A}^{1-\vartheta} (x+y)^{1+\vartheta}. \tag{4.21}$$

Thus, in order to prove the existence and uniqueness of a solution to the problem (G_N), it is sufficient to show that the integro–differential equation (4.18) has a unique solution. We shall apply the Banach fixed point theorem to the complete metric space $\mathbf{Z}$ and, in view of (4.18), we map $\mathbf{Z}$ by the

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*) And only if, provided that the first derivative of the function φ in (4.1) belongs to the class $\mathbf{K}_1$.

transformation

(4.22) $w(x,y) = Tu(x,y) := R_u(x,y) + \varphi_u(x) + \psi_u(y), \quad (x,y) \in \Omega;\ u \in \mathbf{Z}.$

Evidently, w satisfies condition (4.5). Moreover, it follows from (4.20)–(4.22) that

$$(4.23) \qquad \max\left(\left\|\frac{\partial w}{\partial x}(x,y)\right\|, \left\|\frac{\partial w}{\partial y}(x,y)\right\|\right) \le C_*(1+\varrho)\widehat{A}^{1-\vartheta}(x+y)^{1+\vartheta}$$

(where C_* is a positive constant) and hence w satisfies inequality (4.6), provided that the condition

$$(4.24) \qquad C_*(1+\varrho)\widehat{A}^{1-\vartheta} \le \varrho$$

is fulfilled. Evidently, (4.24) holds good, if $\widehat{A}$ is sufficiently small, so that

$$\widehat{A}^{1-\vartheta} \le \frac{1}{3C_*} \le \frac{1}{C_*}\frac{\varrho}{\varrho+1}$$

that is if

$$(4.25) \qquad \widehat{A} \le (3C_*)^{-1/(1-\vartheta)}.$$

Thus, we can assert that the inclusion $T(\mathbf{Z}) \subset \mathbf{Z}$ is valid, if $\widehat{A}$ satisfies (4.25).

It is still to be proved that T is a contraction. Let us first observe that, by (3.7), (4.3) and (4.4), we have

$$(4.26) \qquad \|R_u(x,y) - R_{\overline{u}}(x,y)\| \le \text{const } (1+\varrho)(x+y)^{2+\vartheta} d(u,\overline{u}),$$

$$(4.27) \begin{cases} \left\|\frac{d}{dn}R_u(x,y) - \frac{d}{dn}R_{\overline{u}}(x,y)\right\| \le \text{const } (1+\varrho)(x+y)^{1+\vartheta} d(u,\overline{u}), \\ \left\|\frac{d}{d\widetilde{n}}R_u(x,y) - \frac{d}{d\widetilde{n}}R_{\overline{u}}(x,y)\right\| \le \text{const } (1+\varrho)(x+y)^{1+\vartheta} d(u,\overline{u}) \end{cases}$$

for $u, \overline{u} \in \mathbf{Z}$. Furthermore (cf. (3.4))

$$(4.28) \qquad \begin{cases} \|M_u(x) - M_{\overline{u}}(x)\| \le \text{const } (1+\varrho)x^{2\vartheta} d(u,\overline{u}), \\ \|N_u(y) - N_{\overline{u}}(y)\| \le \text{const } (1+\varrho)y^{2\vartheta} d(u,\overline{u}). \end{cases}$$

It follows from the estimates (4.27), (4.28) and from the inequality $2\vartheta > 1$ that (cf. the proof of (2.23))

$$(4.29) \qquad \|S_u(x) - S_{\overline{u}}(x)\| \le \text{const } (1+\varrho)x^{2\vartheta} d(u,\overline{u}),$$

whence, and by (4.9), (4.13) and (4.19), we obtain

$$(4.30) \quad \|\varphi_u(x) + \psi_u(y) - \varphi_{\overline{u}}(x) - \psi_{\overline{u}}(y)\| \le \text{const } (1+\varrho)(x+y)^{1+2\vartheta} d(u,\overline{u}).$$

Relations (4.22), (4.26) and (4.30) yield

$$(4.31) \qquad \sup_{\Omega} \|w(x,y) - \overline{w}(x,y)\| \le \text{const } (1+\varrho)\widehat{A}^{1+2\vartheta} d(u,\overline{u}),$$

where $\overline{w} = T\overline{u}$. In a similar way we prove (cf. (4.22), (4.27), (4.28)) that

$$(4.32)\qquad \begin{cases} \sup\limits_{\Omega}\left\|\dfrac{\partial w}{\partial x} - \dfrac{\partial \overline{w}}{\partial x}(x,y)\right\| \leq \text{const } (1+\varrho)\widehat{A}^{2\vartheta} d(u,\overline{u}), \\ \sup\limits_{\Omega}\left\|\dfrac{\partial w}{\partial y}(x,y) - \dfrac{\partial \overline{w}}{\partial y}(x,y)\right\| \leq \text{const } (1+\varrho)\widehat{A}^{2\vartheta} d(u,\overline{u}). \end{cases}$$

Thus (cf. (4.4))

$$(4.33)\qquad d(w,\overline{w}) \leq C_{**}(1+\varrho)\widehat{A}^{2\vartheta} d(u,\overline{u}),$$

where C_{**} is a positive constant, and hence we can assert that the transformation T (cf. (4.22)) is a contraction, if the inequality

$$(4.34)\qquad C_{**}(1+\varrho)\widehat{A}^{2\vartheta} < 1$$

is satisfied, which takes place, if

$$\widehat{A}^{2\vartheta} < \frac{1}{2C_{**}} < \frac{1}{C_{**}(1+\varrho)}$$

that is

$$(4.35)\qquad \widehat{A} < (2C_{**})^{-1/2\vartheta}.$$

Bearing in mind the above-obtained results and basing on the Banach fixed point theorem, we conclude that, if inequalities (4.25) and (4.35) are satisfied, then the transformation T (cf. (4.22)) has a unique fixed point $u_* \in \mathbf{Z}$. The function u_* satisfies equation (4.18) and hence it is a solution of problem (G_N).

As a result, we can formulate the following theorem.

THEOREM 4.1. *If Assumptions* **I**, **II**$'$, *and* **III**$'$ *are satisfied, and if the value of* $\widehat{A} = \max(A,B)$ *is sufficiently small, so that inequalities* (4.25) *and* (4.35) *hold good, then problem* (G_N) *has a solution* $u_* \in \mathbf{Z}$. *This solution is unique in the class of all functions* $u \in \mathbf{Z}$ *of the form* (4.1), *where* $\varphi' \in \mathbf{K}_1$ (see Proposition 2.1).

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INSTITUTE OF MATHEMATICS
WARSAW UNIVERSITY OF TECHNOLOGY
Pl. Politechniki 1
00-661 WARSAW, POLAND

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