DEMONSTRATIO MATHEMATICA
Vol. XXX No 4 1997

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THE KER-COKER SEQUENCE IN BCI-ALGEBRAS

I. Introduction

In 1966, K. Iséki introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. We recall that an algebra $(X;*,0)$ of type $(2,0)$ is said to be a *BCI-algebra* if it satisfies

(I) $((x*y)*(x*z))*(z*y)=0$;
(II) $(x*(x*y))*y=0$;
(III) $x*x=0$;
(IV) $x*y=0$ and $y*x=0$ imply $x=y$.

W. A. Dudek ([2]) defined the concept of a medial BCI-algebra and studied various properties of it. A BCI-algebra X is said to be *medial* if $(x*y)*(z*u)=(x*z)*(y*u)$ for any $x,y,z,u\in X$. A BCI-algebra X is said to be *p-semisimple* if its BCK-part $M=\{x\in X \mid 0*x=0\}=\{0\}$. C. S. Hoo ([4]) proved that a BCI-algebra X is medial if and only if it is p-semisimple. W. A. Dudek ([3]) showed that p-semisimple BCI-algebras are precisely medial quasigroups completely described via abelian groups. This means that any discussions on p-semisimple BCI-algebras can be derived easily from group theory ([3]). C. Z. Mu and W. H. Xiong ([7,8]) and Y. Liu ([6]) studied some properties of an exact sequence in BCI-algebras. In this paper we obtain some interesting properties of the Ker-Coker sequence in BCI-algebras which is an exact analogue of the Snake Lemma in commutative algebras ([9]). It is well known that any ideal is a subalgebra in BCK-algebras, while it fails in BCI-algebras ([8]). We refer definitions and properties mainly to [1, 7, 8].

1991 *Mathematics Subject Classification*: 03F35.

Key words and phrases: BCI-algebras, p-semisimple, regular, exact.

The paper has been partially supported by grant G-224/94 of the Slovak Academy of Sciences, Slovakia.

Let $(X; *, 0)$ be a BCI-algebra and let I be a subset of X with $0 \in I$. Then I is called an *ideal* of X if $x * y \in I$ and $y \in I$ imply $x \in I$ for any x, y in X.

DEFINITION 1.1. Let X and Y be BCI-algebras. A BCI-homomorphism $f : X \to Y$ is said to be *regular* if $\operatorname{Im} f$ is an ideal of Y.

By the definition, for any subalgebra A of X, A is a regular ideal if and only if the inclusion $\iota : A \to X$ is a regular homomorphism.

DEFINITION 1.2. An ideal I of a BCI-algebra X is said to be *closed* if $0 * x \in I$ for every $x \in I$.

In BCI-algebra, $\{0\}$ and X itself are clearly regular ideals and closed ideals.

DEFINITION 1.3. Let $A_0, A_1, \ldots, A_{n+1}$ be BCI-algebras and let $f_i : A_i \to A_{i+1}$ be a BCI-homomorphism for any $i = 1, \ldots, n$ $(n \geq 1)$. The sequence $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} A_n \xrightarrow{f_{n+1}} A_{n+1}$ is *exact* at $A_1, \ldots, A_n$ if $\operatorname{Ker} f_{i+1} = \operatorname{Im} f_i$ for $i = 1, \ldots n$. If $f_1, \ldots f_{n+1}$ are known then $A_0 \to A_1 \to \ldots \to A_{n+1}$.

EXAMPLES 1.4. (a) $0 \to A \xrightarrow{f} B$ is exact (at A) if and only if f is injective.

(b) $A \xrightarrow{f} B \to 0$ is exact (at A) if and only if f is surjective.

(c) The sequence $0 \to A' \xrightarrow{\mu} A \xrightarrow{\varepsilon} A'' \to 0$ is exact (at A', A, A'') if and only if μ induces an isomorphism $A' \xrightarrow{\sim} \mu A'$ and ε induces an isomorphism $A / \operatorname{Ker} \varepsilon = A/\mu A' \xrightarrow{\sim} A''$. Essentially A' is then a regular ideal of A and A'', the corresponding quotient algebra. Such an exact sequence is called *short exact*.

THEOREM 1.5. *Let $f : A \to B$ and $g : B \to C$ be two BCI-homomorphisms. Then $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ is an exact sequence if and only if*

(i) $gf = 0$,

(ii) *if there is a BCI-homomorphism $h : X \to B$ with $gh = 0$ then there exists a unique BCI-homomorphism $\sigma : X \to A$ such that $h = f\sigma$.*

THEOREM 1.6. *Let $f : A \to B$ and $g : B \to C$ be two regular BCI-homomorphisms. Then $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an exact sequence if and only if*

(i) $gf = 0$,

(ii) *if there is a BCI-homomorphism $h : B \to Y$ with $hf = 0$ then there exists a unique BCI-homomorphism $\tau : C \to Y$ such that $h = \tau g$.*

Using Theorem 1.5 and Theorem 1.6 we obtain the following useful corollary:

COROLLARY 1.7. *Let $f : A \to B$ be a regular homomorphism of BCI-algebras. Then each of the following sequences is exact:*

(i) $0 \to \operatorname{Ker} f \to \operatorname{Coim} f (= A/\operatorname{Ker} f) \to 0$,
(ii) $0 \to \operatorname{Im} f \to B \to \operatorname{Coker} f (= B/\operatorname{Im} f) \to 0$,
(iii) $0 \to \operatorname{Ker} f \to A \to B \to \operatorname{Coker} f \to 0$.

II. Main results

In this section, we study the Ker-Coker sequence in BCI-algebras and obtain some properties of BCI-algebras. We use the following useful lemma and omit its proof.

LEMMA 2.1. *Let $f : A \to B$ be a BCI-homomorphism. Then f is a monomorphism if and only if* $\operatorname{Ker} f = \{0\}$.

We note that, given a commutative diagram of BCI-algebras and BCI-homomorphisms such that ϕ, ψ are regular homomorphisms:

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & & \downarrow \\ A' & \xrightarrow{\psi} & B' \end{array}$$

there exist unique BCI-homomorphisms $\operatorname{Ker}\phi \to \operatorname{Ker}\psi$ and $\operatorname{Coker}\phi \to \operatorname{Coker}\psi$, which make the enlarged configuration:

$$\begin{array}{ccccccc} \operatorname{Ker}\phi & \longrightarrow & A & \xrightarrow{\phi} & B & \longrightarrow & \operatorname{Coker}\phi \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \operatorname{Ker}\psi & \longrightarrow & A' & \xrightarrow{\psi} & B' & \longrightarrow & \operatorname{Coker}\psi \end{array}$$

commutative.

LEMMA 2.2. *Let*

$$\begin{array}{ccccc} A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow \\ A' & \xrightarrow{\phi'} & B' & \xrightarrow{\psi'} & C' \end{array}$$

be a commutative diagram of BCI-algebras and BCI-homomorphisms such that each row is exact, B' is p-semisimple, and f, g, h are regular homomorphisms. If $\phi' : A' \to B'$ is monic, then the induced sequence $\operatorname{Ker} f \to \operatorname{Ker} g \to \operatorname{Ker} h$ *is exact. On the other hand, if $\psi : B \to C$ is epic, then the resulting sequence* $\operatorname{Coker} f \to \operatorname{Coker} g \to \operatorname{Coker} h$ *is exact.*

Proof. Assume that $\phi' : A' \to B'$ is monic. Obviously, the product of $\operatorname{Ker} f \to \operatorname{Ker} g$ and $\operatorname{Ker} g \to \operatorname{Ker} h$ is null, since $f(0) = 0'$, i.e., $\operatorname{Ker} f \to$

$\operatorname{Ker} g \to \operatorname{Ker} h$ is a zero map. Hence $\operatorname{Im}(\operatorname{Ker} f \to \operatorname{Ker} g) \subseteq \operatorname{Ker}(\operatorname{Ker} g \to \operatorname{Ker} h)$.

We show that

$$\operatorname{Ker}(\operatorname{Ker} g \to \operatorname{Ker} h) \subseteq \operatorname{Im}(\operatorname{Ker} f \to \operatorname{Ker} g).$$

Let b in $\operatorname{Ker} g$ become zero in $\operatorname{Ker} h$. Since $\operatorname{Im}\phi = \operatorname{Ker}\psi$, b is the image with respect to $\phi : A \to B$ of some element $a \in A$, i.e. $\phi(a) = b$ for some $a \in A$. By commutativity

$$\phi' f(a) = g\phi(a) = g(b) = 0$$

and therefore $f(a) = 0$, i.e. $a \in \operatorname{Ker} f$, because $\phi' : A' \to B'$ is monic. This proves that $\operatorname{Ker} f \to \operatorname{Ker} g \to \operatorname{Ker} h$ is exact.

Also, it is clear that the product of $\operatorname{Coker} f \to \operatorname{Coker} g$ and $\operatorname{Coker} g \to \operatorname{Coker} h$ is null. Hence $\operatorname{Im}\{\operatorname{Coker} f \to \operatorname{Coker} g\} \subseteq \operatorname{Ker}\{\operatorname{Coker} g \to \operatorname{Coker} h\}$.

It is enough to show that

$$\operatorname{Ker}\{\operatorname{Coker} g \to \operatorname{Coker} h\} \subseteq \operatorname{Im}\{\operatorname{Coker} f \to \operatorname{Coker} g\}.$$

Now assume that ψ is epic. Suppose $b' \in B'$ and let $[b']$ denote its image in $\operatorname{Coker} g$. If $[b']$ maps into zero in $\operatorname{Coker} h$, then $\psi'(b') = h(c)$ for some $c \in C$, and hence $\psi(b) = c$ for some suitable $b \in B$, because ψ is epic. So, $\psi'(b') = h(c) = h\psi(b) = \psi' g(b)$. Thus

$$\psi'(b' * g(b)) = 0$$

and $b' * g(b) \in \operatorname{Ker}\psi' = \operatorname{Im}\phi'$ by exactness. Hence $b' * g(b) = \phi'(a')$, for some $a' \in A'$

Consider $\phi'(a') * (0 * g(b)) \in B'$. Since B' is p-semisimple,

$$\begin{aligned}\phi'(a') * (0 * g(b)) &= (b' * g(b)) * (0 * g(b)) \\ &= (b' * 0) * (g(b) * g(b)) = b' * 0 = b'.\end{aligned}$$

Hence $b' = \phi'(a') * (0 * g(b))$ and $[b'] = [\phi'(a')]$ is the image of the element $[a']$ of $\operatorname{Coker} f$, corresponding to a'. This shows that

$$\operatorname{Ker}\{\operatorname{Coker} g \to \operatorname{Coker} h\} \subseteq \operatorname{Im}\{\operatorname{Coker} f \to \operatorname{Coker} g\}. \ \blacksquare$$

THEOREM 2.3. *Suppose that the diagram of BCI-algebras and BCI-homomorphisms*:

$$\begin{array}{ccccccc} & & A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \\ 0 & \longrightarrow & A' & \xrightarrow{\phi'} & B' & \xrightarrow{\psi'} & C' & & \end{array}$$

is commutative and has exact rows such that B' is p-semisimple, where f, g, h are regular homomorphisms. Then this diagram can be extended to a diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{Ker} f & \xrightarrow{\phi_1} & \operatorname{Ker} g & \xrightarrow{\psi_1} & \operatorname{Ker} h & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A & \xrightarrow{\phi} & B & \xrightarrow{\psi} & C & \longrightarrow & 0 \\
 & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \\
0 & \longrightarrow & A' & \xrightarrow{\phi'} & B' & \xrightarrow{\psi'} & C' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{Coker} f & \xrightarrow{\phi_2} & \operatorname{Coker} g & \xrightarrow{\psi_2} & \operatorname{Coker} h & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

which is also commutative and has exact rows and columns.

Also, there is a 'connecting BCI-homomorphism' $\Delta : \operatorname{Ker} h \to \operatorname{Coker} f$ *such that*

$$\operatorname{Ker} f \xrightarrow{\phi_1} \operatorname{Ker} g \xrightarrow{\psi_1} \operatorname{Ker} h \xrightarrow{\Delta} \operatorname{Coker} f \xrightarrow{\phi_2} \operatorname{Coker} g \xrightarrow{\psi_2} \operatorname{Coker} h$$

is exact.

Proof. We define a BCI-homomorphism $\Delta : \operatorname{Ker} h \to \operatorname{Coker} f$ as follows: Let $c \in \operatorname{Ker} h \subseteq C$. Since ψ is an epimorphism, $\psi(b) = c$ for some element $b \in B$ and then $\psi' g(b) = h\psi(b) = h(c) = 0$. Hence $g(b) \in \operatorname{Ker} \psi' = \operatorname{Im} \phi'$ and $g(b) = \phi'(a')$ for some $a' \in A'$ and a' itself has a natural image, say $[a']$, in $\operatorname{Coker} f$. The mapping Δ can be now defined by $\Delta(c) = [a']$. In this construction, the element b is not unique. However, if we change it then a' has to be replaced it by an element of the form $a' * f(a)$, where $a \in A$. This does not alter $[a']$. Thus Δ is well defined and due to the above observation, it is easily seen to be a BCI-homomorphism.

By Lemma 2.2, it is enough to show that both

$$\operatorname{Ker} g \to \operatorname{Ker} h \xrightarrow{\Delta} \operatorname{Coker} f \qquad \text{and} \qquad \operatorname{Ker} h \xrightarrow{\Delta} \operatorname{Coker} f \to \operatorname{Coker} g$$

are exact. Let $b \in \operatorname{Ker} g$ and consider $\Delta(\psi(b))$. Since $g(b) = 0 = \phi'(0)$, it follows that $\Delta(\psi(b))$ is the image of zero in $\operatorname{Coker} f$. Thus $\Delta(\psi(b)) = 0$ and therefore the result of combining $\operatorname{Ker} g \to \operatorname{Ker} h$ with Δ is null. Hence

$$\operatorname{Im}\{\operatorname{Ker} g \to \operatorname{Ker} h\} \subseteq \operatorname{Ker}\{\operatorname{Ker} h \xrightarrow{\Delta} \operatorname{Coker} f\}.$$

Now suppose that $c \in \operatorname{Ker} h$ and $\Delta(c) = 0$. Since ψ is epic, there exists $b \in B$ such that $\psi(b) = c$ and $h(c) = 0$. Let b and a' be defined as in the construction of $\Delta(c)$. Then $a' = f(a)$ for some $a \in A$. It follows that $g(b) = \phi'(a') = \phi' f(a) = g\phi(a) = 0$ and therefore $b * \phi(a) \in \operatorname{Ker} g$. Thus $c = \psi(b) = \psi(b * \phi(a)) \in \psi(\operatorname{Ker} g)$. This proves that

$$\operatorname{Ker} g \to \operatorname{Ker} h \xrightarrow{\Delta} \operatorname{Coker} f$$

is exact.

It remains to show that

$$\operatorname{Ker} h \xrightarrow{\Delta} \operatorname{Coker} f \to \operatorname{Coker} g.$$

Suppose therefore that $c \in \operatorname{Ker} h$. Since ψ is epic, there exists $b \in B$ such that $\psi(b) = c$ and then $\psi' g(b) = h\psi(b) = h(c) = 0$. Hence $g(b)$ is in $\operatorname{Ker} \psi' = \operatorname{Im} \phi'$, i.e., $g(b) = \phi'(a')$ for some a' in A'. By the definition of the connecting BCI-homomorphism, $\Delta(c) = [a']$, where the notation is the same as in the construction of $\Delta(c)$.

But $[\phi'(a')] = [g(b)] = 0$ and therefore c maps into zero under the composite homomorphism

$$\operatorname{Ker} h \xrightarrow{\Delta} \operatorname{Coker} f \to \operatorname{Coker} g.$$

Hence $\operatorname{Im}\{\operatorname{Ker} h \xrightarrow{\Delta} \operatorname{Coker} f\} \subseteq \operatorname{Ker}\{\operatorname{Coker} f \to \operatorname{Coker} g\}$.

Suppose next that $[a']$ maps into zero under $\operatorname{Coker} f \to \operatorname{Coker} g$. Thus $\phi'(a') = g(b)$ for some b in B and $h\psi(b) = \psi' g(b) = \psi'\phi'(a') = 0$. Thus $\psi(b) \in \operatorname{Ker} h$ and $\Delta\psi(b) = [a']$. It follows that $[a']$ is in $\operatorname{Im} \Delta$. ∎

PROPOSITION 2.4 ([2]). *Let $f : X \to Y$ be a homomorphism of BCI-algebras, where Y is p-semisimple. If I is a closed ideal of X, then $f(I)$ is a closed ideal of Y.*

LEMMA 2.5. *Let $f : X \to Y$ be a homomorphism of BCI-algebras where Y is p-semisimple. Then f is a regular BCI-homomorphism.*

Proof. Since X itself is a closed ideal of a BCI-algebra X, $f(X)$ is a closed ideal of Y by Proposition 2.4. Hence $f(X)$ is an ideal of Y, i.e. $\operatorname{Im} f$ is an ideal of Y. Thus f is a regular BCI-homomorphism.

PROPOSITION 2.6. *Let $f : A \to B$ and $g : B \to C$ be regular BCI-homomorphisms of BCI-algebras, where B, C are p-semisimple. Then there exists an exact sequence:*

$$0 \to \operatorname{Ker} f \to \operatorname{Ker}(gf) \to \operatorname{Ker} g \to \operatorname{Coker} f \to \operatorname{Coker}(gf) \to \operatorname{Coker} g \to 0.$$

Proof. We claim that $B \oplus C$ is a p-semisimple BCI-algebra. Indeed, suppose $(0,0) * (b,c) = (0,0)$. Then $(0 * b, 0 * c) = (0,0)$ and so $0 * b = 0$ and $0 * c = 0$. Hence $b = 0$ and $c = 0$, since B, C are p-semisimple. Define a map $h : A \oplus B \to B \oplus C$ by $h(a,b) = (f(a) * b, g(b))$, and define maps $\phi : A \to A + B$ via $\phi(a) = (a, 0), a \in A$ and $\psi : A \oplus B \to B$ via $\psi(a,b) = b,\ a \in A, b \in B$.

By a similar way we define maps $\phi' : B \to B \oplus C$ and $\psi' B \oplus C \to C$. Then h is a BCI-homomorphism, since

$$\begin{aligned} h(a * a', b * b') &= (f(a * a') * (b * b'), g(b * b')) \\ &= ((f(a) * f(a')) * (b * b'), g(b) * g(b')) \\ &= ((f(a) * b) * (f(a') * b'), g(b) * g(b')) \\ &= (f(a) * b, g(b)) * (f(a') * b', g(b')) = h(a,b) * h(a',b'). \end{aligned}$$

Also, by Lemma 2.5, h is a regular homomorphism, since $B \oplus C$ is a p-semisimple BCI-algebra. Then

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\phi} & A \oplus B & \xrightarrow{\psi} & B & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow h & & \downarrow g & & \\ 0 & \longrightarrow & B & \xrightarrow{\phi'} & B \oplus C & \xrightarrow{\psi'} & C & \longrightarrow & 0 \end{array}$$

is a commutative diagram with exact rows, where the horizontal mappings are regular BCI-homomorphisms. By Theorem 2.3, we have an exact sequence

$$0 \to \operatorname{Ker} f \to \operatorname{Ker}(h) \to \operatorname{Ker} g \to \operatorname{Coker} f \to \operatorname{Coker}(h) \to \operatorname{Coker} g \to 0.$$

It is enough to show that $\operatorname{Ker} h$ and $\operatorname{Coker} h$ are isomorphic to $\operatorname{Ker} gf$ and $\operatorname{Coker} gf$, respectively. If $h(a,b) = 0$, then $g(b) = 0$ and $f(a) * b = 0$, and hence $g(f(a)*b) = gf(a)*g(b) = gf(a)*0 = gf(a)$ and $g(f(a)*b) = g(0) = 0$. Hence $gf(a) = 0$ and consequently $a \in \operatorname{Ker}(gf)$. We can therefore define a BCI-homomorphism

$$\phi : \operatorname{Ker} h \to \operatorname{Ker}(gf)$$

by $\phi(a,b) = a$. Clearly, ϕ is monic. Also, if $\alpha \in \operatorname{Ker}(gf)$ then $(\alpha, f(\alpha)) \in \operatorname{Ker} h$ and i.e., $\phi(\alpha, f(\alpha)) = \alpha$. This shows that ϕ maps $\operatorname{Ker} h$ isomorphically onto $\operatorname{Ker}(gf)$.

A map $\mu : B \oplus C \to C$ defined by $\mu(b,c) = g(b) * (0 * c)$ is a BCI-homomorphism, since

$$\begin{aligned} \mu(b * b', c * c') &= g(b * b') * (0 * (c * c')) \\ &= (g(b) * g(b')) * ((0 * c) * (0 * c')) \\ &= (g(b) * (0 * c)) * (g(b') * (0 * c')) \\ &= \mu(b,c) * \mu(b',c'). \end{aligned}$$

If $(b,c) \in \operatorname{Im} h$ then there exist $\alpha \in A$ and $\beta \in B$ such that $(b,c) = (f(\alpha) * \beta, g(\beta))$ and therefore

$$\begin{aligned} \mu(b,c) = \mu(f(\alpha) * \beta, g(\beta)) &= gf(\alpha) * \beta) * (0 * g(\beta)) \\ &= (gf(\alpha) * g(\beta)) * (0 * g(\beta)) = (gf(\alpha) * 0) * (g(\beta) * g(\beta)) \\ &= (gf(\alpha) * 0) * 0 = gf(\alpha) * 0 = gf(\alpha), \end{aligned}$$

since C is p-semisimple. Thus $\mu(\operatorname{Im} h) \subseteq \operatorname{Im}(gf)$. Therefore, μ induces a BCI-homomorphism ψ : $\operatorname{Coker} h \to \operatorname{Coker}(gf)$ which is epic since μ is epic.

We shall now show that ψ is also monic. Suppose that the image of (b, c) in $\operatorname{Coker} h$ is mapped into zero by ψ, i.e., $\mu(b, c) = gf(a)$ for some $a \in A$. It is sufficient to show that $(b, c) \in \operatorname{Im} h$. But $gf(a) = \mu(b, c) = g(b) * (0 * c)$, whence

$$\begin{aligned} gf(a) * g(b) &= (g(b) * (0 * c)) * g(b) \\ &= (g(b) * g(b)) * (0 * c) = 0 * (0 * c) = c, \end{aligned}$$

since C is p-semisimple. Thus $h(a, f(a)*b) = (b, c)$, since $f(a)*(f(a)*b) = b$. Consequently $(b, c) \in \operatorname{Im} h$ and this complete the proof. ∎

Acknowledgement. The authors are very indebted to the referee for his valuable suggestions which improved the present paper.

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Received October 14, 1996.