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PERTURBATION OF NORMAL OPERATORS, VANISHING ON A GIVEN SUBSPACE

Abstract. For a normal bounded operator A in a Hilbert space, a normal and diagonal perturbed operator $A+Y$ is constructed in such a way that $\|Y\|_p < \varepsilon$ for a given $\varepsilon > 0$, $p > 2$ and $PYP = 0$ for a given finite-dimensional orthogonal projection P.

1. Introduction and main results

Let A be any normal operator in a Hilbert space H. For example, A can be given as an integral operator in $L_2(0,1)$ with a kernel $K(x,y)$. For a number of theoretical and computational problems, it is natural to use a perturbed operator $A+Y$ such that

$$A+Y = \sum \lambda_i \widehat{e}_i$$

with $\lambda_i \in C$ and $\{e_i\}$ being an orthonormal system in H, $\widehat{e}_i = \langle \cdot, e_i \rangle e_i$, i.e., $A+Y$ is a diagonal operator. Moreover, one usually requires that Y be small in a metric as strong as possible. Thus the requirement that $\|Y\|_2 < \varepsilon$ or $\|Y\|_p < \varepsilon$ (for a Schatten norm $\|\cdot\|_p, p \geq 1$) is more desirable than $\|Y\| < \varepsilon$. A number of monographs can be recommended: [1], [3].

On other hand, some vectors in H can be especially important and the approximation $A+Y$ of A should be exact on these vectors. For example, it could be important that $Yx^k = 0$ for some monomials $x, x^2, ..., x^n$. As a partial solution of the problem of such an approximation we shall prove the following.

1.1. THEOREM. *Let H be a separable, infinite-dimensional Hilbert space, A — a normal bounded operator in H. For any finite-dimensional projection P in H, for any $\varepsilon > 0$ and $p > 2$, there exists a compact operator Y satisfying:*

(i) *$A+Y$ is normal and diagonal;*

(ii) $PYP = 0$;
(iii) $\|Y\|_p < \varepsilon$.

For a selfadjoint operator A, the approximation described in the theorem was shown in [2]. The case of any normal operator A is more complicated. As a preparation, we need the construction of a perturbation Y such that $A+Y$ is a normal operator with $(A+Y)T = \lambda_0 T$ for an orthogonal projection T of an arbitrary finite dimension and $\|Y\|_p < \delta$ for any given $\delta > 0$, $p > 1$, cf. Lemma 2.1. Then we can obtain the following result which seems to be quite interesting.

1.2. THEOREM. *For any normal bounded operator A in a separable infinite-dimensional Hilbert space H and for a finite-dimensional orthogonal projection P, $\varepsilon > 0$ and $p > 2$, there exist an operator X and a finite-dimensional orthogonal projection T, such that:*

(i) $PXP = 0$,
(ii) $T^\perp P = 0$,
(iii) $\|X\|_p < \varepsilon/2$,
(iv) $T(A+X) = (A+X)T$.

Then our main result 1.1 can be obtained by using the following (partial case) of the Voiculescu theorem.

1.3. THEOREM. [4], [5]. *For any normal operator A, for $p \geq 2$ and $\varepsilon > 0$, there exists a perturbation Y such that:*

(i) *$A+Y$ is normal and diagonal,*
(ii) $\|Y\|_p < \varepsilon$.

The present form of Theorem 1.1 is not satisfactory mostly because we have not obtained the equality $YP = 0$ (but only $PYP = 0$). We also do not know if the Schatten norm $\|\cdot\|_p$ for $p = 2$ can be used in Theorem 1.2 and in the main result Theorem 1.1.

The problem of approximating of one normal operator is obviously a special case of the approximation of a system of commuting selfadjoint operators $A_1, \dots, A_n$, cf. Voiculescu papers [4], [5]. Anyway, the main difficulties can be explained for $n = 2$ and we discuss just this case in our paper.

2. Proof of main results

2.1. LEMMA. *Let A be a normal operator. Assume that $T_1, T_2, \dots$ are orthogonal projections of the same finite dimension* $\dim T_k = n$, *mutually orthogonal and satisfying*

$$\|AT_k\| \leq \frac{\delta}{2^k}, \quad \|A^*T_k\| \leq \frac{\delta}{2^k}, \quad k = 1, 2, \dots, \quad \textit{for some } \delta > 0.$$

Then there exists an operator Y *satisfying*:

$$A + Y \quad \text{is normal}, \qquad (A+Y)T_1 = 0, \qquad (A+Y)^* T_1 = 0,$$

(1) $\|Y\|_p \le 4\delta n^{\frac{1}{p}}$ *for any* $p \ge 1$,

(2) $Y(SH + A^*SH)^{\perp} = 0$

for $S = \sum T_k$ *and* $K^{\perp}$ *denoting the subspace orthogonal to* K *for* $K \subset H$.

Proof. Let U_j be a partial isometry satisfying $T_j = U_j^* U_j, T_{j+1} = U_j U_j^*$. Denote

$$W = \sum_j U_j + S^{\perp} \quad \text{for} S = \sum_j T_j.$$

Then

$$\begin{aligned} W^*W &= 1, \\ WW^* &= 1 - T_1, \qquad (3) \\ WT_jW^* &= T_{j+1}. \end{aligned}$$

It is enough to take $Y = -A + WAW^*$.

Note that $A^1A^2 = A^2A^1$ implies $WA^1W^*WA^2W^* = WA^2W^*WA^1W^*$ for any operators A^1, A^2, and $W^*AW = A + Y$ is normal for any normal A. Moreover,

$$\begin{aligned} (A+Y)T_1 &= WAW^*T_1 = 0 \quad \text{by (3)}, \\ (A+Y)^*T_1 &= WA^*W^*T_1 = 0. \end{aligned}$$

We shall prove (2). Observe that, for $x, y \in H$, $x \perp SA^*Sy$ and $x \perp A^*Sy$ imply $x \perp S^{\perp}A^*Sy$. Thus $x \perp SH$ and $x \perp A^*SH$ imply $x \perp S^{\perp}A^*SH$. Assume that $x \in H$ is any vector orthogonal to SH and to A^*SH; then we have

$$\begin{aligned} &x \perp S^{\perp}WA^*SH && (\text{as } S^{\perp}W = S^{\perp}), \\ &x \perp S^{\perp}WA^*W^*SH && (\text{as } W^*SH = SW^*H \subset SH), \\ &x \perp WA^*W^*SH && (\text{as } x \perp SWA^*W^*SH). \end{aligned}$$

Thus, for $x \perp SH$ and $x \perp A^*SH$, we have

$$\begin{aligned} (A^*S)^*x &= 0, \\ (WA^*W^*S)^*x &= 0, \\ S(-A + WAW^*)x &= 0, \\ (-A + WAW^*)x &= S^{\perp}(-A + WAW^*)x \\ &= (-S^{\perp}A + S^{\perp}AW^*)x = 0 \quad (\text{as } W^*x = x). \end{aligned}$$

Equality (2) is proved.

To prove inequality (1), let us notice that

$$\|AT_k\|_p^p \le \left(\frac{\delta}{2^k}\right)^p, \quad \|T_kA\|_p^p = \|A^*T_k\|_p^p \le n\left(\frac{\delta}{2^k}\right)^p$$

and

$$\|WAW^*T_k\|_p^p, \|T_kWAW^*\|_p^p \le n\left(\frac{\delta}{2^{k-1}}\right)^p \quad (\text{as } W^*T_k = T_{k-1}W^*).$$

Moreover

$$Y = -A + S^\perp AS^\perp + WAW^* - S^\perp WAW^*S^\perp \quad (\text{as } S^\perp W = WS^\perp = S^\perp).$$

In consequence,

$$Y = -\left(\sum AT_k + \sum T_kAS^\perp\right) + \sum WAW^*T_k + \sum T_kWAW^*S^\perp$$

and

$$\|Y\|_p \le \sum_{k=1}^{\infty}(\|AT_k\|_p + \|T_kA\|_p) + \sum_{k=2}^{\infty}(\|WAW^*T_k\|_p + \|T_kWAW^*\|_p) \le 4n^{1/p}\delta.$$

2.2. Remark. *Let P,T be orthogonal projections satisfying $P \le T$, $5\dim P \le \dim T$. For any operator A satisfying $A = PAP$, there exists a normal operator satisfying*

$$PFP = A, \quad TF + FT = T, \quad \|F\| \le 4\|A\|.$$

Proof. The construction of F can be done in a number of ways. We suggest the use of the Naymark dilation theorem in a Hilbert space $K = PH$. We can assume that operators A^η, $\eta = 1, 2$, where

$$A^1 = \frac{1}{2}(A + A^*), \quad A^2 = \frac{1}{2i}(A - A^*),$$

act in K, and

$$\|A^\eta\| \le \|A\|, \quad \eta = 1,2,$$
$$A^1 = C_1 - C_2,$$
$$A^2 = C_3 - C_4$$

for some $C_i \ge 0$, $\|C_i\| \le \|A\|$, $i = 1,2,3,4$. Thus $C_5 \ge 0$ for $C_5 = 4\|A\|1_K - (C_1 + C_1 + C_3 + C_4)$. Let a dilation $\boldsymbol{K}$ of K and projections $P_1, \ldots, P_5$ be given by the Naymark theorem in such a way that $C_i = 4\|A\|\widehat{K}P_i/K$ where $\widehat{K}$ is an orthogonal projection in $\boldsymbol{K}$ onto the subspace K. Now, it is enough to identify the space $\boldsymbol{K}$ with the subspace $PH + TH$ and to denote

$$F = 4\|A\|(P_1 - P_2 + i(P_3 - P_4)).$$

2.3. LEMMA. *Let H be a separable infinite-dimensional Hilbert space, and let B be a normal operator with the spectral representation*

$$B = \int_{u(\alpha,\beta,\varepsilon)} \lambda \, dE(\lambda), \tag{4}$$

$$U(\alpha,\beta,\varepsilon) = \{z \in C;\ \operatorname{Re} z \in [\alpha, \alpha+\varepsilon],\ \operatorname{Im} z \in [\beta, \beta+\varepsilon]\}$$

for some $\alpha, \beta \in R$, $\varepsilon > 0$. For any n-dimensional orthogonal projection P and for $\delta > 0$, there exist operators $Z, Y \in B(H)$ and an orthogonal projection T, satisfying:

(i) $\dim Z \le 7n$,
(ii) $\|Z\| = 7\sqrt{2\varepsilon}$,
(iii) $\|Y\|_p \le \delta$, *for any* $p \ge 1$,
(iv) $P(Z+Y)P = 0$,
(v) $B + Y + Z$ *is normal,*
(vi) $\dim T \le 6n$,
(vii) $T^{\perp} P = 0$,
(viii) $(B+Y+Z)T = T(B+Y+Z)$.

Proof. One can find a decreasing subset U_j in $U(\alpha,\beta,\varepsilon)$ satisfying

$$\operatorname{diam} U_j < \frac{\delta}{4n^{1/p}2^j}, \qquad \lambda_0 \in \bigcap U_j$$

for some λ_0, such that $E(U_j)$ are infinite-dimensional projections.

By induction, some mutually orthogonal projections $T_j \le E(U_j)$ can be taken in such a way that $\dim T_j = 6 \dim P$ and

$$\begin{aligned} &T_j P = 0 \quad \text{for } j = 1, \ldots, i, \\ &T_j B P = T_j B^* P = 0 \quad \text{for } j = 1, \ldots, i. \end{aligned} \tag{5}$$

We shall examine an operator $A = B - \lambda_0 1_H$. Obviously, A is normal and

$$\|AT_j\| \le \frac{\delta}{4n^{\frac{1}{p}}2^j}, \qquad \|A^*T_j\| \le \frac{\delta}{4n^{\frac{1}{p}}2^j}.$$

Now, we can use lemma 2.1 for the operator A and the projections $T_1, T_2, \ldots$. We obtain an operator Y such that $B + Y$ is normal,

$$(A+Y)T_1 = 0, \qquad (A+Y)^*T_1 = 0,$$
$$\|Y\|_p \le \delta, \qquad \text{for any } p \ge 1,$$
$$YP = 0 \quad \text{(in virtue of (5), (2))}.$$

Let $T_1 = T' + T''$ for orthogonal projections T', T''; $\dim T' = 5 \dim P$, $\dim T'' = \dim P$. By Remark 2.2

$$PAP = PFP \tag{6}$$

for some normal operator F satisfying $T'F = FT'$, $\|F\| \le 4\|A\|$.

Now, we are in a position to describe a perturbed operator $A + Y + Z$. Denote by V a partial isometry satisfying

$$V^*V = P, \qquad VV^* = T'',$$

and take $U = V + (P + T_1)^{\perp}$. It is sufficient to define $A + Y + Z = U(A + Y)U^* + F$ and $T = P + T'$.

Conditions (vi), (vii) are obvious. Properties (v), (viii) of an operator $B + Y + Z = A + Y + Z + \lambda_0 1_H$ can also be easily verified. Inequality (iii) has already been proved. To obtain (iv), observe that

$$\begin{aligned} Y + Z &= U(A + Y)U^* + F - A, \\ PU(A + Y)U^*P &= 0 \quad (\text{as } U^*P = 0), \\ PFP &= PAP \quad \text{by (6)}. \end{aligned}$$

Conditions (i), (ii) can be obtained by using of a rather long explicit formula for Z:

$$\begin{aligned} Z &= U(A + Y)U^* + F - A - Y \\ &= F + [V + (P + T_1)^{\perp}](A + Y)[V^* + (P + T_1)^{\perp}] \\ &\quad - T_1(A + Y)T_1 - P_1^{\perp}(A + Y)T_1^{\perp} \\ &= F + V(A + Y)V^* + (P + T_1)^{\perp}(A + Y)(P + T_1)^{\perp} - T_1^{\perp}(A + Y)T_1^{\perp} \\ &= F + VAV^* + (P + T_1)^{\perp}(A + Y)(P + T_1)^{\perp} \\ &\quad - [(P + T_1)^{\perp} + P](A + Y)[(P + T_1)^{\perp} + P] \\ &= F + VAV^* - (P + T_1)^{\perp}(A + Y)P - P(A + Y)(P + T_1)^{\perp} - P(A + Y)P \\ &= F + VAV^* - P^{\perp}AP - PAP^{\perp} - PAP \\ &= F + VAV^* - P^{\perp}AP - PA. \end{aligned}$$

Thus

$$\begin{aligned} \|Z\| &\le \|F\| + 3\|A\| \le 4\|A\| + 3\|A\| \le 7\sqrt{2}\varepsilon, \\ \dim Z &\le \dim F + 3\dim P \le 7n. \end{aligned}$$

2.4. Proof of 1.2. Without loss of generality one may assume that

$$A = \int_D \lambda dE(\lambda), \qquad D = \{\lambda \in C,\ \operatorname{Re}\lambda,\ \operatorname{Im}\lambda \in \langle -1, 1\rangle\}, \tag{7}$$

D is a union of sets $U(\alpha_i, \beta_i, (2M)^{-1})$, $i = 1, \ldots, 16M^2$, according to notation (4).

Lemma 2.3 can be used for an operator $B_i = AE(U(\alpha_i, \beta_i, (2M)^{-1}))$ acting in the space $H_i = E(U(\alpha_i, \beta_i, (2M)^{-1}))H$ and for an orthogonal projection P_i on a subspace $E(U(\alpha_i, \beta_i, (2M)^{-1}))PH$. By Lemma 2.3, there exist operators Z_i, Y_i in H_i such that

$$P_i(Z_i + Y_i)P_i = 0,$$
$$B_i + Z_i + Y_i \quad \text{is normal,}$$
$$\dim Z_i \le 7n,$$
$$\|Z_i\| \le 7\sqrt{2}(2M)^{-1},$$
$$\|Y_i\|_p \le \frac{\varepsilon}{32M^2} \quad \text{for any } p \ge 1.$$

It suffices to take $X = \sum_{i=1}^{16M^2}(Z_i + Y_i)$ according to the representation $H = \oplus_{i=1}^{16M^2} H_i$. Indeed,

$$\|X\|_p \le \| \oplus Z_i\|_p + \| \oplus Y_i\|_p,$$
$$\| \oplus Y_i\|_p \le 16M^2 \cdot \frac{\varepsilon}{32M^2} < \frac{\varepsilon}{2},$$
$$\| \oplus Z_i\|_p^p = \sum_{i=1}^{16M^2} \|Z_i\|_p^p \le 16M^2\|Z_i\|^p 7n \le 16M^2(7\sqrt{2})^p(2M)^{-p}7n \le \frac{\varepsilon}{2}$$

for M large enough whenever $p > 2$.

2.5. Proof of 1.1. By the Voiculescu theorem, see 1.3, it is enough to take a compact operator X and a finite-dimensional projection T, given in theorem 1.2. This means that

$$PXP = 0;$$
$$T^{\perp}P = 0;$$
$$\|Y\|_p < \frac{\varepsilon}{2};$$
$$A + X \quad \text{commutes with } T.$$

Thus $T(A+X)T$ and $T^{\perp}(A+X)T^{\perp}$ are normal. Then the Voiculescu theorem can be used for the operator $T^{\perp}(A+X)T^{\perp}$ in the space $T^{\perp}H$.

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