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A TARGET PROBLEM FOR DIFFERENTIAL INCLUSIONS WITH STATE-SPACE CONSTRAINTS

Abstract. We study the existence of solutions to a nonstandard target problem for set-valued flows generated by a vector field as a Marchaud map. The proof is performed by means of the generalized Lefschetz trace formula (see [G]) which can be reduced, under mild assumptions imposed on a constraint, to computation of the sum of local indices. An illustrating example is given in $\mathbb{R}^3$.

1. Introduction

Recently, various aspects of the target problems have been intensively studied (see e.g. [CQS1], [CQS2], [CFM], [Q], [SOGY] and the references therein). Usually, the problem is solved in the frame of the optimal control theory and consists in the characterization of the minimal-time function, i.e. the first time such that the system can reach the target and satisfies the constraints before it (see [CQS1], [Q]). The system means basicly the first-order differential inclusions determined by the Marchaud map vector field.

Here, we would like to discuss a rather nonstandard form for a system of differential inclusions

$$\theta' \in f(t,\theta), \tag{1}$$

where $\theta = (\theta_1,\dots,\theta_n)$, $\theta' = (\theta'_1,\dots,\theta'_n)^T$ and $f(t,\theta) = (f_1(t,\theta),\dots, f_n(t,\theta))^T$, posed as follows.

TARGET PROBLEM: Given two codimension one manifolds $\Sigma_1 \subset \mathbb{R}^n$, $\Sigma_2 \subset \mathbb{R}^n$ and a constraint diffeomorphism $C : \Sigma_1 \to \Sigma_2$. Does there exist a solution $\theta(t)$ of (1), where $\theta(t_1) \in \Sigma_1$, such that $C(\theta(t_1)) = \theta(t_2) \subset \Sigma_2$, for some $t_2 > t_1$?

For our convenience, a schematical sketch of the planar problem can be seen in Fig. 1,

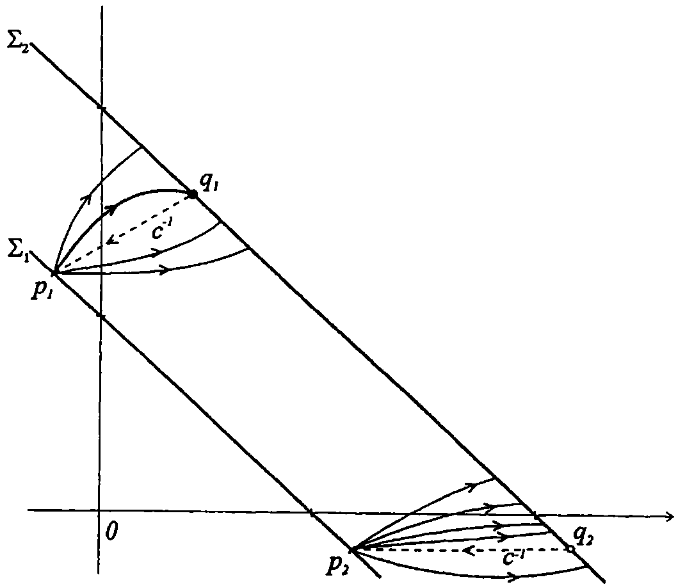

Fig. 1

where,

- the set of admissible (under a constraint C) related initial points $\ldots\{p_1, p_2, \ldots\}$,
- the target set $\ldots\{q_1, q_2, \ldots\}$,
- the admissible trajectory reaching the target (a solution of the problem) ...the bold curve.

As we will see in the following, we can give at least a partial answer to the above problem on the torus. Thus, for example, we will be only able to consider the situation in Fig. 1 mod $\sqrt{2}d$–like, where $|d|$ denotes the distance between Σ_1 and Σ_2.

2. Main (existence) result

Hence, consider (1) and assume that the set-valued map $f(t,\theta) : \mathbb{R}_0^+ \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ has maximally a linear growth in θ and is bounded in t. Assume, moreover, it is upper semi-continuous with nonempty, convex, compact values (in this case, f is called the Marchaud map). It is well-known (see e.g. [F, p. 56] and [BGP], where the appropriate definitions can also be found) that then all solutions of (1) entirely exist in the Carathéodory sense (i.e. are locally absolutely continuous and satisfy (1) a.e.) and are with R_δ-values on any compact interval.

Let, furthermore,

$$\text{(2)} \qquad \sum_{i=1}^{n} f_i(t,\theta) \geq \varepsilon > 0 \qquad \left(\text{or } \sum_{i=1}^{n} f_i(t,\theta) \leq -\varepsilon < 0\right),$$

by which

$$\lim_{t\to\infty} \sum_{i=1}^{n} \theta_i(t) = \infty \quad \left(\text{or } \lim_{t\to\infty} \sum_{i=1}^{n} \theta_i(t) = -\infty, \text{ respectively}\right).$$

Consider still the $(n-1)$-dimensional torus $\Sigma \subset T^n = \mathbb{R}^n/\mathbb{Z}_b^n$ given by

$$\text{(3)} \qquad \sum_{i=1}^{n} \theta_i = 0 \bmod b,$$

where $\mathbb{Z}_b$ denotes the set of all integer multiples of b and $b \neq 0$ is an arbitrary constant. For better understanding of this—see Fig. 2, where the definition of two-dimensional torus $\Sigma \subset T^3$ is illustrated.

Since (because of convenience) (1) will be considered on the cylinder $\mathbb{R}_0^+ \times T^n$, the natural restriction imposed on f is still

$$\text{(4)} \qquad f(t,\ldots,\theta_j + b,\ldots) \equiv f(t,\ldots,\theta_j,\ldots) \text{ for } j = 1,\ldots,n.$$

Remark 1. The particular form of f (see (2) and (4)) will play an important role in the definition of the Poincaré set-valued map (6) below. Under (2) and (4), this map has been proved in [A] to be admissible in the sense of [G].

Now, we are in position to give the main statement of our paper.

THEOREM. *Let the above assumptions be satisfied jointly with* (2) *and* (4). *Assume* $C : \Sigma \to \Sigma$ *is a diffeomorphism having finitely many, but at least one, simple fixed points,* $\gamma_1,\ldots,\gamma_r$, *on the torus* $\Sigma \subset T^n$ (*see* (3)) *and*

$$\text{(5)} \qquad \sum_{k=1}^{r} \operatorname{sgn} \det(I - dC_{\gamma_k}^{-1}) \neq 0,$$

where $dC_{\gamma_k}^{-1}$ *denotes the derivative of* C^{-1} *at* $\gamma_k \in \Sigma$.

Then, for a given constant $b > 0$ *or* $b < 0$, *respectively* (*see* (2)), *there always exists a solution* $\theta(t)$ *of* (1) *such that*

$$C(\theta(0)) = \theta(t^*) \text{ on the torus } \Sigma, \quad \text{for some } t^* > 0.$$

Proof. Take into account only the first inequality in (2); the second one can be used quite analogously.

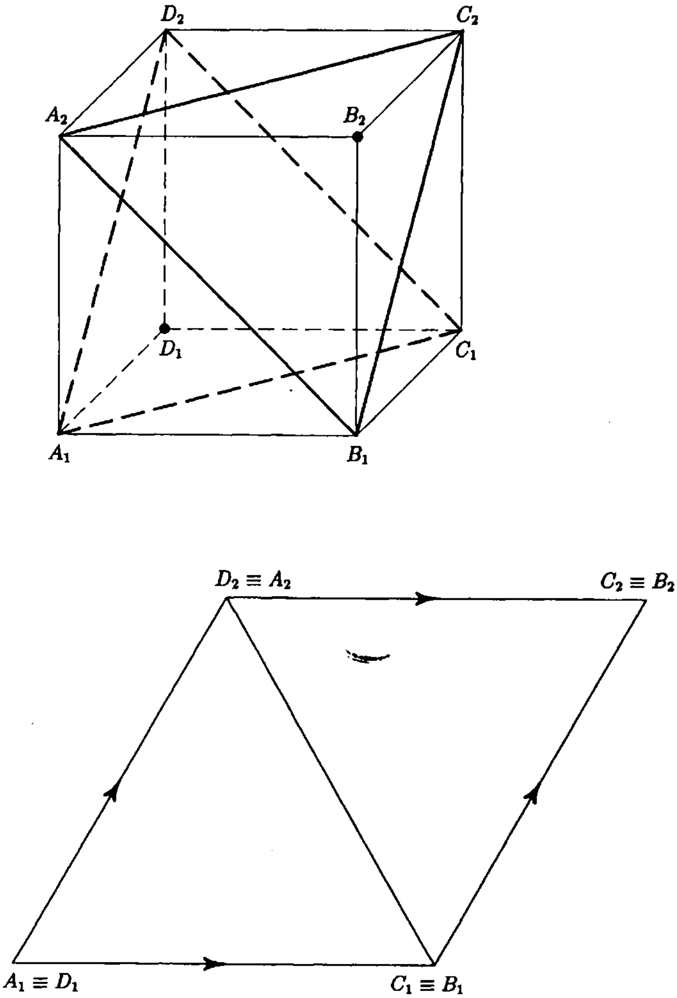

Fig. 2

Because of (2) and (4), there is a well-defined (cf. Remark 1) set-valued admissible Poincaré map Φ on Σ, namely

$$\Phi_{\{\tau(p)\}}(p) : \Sigma \rightsquigarrow \Sigma, \quad \Phi_{\{\tau(p)\}}(p) := \{\theta(\tau(p)\}, \tag{6}$$

where $\tau(p)$ denotes the least time for p to return back to Σ, when taking into account each branch of a solution $\theta(t)$ of (1) with $\Phi_0(p) = \theta(0) = p \in \Sigma$.

Our problem is obviously solvable, if we show $p \in \Sigma$ such that $C(p) \in \Phi(p)$ on Σ or, because of a diffeomorphism C, when we find a fixed-point of $C^{-1}(\Phi(p))$ on Σ.

For this purpose, we apply the "multi-valued analogy" of the Lefschetz trace formula in [G]. First of all, we show that the map $\Phi : \Sigma \rightsquigarrow \Sigma$ is homotopic, in the sense of multi-valued admissible maps, to identity I. We exhibit such homotopy as follows.

Taking

$$(\theta_1(p,s),\ldots,\theta_n(p,s)) = \theta(st(p)), \quad \text{where } 0 \le s \le 1,$$

set

$$H_s(p) = \Big\{\Big(\theta_1(p,s),\ldots,\theta_{n-1}(p,s), -\sum_{i=1}^{n-1}\theta_i(p,s)\Big)\Big\}.$$

Observe that $H_s(p)$ represents indeed the required homotopy, because for $s=0$, $H_0(p)=p$, and for $s=1$, $H_1(p)=\Phi(p)$. The crucial step in application of the generalized Lefschetz fixed-point theorem for the admissible (multi-valued) map $C^{-1}(\Phi(p))$ on a compact $(n-1)$-dimensional manifold Σ consists in verifying the inequality $\Lambda(C^{-1}(\Phi(p))) \neq \{0\}$, where $\Lambda(.)$ is the generalized Lefschetz number (for the definitions see [G]). Because of the invariance under homotopy, it is however sufficient to show that $\Lambda(C^{-1}) \neq 0$.

Since C^{-1} is, by the hypothesis, smooth with finitely many simple fixed-points, $\gamma_1,\ldots,\gamma_r$, on a compact manifold Σ, the Lefschetz number $\Lambda(C^{-1})$ can be simply calculated (see [B], [G]) as the sum of the local indices, namely

$$\Lambda(C^{-1}) = \sum_{k=1}^{r} \operatorname{sgn}\det(I - dC_{\gamma_k}^{-1}),$$

where $dC_{\gamma_k}^{-1}$ denotes the derivative of C^{-1} on the tangent space $T_{\gamma_k}\Sigma$, which coincides here with Σ.

So, because of (5), the proof is complete.

Remark 2. The time t^* to reach the target can be obviously estimated from above as $t^* \le \frac{|b|}{\varepsilon}$. A lower estimate, $t^* \ge \frac{|b|}{E}$, holds provided, additionally,

$$\sum_{i=1}^{n} f_i(t,\theta) \le E \qquad \Big[\text{or } \sum_{i=1}^{n} f_i(t,\theta) \ge -E\Big].$$

In particular, for $E = \varepsilon \equiv |\sum_{i=1}^{n} f_i(t,\theta)|$, we have the exact time, $t^* = \frac{|b|}{\varepsilon}$.

Remark 3. In the single-valued case, it is well-known (see [BBPT]) that $N(C^{-1}\Phi) = |\Lambda(C^{-1}\Phi)|$ on the torus Σ. Therefore, since the Nielsen index $N(C^{-1}\Phi)$ determines the lower estimate of fixed-points of the map $C^{-1}\Phi$ on Σ, the absolute value of the nonzero number in (5) designates at the same time the lower estimate of desired solutions. The Nielsen index has been also generalized for admissible (in the sense of [G]) set-valued maps (see [AGJ]) and the same is true.

An illustrating example in $\mathbb{R}^3$

Suppose, for simplicity, $n = 3$, and C is linear, $C(\theta) := L\theta + c$, where

$$L = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$$

is a regular matrix such that $\mathcal{L} := u_1+v_1+w_1 = u_2+v_2+w_2 = u_3+v_3+w_3$, $\mathcal{L} \in \mathbb{Z}$, and $c = (c_1, c_2, c_3)^T$ is a nonzero vector such that $c_1 + c_2 + c_3 = 0 \bmod b$. Observe that the particular forms of L and c allow us to operate on Σ, where $\theta_1 + \theta_2 + \theta_3 = 0 \bmod b$.

Because of the regularity, there exists an inverse operator, namely $C^{-1}(\theta) = L^{-1}(\theta - c)$.

$C^{-1}(\theta)$ has exactly one fixed-point γ_1 on Σ as far as $C(\theta)$ has, which is true iff

$$\begin{aligned} 0 \neq & -u_1v_2w_3 + u_1v_3w_2 + u_2v_1w_3 + u_2v_3w_1 \\ & + u_3v_1w_2 + u_3v_2w_1 + u_1v_2 + u_1w_3 - u_2v_1 \\ & - u_3w_1 + v_2w_3 - v_3w_2 - u_1 - v_2 - w_3 + 1. \end{aligned} \tag{7}$$

Then, under all the above assumptions,

$$|\Lambda(C^{-1}(\theta))| = |\text{sgn}\ \det(I - dC_{\gamma_1}^{-1}(\theta))| = |\text{sgn}\ \det(I - L^{-1})| = 1,$$

and we arrived at (5). So, Theorem can be applied, provided still (2) and (4) for a suitable Marchaud map $f(t, \theta) = (f_1(t,\theta), f_2(t,\theta), f_3(t,\theta))^T$.

In Fig. 3, a trial of shooting to a target is demonstrated for $b = 6$ and $L = 2I, c = (1, 2, 3)^T$, which evidently satisfies (7) ($\Rightarrow$ (5)). Exactly one fixed point $\gamma_1 = -c$ of C^{-1} belongs to Σ.

The targets related to the admissible initial states are indicated by the same marks. The trajectories starting at the initial states are generated here by a system of, for simplicity, differential equations

$$\begin{aligned} \theta_1' &= 10 + \sin\frac{\pi}{3}\theta_2 + \sin\frac{\pi}{3}\theta_3 + \cos 2\pi t, \\ \theta_2' &= -3 + \sin\frac{\pi}{3}\theta_1 - \sin\frac{\pi}{3}\theta_3 + 2\cos 2\pi t, \\ \theta_3' &= -1 - \sin\frac{\pi}{3}\theta_1 - \sin\frac{\pi}{3}\theta_2 - 3\cos 2\pi t. \end{aligned}$$

Since $\varepsilon = E = 6$, the time t interval is chosen as $[0, 2]$. A special attention, however, should be paid to the value $t^* = 1$, when the first hits to the target are expected. The bold trajectory, starting at the point $[9.5, -5.8, -3.7]$, hits

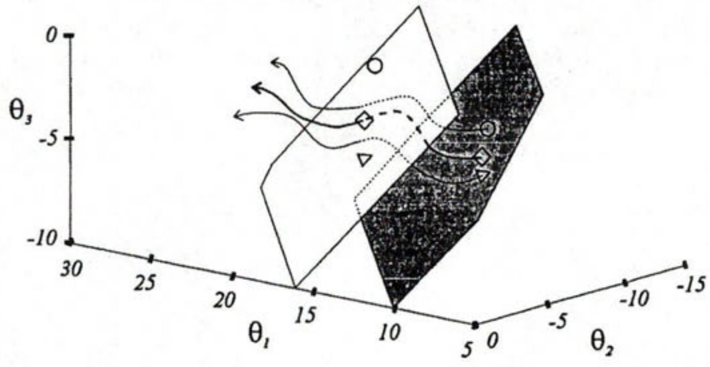

Fig. 3

the associated target $[20, -9.6, -4.4]$ approximately for one correct decimal digit.

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