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ON SOME BOUNDED SOLUTIONS OF A NONLINEAR DIFFERENTIAL EQUATIONS

1. Introduction

In this paper we will prove an existence theorem for bounded weak and pseudo-solutions of nonlinear differential equations

$$x' = A(t)x + f(t,x) \tag{1}$$

on the real line $\mathbb{R}$.

An exponential trichotomy of A in our sense was introduced by Elaydi and Hajek [7]. This problem was also studied by many authors ([2], [5], [8], [12], [13], [15], [16], for instance).

While in all these papers the continuity of the function f or Carathéodory conditions were assumed, in our papers f is only assumed to be weakly-weakly sequentially continuous or Pettis-integrable. Some additional assumptions we imposed on f, are expressed in terms of a measure of weak noncompactness. We assume also that the linear part of our equation is trichotomic.

Throughout this paper $(E, \|\cdot\|)$ will denote a real Banach space, E^* the dual space, $(E,w) = (E, \sigma(E,E^*))$ the space E with its weak topology and $B(a,r) = \{y \in E : \|y-a\| \leq r\}$. Moreover, we introduce the following notations: $L(E)$ is the algebra of continuous linear operators from E into itself with induced standard norm $|\cdot|$; $C(\mathbb{R}, E)$ is the space of all continuous function from $\mathbb{R}$ into E , endowed with the topology of almost uniform convergence on $\mathbb{R}$.

Let $A : \mathbb{R} \mapsto L(E)$ be strongly measurable and Bochner integrable on every finit subinterval of $\mathbb{R}$.

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Consider the equation

$$x'(t) = A(t)x(t). \tag{2}$$

By $U(t)$ we denote the fundamental solution of the equation

$$U'(t) = A(t)U(t) \quad \text{with} \quad \mathrm{U}(0) = \mathrm{Id}.$$

Following Elaydi and Hajek [7] we introduce

DEFINITION 1. A linear equation (1) is said to have a trichotomy on $\mathbb{R}$ if there exist linear projections P, Q such that

$$PQ = QP, \qquad P + Q - PQ = Id \tag{3}$$

and constants $\alpha \geq 1, \sigma > 0$ such that

$$\begin{aligned} |U(t)PU^{-1}(s)| &\leq \alpha e^{-\sigma(t-s)} && \text{for } 0 \leq s \leq t, \\ |U(t)(Id - P)U^{-1}(s)| &\leq \alpha e^{-\sigma(s-t)} && \text{for } t \leq s,\ s \geq 0, \\ |U(t)QU^{-1}(s)| &\leq \alpha e^{-\sigma(s-t)} && \text{for } t \leq s \leq 0, \\ |U(t)(Id - Q)U^{-1}(s)| &\leq \alpha e^{-\sigma(t-s)} && \text{for } s \leq t, \leq 0. \end{aligned} \tag{4}$$

Define the integral kernel $G(t,s) = U(t)L(t,s)U^{-1}(s)$, where

$$L(t,s) = \begin{cases} Id - Q & \text{for } 0 < s \leq \max(t,0), \\ -Q & \text{for } \max(t,0) < s, \\ P & \text{for } s \leq \min(t,0), \\ P - Id & \text{for } \min(t,0) < s \leq 0. \end{cases}$$

Then $|G(t,s)| \leq \alpha e^{-\sigma|t-s|}$ for $t, s \in \mathbb{R}$ ([8], Lemma 7).

Now we recall the notion of the pseudo-solution, which is similar to the notion of Carathéodory-type strong solution (strong C-solution) [9]. For such solutions problem (1) is equivalent to the integral problem

$$x(t) = \int_A G(t,s)f(s,x(s))\,ds, \qquad t \in I. \tag{5}$$

Fix $x^* \in E^*$, compact set $A \in \mathbb{R}$ and consider the equation

$$(x^*x)' = x^*(A(t)x + f(t,x)), A \in R \tag{5'}$$

Now we are ready to introduce the following

DEFINITION 2. A function $x : \mathbb{R} \mapsto E$ is said to be a pseudo-solution of the equation (1) if it satisfies the following conditions:

(i) $x(\cdot)$ is absolutely continuous,

(ii) for each $x^* \in E^*$ there exists a negligible set $A(x^*)$ (i.e. $mes(A(x^*) = 0)$, such that for each $t \notin A(x^*)$

$$x^*(x'(t)) = x^*(A(t)x + f(t,x(t)).$$

Now recall the definition of De Blasi's measure of weak noncompactness:

$$\beta(A) = \inf\{\varepsilon > 0 : \text{there exists a weakly compact subset } K \text{ of } E, \text{ such that } A \subset K + \varepsilon B_1\}.$$

where A is a nonempty bounded subset of E and B_1 the closed unit ball in E.

For the properties of β the reader is refered to [6] or [14]. We will use also the following lemmas:

LEMMA 1 [14]. *Let $H \subset C(I,E)$ be a family of strongly equicontinuous function. Then*

$$\beta_c(H) = \sup_{t\in I} \beta(H(t)),$$

where $\beta_c(H)$ denote the measure of weak noncompactness in $C(I,E)$ and the function $t \mapsto \beta(H(t))$ is continuous.

LEMMA 2 [3]. *If T is a continuous mapping from a compact interval I to $L(E)$ and W is a bounded subset of E, then*

$$\beta\Big(\bigcup_{t\in I} T(t)W\Big) \leq \sup_{t\in I} |T(t)| \cdot \beta(W).$$

In the proof of the main theorem we will apply the following fixed point theorem:

THEOREM 1 [11]. *Let D be a closed convex subset of E, and let F be a weakly sequentially continuous map from D into itself. If for some $x \in D$ the implication*

(6) $\qquad \overline{V} = \overline{conv}(\{x\} \cup F(V)) \Rightarrow V$ *is relatively weakly compact,*

holds for every subset V of D, then F has a fixed point.

2. Main result

We assume that:

(A1) $A : \mathbb{R} \mapsto L(E)$ is strongly measurable and Bochner integrable on every finite subinterval of $\mathbb{R}$. Moreover suppose that the linear equation

$$(A) \qquad x'(t) = A(t)x(t)$$

has a trichotomy with constants $\alpha \geq 1$ and $\sigma > 0$.

(A2) Let $f : \mathbb{R} \times E \mapsto E$ be a function with the following properties:

(i) for each strongly absolutely continuous function $x : \mathbb{R} \mapsto E$, $f(\cdot, x(\cdot))$ is Pettis-integrable on every compact subset of $\mathbb{R}$, $f(t,\cdot)$ is weakly-weakly sequentially continuous,

(ii) there exist real nonnegative functions a and b locally integrable on $\mathbb{R}$, such that

$$\|f(t,x)\| \leq a(t) + b(t) \cdot \|x\|$$

for each $t \in \mathbb{R}$ and $x \in E$. Assume in addition that

$$(B) \qquad \sup_{t\in\mathbb{R}} \int_t^{t+1} a(s)\, ds \leq M_1,$$

$$(C) \qquad \sup_{t\in\mathbb{R}} \int_t^{t+1} b(s)\, ds \leq M_2,$$

where $0 < M_1 < \infty$ and $0 < M_2 < \frac{1-e^{-\sigma}}{2\alpha}$.

(A3) Let $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be continuous and let $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a nondecreasing function. Assume that

$$(D) \qquad \beta(f(I \times X)) \leq \sup_{t\in I} g(t) \cdot h(\beta(X))$$

for each compact subinterval I of $\mathbb{R}$ and each bounded subset X of E.

(A4) Put

$$L = \sup\Big\{ \int_{\mathbb{R}} |G(t,s)| g(s)\, ds : t \in \mathbb{R} \Big\}.$$

Assume that $0 < L < \infty$ and $L \cdot h(t) < t$ for $t > 0$.

THEOREM. *Under the above assumptions there exists at least one bounded pseudo-solution of the equation*

$$x'(t) = A(t)x + f(t,x)$$

on an arbitrary compact subset A of $\mathbb{R}$.

Proof. (Analogical as in [4]).

Let $\alpha \geq 1$ and $\sigma > 0$ be constants from Definition 1 (assumption (A1)), so $|G(t,s)| \leq \alpha e^{-\sigma|t-s|}$ for all $t, s \in \mathbb{R}$.

By H we denote the set of the form

$$H = \Big\{ x \in C(\mathbb{R}, E) : \|x(t)\| \leq K,$$

$$\|x(t) - x(\tau)\| \leq K \int_\tau^t |A(s)|\, ds + \int_\tau^t a(s)\, ds + K \int_\tau^t b(s)\, ds,\ \tau, t \in \mathbb{R} \Big\}$$

where $K = 2\alpha M_1 / (1 - e^{-\sigma} - 2\alpha M_2)$. Note that $K > 0$.

It is clear that H is nonempty, closed, bounded, almost equicontinuous and convex in $C(\mathbb{R}, E)$. For each $x \in H$ we define

$$F_A(x)(t) = \int_A G(t,s)f(s,x(s))\,ds,$$

where $\int$ denotes the Pettis integral. By the assumption (A2) $G(t,\cdot)f(\cdot,x(\cdot))$ is Pettis integrable on every compact subset of $\mathbb{R}$.

Without loss of the generality, we will assume that $A = [a,b]$ and $0 \notin A$. Fix $x^* \in E^*, \|x^*\| \le 1$. Then

$$\begin{aligned}|x^*(F_A(x)(t))| &= \left|x^*\Big(\int_a^b G(t,s)f(s,x(s))\,ds\Big)\right| \le \int_a^b |x^*G(t,s)f(s,x(s))|\,ds \\ &= \int_a^b |G^*(t,s)|\cdot|x^*f(s,x(s))|\,ds \le \int_a^b |G(t,s)|\cdot|x^*f(s,x(s))|\,ds \\ &\le \int_a^b |G(t,s)|\cdot\|f(s,x(s))\|\,ds \le \alpha e^{-\sigma|t-s|}(a(s)+Kb(s))\,ds.\end{aligned}$$

Similarly as in ([7], Lemma 5.1) one gets a constant K such that

$$\|F_A(x)(t)\| \le \frac{2\alpha(M_1+M_2K)}{1-e^{-\sigma}} = K.$$

Furthermore, since $F_A(x)$ is a solution of

$$y' = A(t)y + f(t,x(t)), \quad \text{for } \tau \le \eta \ (\tau,\eta \in A)$$

we have

$$\begin{aligned}\|F_A(x)(\eta) - F_A(x)(\tau)\| = \|y(\eta)-y(\tau)\| &\le \left\|\int_\tau^\eta y'(s)\,ds\right\| \\ &\le K\cdot\int_\tau^\eta |A(s)|\,ds + \int_\tau^\eta (a(s)+K\cdot b(s))\,ds.\end{aligned}$$

We conclude that $F_A(x) \in H$ and $F_A : H \mapsto H$.

Now we will prove the weakly-weakly sequentially continuity of F_A.

Since H is almost strongly equicontinuous, the sequence (x_n) in H converges weakly to $x \in C(\mathbb{R}, E)$ iff $x_n(t) \mapsto x(t)$ in (E,w) for each $t \in A$ (lemma 1.9 [14]).

Since $f(t,\cdot)$ is weakly-weakly sequentially continuous, then by using the Lebesgue dominated convergence theorem for the Pettis integral [10] we have

$$x^*(F_A(x_n)(t)) \mapsto x^*(F_A(x)(t))$$

for each $x^* \in E^*$ and each $t \in A$ whenever $x_n \mapsto x$ in $(C(\mathbb{R}, E),\omega)$.

Therefore F_A is weakly-weakly sequentially continuous on H.

Suppose that

$$\overline{Y} = \overline{conv}(\{x\} \cup F_A(Y))$$

for some $Y \subset H$. We will prove that Y is relatively weakly compact, thus (6) will be satisfied.

For an arbitrary $\varepsilon_1 > 0$ there exists a $\delta_1 > 0$ such that $|s_1 - s_2| < \delta_1$ with $s_1, s_2 \in \langle a, b\rangle$ implies $|G(t,s_1) - G(t,s_2)| < \varepsilon_1$ and $|g(s_1) - g(s_2)| < \varepsilon_1$. Let $a = t_0 < t_1 < t_2 < t_3 < \ldots < t_k = t < \ldots < t_{2k} = b$ be a partition of $\langle a, b\rangle$ with $t_i - t_{i-1} < \delta_1$ for each $i = 1, 2, \ldots, 2k$.

The interval $[t_{i-1}, t_i]$ will be denoted by I_i.

By the continuity of g and $G(t,\cdot)$ (except $G(t,t)$) there exist points $\tau_i, s_i \in I_i$ such that

$$|G(t,s_i)| = \sup\{|G(t,s)| : s \in I_i\},$$
$$g(\tau_i) = \sup\{g(s) : s \in I_i\}.$$

Let

$$c_1 = \sup\{|G(t,s)| : a \le s \le b\},$$
$$c_2 = \sup\{|g(s)| : a \le s \le b\}.$$

By the mean value theorem for Pettis integral [1] we get

$$(7) \qquad \Big\{ \int_a^b G(t,s) f(s, y(s))\, ds : y \in Y \Big\}$$
$$= \Big\{ \sum_{i=1}^{2k} \int_{t_{i-1}}^{t_i} G(t,s) f(s, y(s))\, ds : y \in Y \Big\}$$
$$\subset \sum_{i=1}^{2k} (t_i - t_{i-1}) \overline{conv}\Big(\bigcup_{s \in I_i} G(t,s) f(I_i \times Y(A))\Big).$$

By Lemmas 1 and 2, we have

$$(8) \qquad \beta\Big(\bigcup_{s \in I_i} G(t,s) f(I_i \times Y(A))\Big) \le \sup_{s \in I_i} |G(t,s)| \beta(f(I_i \times Y(A))).$$

By (7), (8) and our assumptions one gets

$$\beta\Big(\Big\{ \int_a^b G(t,s) f(s, y(s)) ds : y \in Y \Big\}\Big)$$
$$\le \beta\Big(\sum_{i=1}^{2k} (t_i - t_{i-1}) \overline{conv}\big(\bigcup_{s \in I_i} G(t,s) f(I_i \times Y(A))\big)\Big)$$

$$\leq \sum_{i=1}^{2k}(t_i - t_{i-1}) \sup_{s\in I_i} |G(t,s)| \cdot \beta(f(I_i \times Y(A)))$$

$$\leq \sum_{i=1}^{2k}(t_i - t_{i-1})|G(t,s_i)| \cdot g(\tau_i) \cdot h(\beta(Y(A))$$

$$= h(\beta(Y)) \cdot \sum_{i=1}^{2k}(t_i - t_{i-1})|G(t,s_i)| \cdot g(\tau_i)$$

$$\leq h(\beta(Y)) \cdot \sum_{i=1}^{2k} \int_{I_i}((|G(t,s_i) - G(t,s)| \cdot g(\tau_i)$$

$$+ |G(t,s)| \cdot |g(\tau_i) - g(s_i)| + |G(t,s)| \cdot g(s))\, ds$$

$$\leq h(\beta(Y)) \cdot \Big[(b-a)(c_1 + c_2)\varepsilon_1 + \int_a^b |G(t,s)| \cdot g(s)\, ds\Big].$$

Since ε_1 is arbitrarily small, we get

$$\beta\Big(\Big\{\int_a^b G(t,s)f(s,y(s))\,ds : y \in Y\Big\}\Big) \leq h(\beta(Y) \cdot \int_a^b |G(t,s)| \cdot g(s)\, ds.$$

Thus

$$\beta(F_A(Y)(t)) \leq L \cdot h(\beta(Y)). \tag{9}$$

Since

$$\overline{Y} = \overline{conv}(\{x\} \cup F_A(Y))$$

we have $\beta(Y(t)) \leq \beta(F_A(Y(t))$ and so, in view of (9) it follows that

$$\beta(Y(t)) \leq \beta(F_A(Y(t)) \leq L \cdot h(\beta(Y)) < \beta(Y).$$

This implies that $\beta(Y) = 0$, so Y is relatively weakly compact.

By Theorem 1, F_A has a fixed point in H, which is a pseudo-solution of our equation.

Remark 1. We don't know if the solution exists on $\mathbb{R}$ because we can't say if there exists Pettis integral on $\mathbb{R}$.

Remark 2. If f is weakly-weakly continuous then a pseudo-solution is a weak solution.

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