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ON EXISTENCE OF SOLUTIONS OF A SINGULAR CAUCHY-NICOLETTI PROBLEM FOR A SYSTEM OF INTEGRO-DIFFERENTIAL EQUATIONS

Abstract. In the paper a singular problem of the type of Cauchy-Nicoletti for a system of integro-differential equations is considered. The existence of solutions the graph of which remains in a properly choosen domain is proved. Moreover, the theorem about uniqueness of solution in this domain is given. The applicability of results is showed on an illustrative example.

1. Introduction

Consider the following singular problem of the type of Cauchy-Nicoletti for the system of n ordinary integro-differential equations

$$(1)\qquad \begin{cases} y_k'(x) = f_k(x, y(x)) + \int\limits_a^x g_k(x, s, y(x), y(s))\, ds, \\ y_l'(x) = f_l(x, y(x)) + \int\limits_x^b g_l(x, s, y(x), y(s))\, ds, \end{cases}$$

$$k = 1, 2, \ldots, h,\ 1 \le h < n,\ l = h+1, \ldots, n,$$

$$(2)\qquad y_k(a^+) = A_k,\ y_l(b^-) = B_l,$$

where $x \in I = (a, b)$, $y(x) = (y_1(x), y_2(x), \ldots, y_n(x))$, a, b, A_k, B_l are real constants, $a < b$ and $(x, y) \in I \times D$, where the set $D \subset \mathbb{R}^n$ is indicated below.

The Cauchy-Nicoletti problem, the generalized Cauchy problems or the boundary value problems for systems of ordinary differential equations have been considered by many authors. Singular problems of such types have been studied e.g. in works [1]–[9], [11]. Singular initial problem for systems

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of integro-differential equations was considered in [10]. In this paper we give sufficient conditions for solvability and uniqueness of the problem (1), (2) and some estimations of the components of solutions. The solution of (1), (2) is defined in the sense of following definition.

DEFINITION. The solution of the problem (1), (2) is defined as a vector-function $y(x) = (y_1(x), y_2(x), \dots, y_n(x)) \in C^1(I)$ which on I satisfies the system (1), $(x, y(x)) \in D$ on I and $y_k(a^+) = A_k$, $y_l(b^-) = B_l$, $k = 1, 2, \dots, h$, $l = h+1, \dots, n$.

2. Main results

We will consider real functions $\gamma_i(x)$, $\delta_i(x)$, $\zeta_i(x,s)$, $\eta_i(x,s)$, $i = 1, 2, \dots, n$, which satisfy the following conditions (H1)–(H2):
(H1):

$$\gamma_k(x), \delta_k(x) \in C(a,b],\ \gamma_k(x) \le \delta_k(x) \quad \text{on} \quad (a,b],\ k = 1, 2, \dots, h,$$
$$\gamma_l(x), \delta_l(x) \in C[a,b),\ \gamma_l(x) \le \delta_l(x) \quad \text{on} \quad [a,b),\ l = h+1, \dots n$$

and, moreover, there are finite integrals

$$\int_a^b \gamma_i(t)dt, \quad \int_a^b \delta_i(t)dt, \quad i = 1, 2, \dots, n.$$

(H2):

$$\zeta_k(x,s), \eta_k(x,s) \in C\left((a,b] \times (a,b]\right),\ \zeta_k(x,s) \le \eta_k(x,s) \text{ on } (a,b] \times (a,b], \quad k = 1, 2, \dots, h,$$
$$\zeta_l(x,s), \eta_l(x,s) \in C\left([a,b) \times [a,b)\right),\ \zeta_l(x,s) \le \eta_l(x,s) \text{ on } [a,b) \times [a,b), \quad l = h+1, \dots, n$$

and, moreover, there are finite integrals

$$\int_a^b\int_a^b \zeta_i(x,s)dsdx, \quad \int_a^b\int_a^b \eta_i(x,s)dsdx \quad i = 1, 2, \dots, n.$$

Define for $k = 1, 2, \dots, h$; $l = h+1, \dots, n$ on $[a,b]$ continuous functions $\alpha_k, \beta_k, \alpha_l, \beta_l$:

$$\alpha_k(x) \equiv A_k + \int_a^x \gamma_k(t)dt + \int_a^x\int_a^t \zeta_k(t,s)\,ds\,dt;\ \alpha_k(a) = \alpha_k(a+0);$$

$$\beta_k(x) \equiv A_k + \int_a^x \delta_k(t)dt + \int_a^x\int_a^t \eta_k(t,s)\,ds\,dt;\ \beta_k(a) = \beta_k(a+0);$$

$$\alpha_l(x) \equiv B_l - \int_x^b \delta_l(t)dt - \int_x^b\int_t^b \eta_l(t,s)\,ds\,dt;\ \alpha_l(b) = \alpha_l(b-0);$$

$$\beta_l(x) \equiv B_l - \int_x^b \gamma_l(t)dt - \int_x^b\int_t^b \zeta_l(t,s)\,ds\,dt;\ \beta_l(b) = \beta_l(b-0).$$

Denote by D and D_1 the domains

$D = \{(x,y) : x \in I, \alpha_i(x) \le y_i \le \beta_i(x), i = 1,2,\dots,n\}$ and
$D_1 = \{(x,s,y,w) : (x,y) \in D,\ (s,w) \in D,\ s \le x\}$.

THEOREM 1. *Let the functions $\gamma_i(x)$, $\delta_i(x)$, $\zeta_i(x,s)$, $\eta_i(x,s)$, $i = 1,2,\dots,n$ satisfy the condition* (H1)–(H2) *and, moreover,*

a) $f_i(x,y) \in C(D)$, $g_i(x,s,y,w) \in C(D_1)$,
b) $\gamma_i(x) \le f_i(x,y) \le \delta_i(x)$ *where* $(x,y) \in D$,
c) $\zeta_i(x,s) \le g_i(x,s,y,w) \le \eta_i(x,s)$ *where* $(x,s,y,w) \in D_1$,
d) *for arbitrary points* $(x,\overline{y})$, $(x,\overline{\overline{y}}) \in D$

$$|f_i(x,\overline{y}) - f_i(x,\overline{\overline{y}})| \le \sum_{j=1}^{n} M_{ij}(x)|\overline{y}_j - \overline{\overline{y}}_j|$$

where $M_{ij}(x) \in C(I)$ are nonnegative functions such that

$$\int_a^b M_{ij}(x)(\beta_j(x) - \alpha_j(x))dx < \infty,$$

e) *for arbitrary points* $(x,s,\overline{y},\overline{w})$, $(x,s,\overline{\overline{y}},\overline{\overline{w}}) \in D_1$

$$|g_i(x,s,\overline{y},\overline{w}) - g_i(x,s,\overline{\overline{y}},\overline{\overline{w}})| \le \sum_{j=1}^{n} N_{ij}(x,s)|\overline{y}_j - \overline{\overline{y}}_j| + \sum_{j=1}^{n} P_{ij}(x,s)|\overline{w}_j - \overline{\overline{w}}_j|$$

where $N_{ij}(x,s) \in C(I\times I)$, $P_{ij}(x,s) \in C(I \times I)$ are nonnegative functions such that

$$\int_a^b\int_a^b N_{ij}(x,s)(\beta_j(x) - \alpha_j(x))\,ds\,dx < \infty,$$

$$\int_a^b\int_a^b P_{ij}(x,s)(\beta_j(s) - \alpha_j(s))\,ds\,dx < \infty.$$

Then there is a solution $y(x) = (y_1(x), y_2(x), \dots, y_n(x))$ of the Cauchy–Nicoletti problem (1), (2).

Remark 1. Note that the conditions of Theorem 1 are necessary simultaneously. Indeed, if a function $y = \varphi(x) = (\varphi_1(x), \dots, \varphi_n(x))$ is a solution of (1), (2) then there can be put

$$\gamma_i(x) \equiv \delta_i(x) \equiv f_i(x, \varphi(x)), \zeta_i(x,s) \equiv \eta_i(x,s) \equiv g_i(x, s, \varphi(x), \varphi(s))$$

and, consequently, $\alpha_i(x) \equiv \beta_i(x)$, $i = 1, 2, \dots, n$. In this case we put

$$M_{ij}(x) \equiv N_{ij}(x,s) \equiv P_{ij}(x,s) \equiv 0, \qquad i = 1, 2, \dots, n.$$

THEOREM 2. *Let all assumptions of Theorem* 1 *hold and, moreover,*

$$q\Big(\int_a^b M(x)\,dx + \int_a^b\int_a^b N(x,s)\,ds\,dx + \int_a^b\int_a^b P(x,s)\,ds\,dx\Big) < 1$$

where

$$q = \max\{h,\ n-h\}, \qquad M(x) = \max_{ij} M_{ij}(x),$$

$$N(x,s) = \max_{ij} N_{ij}(x,s), \qquad P(x,s) = \max_{ij} P_{ij}(x,s); \qquad i,j = 1, 2, \dots, n.$$

Then the solution

$$y(x) = (y_1(x), y_2(x), \dots, y_n(x))$$

of the Cauchy–Nicoletti problem (1), (2) *with the property* $(x, y(x)) \in D$ *on* I *is unique.*

THEOREM 3. *Let all asumptions of Theorem* 1 *hold and, moreover, there is a constant* $p \in [0,1)$ *such that on* I

$$\sum_{j=1}^{n} \int_a^x \Big[M_{kj}(t)(\beta_j(t) - \alpha_j(t)) + \int_a^t (N_{kj}(t,s)(\beta_j(t) - \alpha_j(t)) + P_{kj}(t,s)(\beta_j(s) - \alpha_j(s)))\,ds\Big]\,dt \le p(\beta_k(x) - \alpha_k(x)) \tag{3}$$

if $k \in \{1, 2, \dots, h\}$ *and*

$$\sum_{j=1}^{n} \int_x^b \Big[M_{lj}(t)(\beta_j(t) - \alpha_j(t)) + \int_t^b (N_{lj}(t,s)(\beta_j(t) - \alpha_j(t)) + P_{lj}(t,s)(\beta_j(s) - \alpha_j(s)))\,ds\Big]\,dt \le p(\beta_l(x) - \alpha_l(x)) \tag{4}$$

if $l \in \{h+1, \dots, n\}$. Then the solution $y(x) = (y_1(x), y_2(x), \dots, y_n(x))$ of the Cauchy-Nicoletti problem (1), (2) *with the property $(x, y(x)) \in D$ on I is unique.*

Proof of Theorem 1. In view of a), b), c) and (H1)–(H2) the problem (1), (2) is equivalent in D_1 with the system of integral equations

$$(5)\qquad y_k(x) = A_k + \int_a^x f_k(t, y(t))\,dt + \int_a^x\int_a^t g_k(t, s, y(t), y(s))\,ds\,dt, \qquad k = 1, 2, \dots, h,$$

$$(6)\qquad y_l(x) = B_l - \int_x^b f_l(t, y(t))\,dt - \int_x^b\int_t^b g_l(t, s, y(t), y(s))\,ds\,dt, \qquad l = h+1, \dots, n.$$

Define, with the aid of (5), (6), the sequences of functions $\{y_i^m(x)\}$, $i = 1, 2, \dots, n$, $m = 0, 1, \dots$, on interval I as follows

$$(7)\qquad \begin{cases} y_i^0(x) = \dfrac{1}{2}(\alpha_i(x) + \beta_i(x)), \quad i = 1, 2, \dots, n, \\ y_k^{m+1}(x) = A_k + \displaystyle\int_a^x f_k(t, y^m(t))\,dt + \int_a^x\int_a^t g_k(t, s, y^m(t), y^m(s))\,ds\,dt, \\ y_l^{m+1}(x) = B_l - \displaystyle\int_x^b f_l(t, y^m(t))dt - \int_x^b\int_t^b g_l(t, s, y^m(t), y^m(s))\,ds\,dt. \end{cases}$$

where $k = 1, 2, \dots, h$; $l = h+1, \dots, n$.

I. By method of induction it may be easily proved (with the aid of (H1)–(H2), a)–c)) that all elements of these sequences can be continued continuously on the closed interval [a,b] and, moreover, $(x, y^m(x)) \in \overline{D}$ if $x \in [a, b]$, $m = 0, 1, 2, \dots$ We will take this into account in the next text.

II. We show by Arzeli's theorem that there are subsequences $\{y_i^{m_r}(x)\}$ of the sequences $\{y_i^m(x)\}$ which converge uniformly on $[a, b]$. It is necessary to prove that all members of these sequences are uniformly bounded and equicontinuous. The uniform boundedness follows from fact that $(x, y^m(x)) \in \overline{D}$, $m = 0, 1, 2, \dots$ on $[a, b]$ and functions $\alpha_i(x), \beta_i(x)$ are bounded on $[a, b]$.

Prove the equicontinuity. Let $k \in \{1, 2, \dots, h\}$, $m = 0, 1, 2, \dots$ Define for $x \in (a, b]$

$$\psi_k(x) = \max\{|\gamma_k(x)|, |\delta_k(x)|\},$$

$$\chi_k(x) = \max\left\{\int_a^x |\zeta_k(x, s)|\,ds, \int_a^x |\eta_k(x, s)|\,ds\right\}.$$

Then from b), c), (H1), (H2)

$$|f_k(x, y^m(x))| \le \psi_k(x),\ x \in (a, b],$$
$$\int_a^x |g_k(x, s, y^m(x), y^m(s))|\, ds \le \chi_k(x),\ x \in (a, b].$$

Define

$$\varphi_k(x) = \max\{|\alpha_k(x) - A_k|, |\beta_k(x) - A_k|\},\ x \in [a, b].$$

Then on $[a, b]$

$$\left| \int_a^x f_k(t, y^m(t))\, dt + \int_a^x \int_a^t g_k(t, s, y^m(t), y^m(s))\, ds\, dt \right| \le \varphi_k(x).$$

Choose arbitrary positive number ε_k. Then, because $\varphi_k(a) = 0$ and $\varphi_k(x)$ is continuous, there is a $\omega_k = \omega_k(\varepsilon_k)$, $0 < \omega_k \le b - a$, such that $\varphi_k(x) < \frac{1}{2}\varepsilon_k$ if $x \in [a; a + \omega_k]$. Let

$$I_k = \left(a + \frac{\omega_k}{2}, b\right],\quad M_k = \sup_{x \in I_k} \psi_k(x),\quad N_k = \sup_{x \in I_k} \chi_k(x),$$
$$0 < \lambda_k^* < \frac{\varepsilon_k}{M_k + N_k},\quad \lambda_k = \min\left\{\lambda_k^*, \frac{1}{2}\omega_k\right\}.$$

We obtain for $x', x'' \in [a, b]$, $m \ge 1$ and $|x' - x''| < \lambda_k$:

α) if $x', x'' \in I_k$, then

$$|y_k^m(x') - y_k^m(x'')| \le$$
$$\le \left| \int_{x'}^{x''} |f_k(t, y^{m-1}(t))\, dt \right| + \left| \int_{x'}^{x''} \int_a^t |g_k(t, s, y^{m-1}(t), y^{m-1}(s))\, ds\, dt \right| \le$$
$$\le \left| \int_{x'}^{x''} \psi_k(t)\, dt \right| + \left| \int_{x'}^{x''} \chi_k(t)\, dt \right| \le$$
$$\le M_k|x' - x''| + N_k|x' - x''| \le \lambda_k(M_k + N_k) \le \lambda_k^*(M_k + N_k) < \varepsilon_k;$$

β) if $x', x'' \notin I_k$, or $x' \in I_k, x'' \notin I_k$, then $x', x'' \in [a, a + \omega_k]$ and

$$|y_k^m(x') - y_k^m(x'')| \le$$

$$\leq \left| \int_a^{x'} |f_k(t, y^{m-1}(t))|\, dt + \int_a^{x'} \int_a^t |g_k(t, s, y^{m-1}(t), y^{m-1}(s))|\, ds\, dt \right| +$$

$$+ \left| \int_a^{x''} |f_k(t, y^{m-1}(t))|\, dt + \int_a^{x''} \int_a^t |g_k(t, s, y^{m-1}(t), y^{m-1}(s))|\, ds\, dt \right| \leq$$

$$\leq \varphi_k(x') + \varphi_k(x'') < \frac{1}{2}\varepsilon_k + \frac{1}{2}\varepsilon_k = \varepsilon_k.$$

For $m = 0$ we can proceed analogously. The equicontinuity for indices $k \in \{1, 2, \ldots, h\}$ is proved. By an analogy we can prove the equicontinuity for indices $l \in \{h+1, \ldots, n\}$. Therefore the equicontinuity is proved and by Arzeli's theorem above-mentioned subsequences $\{y_i^{m_r}(x)\}$ exist. We denote the limits of these subsequences as $y_i(x)$, $i = 1, 2, \ldots, n$. In the next reasonings we will use, without loss of generality, the previous sequences instead of these subsequences. Because $(x, y^m(x)) \in \overline{D}$ for each $m = 0, 1, \ldots$ and $x \in [a, b]$ then $(x, y(x)) \in \overline{D}$ on $[a, b]$ too.

III. Prove that the limit function $y(x) = (y_1(x), y_2(x), \ldots, y_n(x))$ satisfies on I the system (5), (6).

For each positive $\tilde{\varepsilon}$ there is (in view of uniform convergence) an index $m_{\tilde{\varepsilon}}$ such that for $m > m_{\tilde{\varepsilon}}$: $|y_i(x) - y_i^m(x)| < \tilde{\varepsilon}$, $i = 1, 2, \ldots, n$ on $[a, b]$. Because $(x, y(x)) \in \overline{D}$ and $(x, y^m(x)) \in \overline{D}$ on $[a, b]$ then

$$(8) \qquad |y_i(x) - y_i^m(x)| \leq \min\{\tilde{\varepsilon}, \beta_i(x) - \alpha_i(x)\}, \quad i = 1, 2, \ldots, n.$$

From d), e) and (8) we conclude that for each positive ε there are: an index n_ε (sufficiently large) a value $x_\varepsilon^1 \in [a, b]$ (perhaps sufficiently near to the point a) and a value $x_\varepsilon^2 \in [a, b]$, $x_\varepsilon^1 < x_\varepsilon^2$ (perhaps sufficiently near to the point b) such that for $m > n_\varepsilon$, $i, j = 1, 2, \ldots, n$

$$(9) \qquad \int_a^{x_\varepsilon^1} \Big[M_{ij}(t)|y_i(t) - y_i^m(t)| +$$
$$+ \int_a^t (N_{ij}(t, s)|y_j(t) - y_j^m(t)| + P_{ij}(t, s)|y_j(s) - y_j^m(s)|)\, ds \Big]\, dt < \frac{\varepsilon}{3n},$$

$$(10) \qquad \int_{x_\varepsilon^1}^{x_\varepsilon^2} \Big[M_{ij}(t)|y_i(t) - y_i^m(t)| +$$
$$+ \int_a^t (N_{ij}(t, s)|y_j(t) - y_j^m(t)| + P_{ij}(t, s)|y_j(s) - y_j^m(s)|)\, ds \Big] dt < \frac{\varepsilon}{3n},$$

$$(11)\qquad \int_{x_\varepsilon^2}^{b}\Big[M_{ij}(t)|y_i(t)-y_i^m(t)|+ \\ +\int_a^t (N_{ij}(t,s)|y_j(t)-y_j^m(t)|+P_{ij}(t,s)|y_j(s)-y_j^m(s)|)\,ds\Big]\,dt<\frac{\varepsilon}{3n}.$$

The integral in (10):

$$J=\int_{x_\varepsilon^1}^{x_\varepsilon^2}\int_a^t P_{ij}(t,s)|y_j(s)-y_j^m(s)|\,ds\,dt$$

can be made sufficiently small (if n_ε is sufficiently large) because

$$J\le\int_{x_\varepsilon^1}^{x_\varepsilon^2}\Big(\int_a^{t_\varepsilon^1}+\int_{t_\varepsilon^1}^{t_\varepsilon^2}+\int_{t_\varepsilon^2}^{b}\Big)P_{ij}(t,s)|y_j(s)-y_j^m(s)|\,ds\,dt$$

where a value $t_\varepsilon^1\in[a,b]$ is, if necessary, sufficiently near to the point a and a value $t_\varepsilon^2\in[a,b]$ is, if necessary, sufficiently near to the point b. Analogously the correspondence integrals can be considered in (9), (11).

In view of d), e), (9)–(11) for $x\in[a,b]$, $k=1,2,\dots,h$:

$$\Big|\int_a^x f_k(t,y(t))\,dt+\int_a^x\int_a^t g_k(t,s,y(t),y(s))\,ds\,dt- \\ -\int_a^x f_k(t,y^m(t))\,dt-\int_a^x\int_a^t g_k(t,s,y^m(t),y^m(s))\,ds\,dt\Big|\le$$

$$\le\Big|\int_a^x|f_k(t,y(t))-f_k(t,y^m(t))|\,dt\Big|+ \\ +\Big|\int_a^x\int_a^t|g_k(t,s,y(t),y(s))-g_k(t,s,y^m(t),y^m(s))|\,ds\,dt\Big|\le$$

$$\le\int_a^x\sum_{j=1}^n M_{kj}(t)|y_j(t)-y_j^m(t)|\,dt+ \\ +\int_a^x\int_a^t\Big(\sum_{j=1}^n N_{kj}(t,s)|y_j(t)-y_j^m(t)|+\sum_{j=1}^n P_{kj}(t,s)|y_j(s)-y_j^m(s)|\Big)\,ds\,dt\le$$

$$\leq \sum_{j=1}^{n} \int_a^x (M_{kj}(t)|y_j(t) - y_j^m(t)| +$$

$$+ \int_a^t (N_{kj}(t,s)|y_j(t) - y_j^m(t)| + P_{kj}(t,s)|y_j(s) - y_j^m(s)|)\, ds)\, dt \leq$$

$$\leq \sum_{j=1}^{n} \int_a^b (M_{kj}(t)|y_j(t) - y_j^m(t)| +$$

$$+ \int_a^t (N_{kj}(t,s)|y_j(t) - y_j^m(t)| + P_{kj}(t,s)|y_j(s) - y_j^m(s)|)\, ds)\, dt =$$

$$= \sum_{j=1}^{n} \Big(\int_a^{x_\varepsilon^1} \dots + \int_{x_\varepsilon^1}^{x_\varepsilon^2} \dots + \int_{x_\varepsilon^2}^{b} \dots \Big) < \varepsilon.$$

Consequently, if $\varepsilon \to 0$ then

$$\int_a^x \Big(f_k(t, y^m(t)) + \int_a^t g_k(t,s,y^m(t),y^m(s))\, ds \Big)\, dt$$

$$\longrightarrow \int_a^x \Big(f_k(t, y(t)) + \int_a^t g_k(t,s,y(t),y(s)) ds \Big)\, dt.$$

By analogy we can prove that for $l \in \{h+1, \dots, n\}$

$$\int_x^b \Big(f_k(t, y^m(t)) + \int_t^b g_k(t,s,y^m(t),y^m(s))\, ds \Big)\, dt$$

$$\longrightarrow \int_x^b \Big(f_k(t, y(t)) + \int_t^b g_k(t,s,y(t),y(s))\, ds \Big)\, dt$$

if $x \in [a,b]$ and $\varepsilon \to 0$.

Therefore the vector function $y(x)$ is a solution of (5), (6), and, consequently, a solution of the problem (1), (2) with mentioned properties too. The theorem is proved.

Proof of Theorem 2. Let there exist two different solutions $y(x)$ and $u(x) = (u_1(x), u_2(x), \dots, u_n(x))$ of the problem (1), (2) with properties indicated in Theorem 1. Then by the condition d), e) of Theorem 1 and from (5) for $k \in \{1, 2, \dots, h\}$ on I it follows that

$$
\begin{aligned}
(12)\quad & |y_k(x)-u_k(x)| \le \\
&\le \left|\int_a^x |f_k(t,y(t))-f_k(t,u(t))|\,dt\right| + \\
&\qquad + \left|\int_a^x\int_a^t |g_k(t,s,y(t),y(s))-g_k(t,s,u(t),u(s))|\,ds\,dt\right| \le \\
&\le \int_a^x \sum_{j=1}^n M_{ij}(t)|y_j(t)-u_j(t)|dt + \\
&\quad + \int_a^x\int_a^t \Big(\sum_{j=1}^n N_{ij}(t,s)|y_j(t)-u_j(t)| + \sum_{j=1}^n P_{ij}(t,s)|y_j(s)-u_j(s)|\Big)\,ds\,dt \le \\
&\le \int_a^x M(t)\sum_{j=1}^n |y_j(t)-u_j(t)|dt + \\
&\quad + \int_a^x\int_a^t \Big(N(t,s)\sum_{j=1}^n |y_j(t)-u_j(t)| + P(t,s)\sum_{j=1}^n |y_j(s)-u_j(s)|\Big)\,ds\,dt = \\
&\quad = \int_a^x M(t)\Delta(t)\,dt + \int_a^x\int_a^t (N(t,s)\Delta(t)+P(t,s)\Delta(s))\,ds\,dt,
\end{aligned}
$$

where $\Delta(x) = \sum_{j=1}^n |y_j(x)-u_j(x)|$. Analogously from (6) for $l \in \{h+1,\dots,n\}$ we obtain

$$
\begin{aligned}
(13)\quad & |y_l(x)-u_l(x)| \le \\
&\quad \le \int_x^b M(t)\Delta(t)\,dt + \int_x^b\int_t^b (N(t,s)\Delta(t)+P(t,s)\Delta(s))\,ds\,dt.
\end{aligned}
$$

Denote $\Delta = \max_{x\in[a,b]} \Delta(x)$. In view of (12), (13) we have

$$
\begin{aligned}
&\Big(\sum_{k=1}^h |y_k(x)-u_k(x)|\Big) + \Big(\sum_{l=h+1}^n |y_l(x)-u_l(x)|\Big) = \Delta(x) \le \\
&\le \sum_{k=1}^h \Big(\int_a^x M(t)\Delta(t)\,dt + \int_a^x\int_a^t (N(t,s)\Delta(t)+P(t,s)\Delta(s))\,ds\,dt\Big) + \\
&\qquad + \sum_{l=h+1}^n \Big(\int_x^b M(t)\Delta(t)\,dt + \int_x^b\int_t^b (N(t,s)\Delta(t)+P(t,s)\Delta(s))\,ds\,dt\Big) \le
\end{aligned}
$$

$$\le \Delta\Big[\Big(\sum_{k=1}^{h}\int_a^x M(t)\,dt + \sum_{l=h+1}^{n}\int_x^b M(t)\,dt\Big)+$$

$$+\Big(\sum_{k=1}^{h}\int_a^x\int_a^t N(t,s)\,ds\,dt + \sum_{l=h+1}^{n}\int_x^b\int_t^b N(t,s)\,ds\,dt\Big)+$$

$$+\Big(\sum_{k=1}^{h}\int_a^x\int_a^t P(t,s)\,ds\,dt + \sum_{l=h+1}^{n}\int_x^b\int_t^b P(t,s)\,ds\,dt\Big)\Big] \le$$

$$\le \Delta q\Big(\int_a^b M(t)dt + \int_a^b\int_a^b N(t,s)\,ds\,dt + \int_a^b\int_a^b P(t,s)\,ds\,dt\Big)$$

We obtain a contradiction because

$$0 < \Delta \le \Delta q\Big(\int_a^b M(t)dt + \int_a^b\int_a^b (N(t,s)+P(t,s))\,ds\,dt\Big) < \Delta.$$

The theorem is proved.

Proof of Theorem 3. Let there exist two different solutions $y(x)$ and $u(x)$ of the problem (1), (2) with properties indicated in Theorem 1. Then by (5) and (3) for $x \in I, k \in \{1,2,\dots,h\}$

$$(14)\qquad |y_k(x) - u_k(x)| \le$$

$$\le \sum_{j=1}^{n}\int_a^x \Big[M_{kj}(t)(y_j(t)-u_j(t))+$$

$$+\int_a^t (N_{kj}(t,s)(y_j(t)-u_j(t)) + P_{kj}(t,s)(y_j(s)-u_j(s)))\,ds\Big]\,dt \le$$

$$\le \sum_{j=1}^{n}\int_a^x \Big[M_{kj}(t)(\beta_j(t)-\alpha_j(t))+$$

$$+\int_a^t (N_{kj}(t,s)(\beta_j(t)-\alpha_j(t)) + P_{kj}(t,s)(\beta_j(s)-\alpha_j(s)))\,ds\Big]\,dt \le$$

$$\le p(\beta_k(x)-\alpha_k(x)),$$

i.e.

$$(15)\qquad |y_k(x) - u_k(x)| \le p(\beta_k(x)-\alpha_k(x)).$$

If we repeat again reasonings (14) taking into account (15), we can prove the inequality

$$|y_k(x) - u_k(x)| \le p^s(\beta_k(x) - \alpha_k(x))$$

for $s = 2$ and, consequently, for each $s \in \mathbb{N}, s > 2$. Therefore, if $s \to \infty$, then $p^s \to 0$ and $y_k(x) \equiv u_k(x)$. By an analogy, using (6) and (4), we can prove that $y_l(x) \equiv u_l(x)$ on I for each $l \in \{h+1, \dots, n\}$. The theorem is proved.

3. Example

Consider the Cauchy-Nicoletti problem (16), (17)

$$\begin{aligned} y_1' &= f(x)y_1 + F(x, y_1, y_2), \\ y_2' &= \int_x^T y_1(s)\,ds, \end{aligned} \tag{16}$$

$$y_1(0+) = 0, \quad y_2(T-) = -\alpha, \tag{17}$$

where $0 < T$, $0 < \alpha$. Let there exist positive function $\chi(x) \in C^1(I_1)$, $I_1 = (0, T)$, and negative function $\omega(x) \in C^2(I_1)$, such that $\chi'(x) > 0$ on I_1, $\chi(0+) = 0$, $\omega(x) < -\alpha$ on I_1, $\omega(T-) = -\alpha$, $\omega'(T-) = 0$, $\chi(x) \le -\omega''(x)$, and let there exist integrals $\int_0^T \chi'(s)\,ds$, $\int_0^T \omega''(s)\,ds$.

Introduce a domain

$$D_2 = \{(x, y_1, y_2) : x \in I_1, 0 \le y_1 \le \chi(x),\ \omega(x) \le y_2 \le -\alpha\}.$$

THEOREM 4. *Let $f(x) \in C(I_1)$ be a nonnegative function, $\int_0^T f(s)\chi(s)\,ds < \infty$, $f(x)\chi(x) + \pi(x) \le \chi'(x)$, $x \in I_1$ where $\pi(x) \in C(I_1)$ is a nonnegative function, $F(x, y_1, y_2) \in C(D_2)$, $0 \le F(x, y_1, y_2) \le \pi(x)$ on D_2, $\int_0^T \pi(s)\,ds < \infty$,*

$$|F(x, \overline{y}_1, \overline{y}_2) - F(x, \overline{\overline{y}}_1, \overline{\overline{y}}_2)| \le M_1(x)|\overline{y}_1 - \overline{\overline{y}}_1| + M_2(x)|\overline{y}_2 - \overline{\overline{y}}_2|,$$

where $M_1(x), M_2(x) \in C(I_1)$, $(x, \overline{y}_1, \overline{y}_2), (x, \overline{\overline{y}}_1, \overline{\overline{y}}_2) \in D_2$, and

$$\int_0^T M_1(x)\chi(x)\,dx < \infty, \quad \int_0^T M_2(x)(-\alpha - \omega(x))\,dx < \infty.$$

Then there is a solution $y = y(x)$ of the problem (16), (17) such that $(x, y_1(x), y_2(x)) \in D_2$ on I_1. Such solution is unique if, moreover, there is a constant $p \in [0, 1)$ such that on I_1

$$\int_0^x [(f(t) + M_1(t))\chi(t) + M_2(t)(-\alpha - \omega(t))]\,dt \le p\chi(x),$$

and

$$\int_x^T\int_t^T \chi(s)\,ds\,dt \le p(-\alpha - \omega(x)).$$

Proof of Theorem 4. It is not difficult to verify all assumptions of Theorems 1 and 3 if $a = 0$, $b = T$, $n = 2$, $h = 1$, $A_1 = 0$, $B_2 = -\alpha$, $D \equiv D_2$, $\gamma_1(x) = \gamma_2(x) = \delta_2(x) \equiv 0$, $\delta_1(x) = \chi'(x)$, $\zeta_1(x,s) = \zeta_2(x,s) = \eta_1(x,s) \equiv 0$, $\eta_2(x,s) = -\omega''(s)$, $f_1(x,y_1,y_2) = f(x)y_1 + F(x,y_1,y_2)$, $g_1 \equiv f_2 \equiv 0$, $g_2 = y_1$, $M_{11}(x) = f(x) + M_1(x)$, $M_{12}(x) \equiv M_2(x)$, $M_{21}(x) \equiv M_{22}(x) \equiv N_{11}(x,s) \equiv N_{12}(x,s) \equiv N_{21}(x,s) \equiv N_{22}(x,s) \equiv P_{11}(x,s) \equiv P_{12}(x,s) \equiv P_{22}(x,s) \equiv 0$, $P_{21}(x,s) = 1$ and $\alpha_1(x) \equiv 0$, $\beta_1(x) = \chi(x)$, $\alpha_2(x) = \omega(x)$, $\beta_2(x) = -\alpha$. From its conclusions follows the conclusion of Theorem 4. The theorem is proved.

Consider the concrete singular problem of the type of (16), (17):

$$y_1' = \frac{1}{x}y_1 + \frac{x^4y_2^2}{4}, \quad y_2' = \int_x^1 y_1(s)\,ds,$$

$$y_1(0+) = 0, \; y_2(1-) = -1.$$

The conditions of Theorem 4 are valid for $\alpha = 1$, $T = 1$, $f(x) = x^{-1}$, $F(x,y_1,y_2) = \frac{x^4y_2^2}{4}$, $\chi(x) = \pi(x) = x^4$, $\omega(x) = -1-(1-x)^2$, $M_1 = 0$, $M_2(x) = x^4$, $p \in [\frac{1}{2}, 1)$. Therefore, there is a unique solution $y(x) = (y_1(x), y_2(x))$ of this problem on $(0,1)$ for which $0 \le y_1(x) \le x^4$, $-1-(1-x)^2 \le y_2(x) \le -1$.

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References

[1] J. Baštinec and J. Diblík, *On existence of solutions of singular Cauchy-Nicoletti problem for system of two differential equations*, Fasc. Math. 25 (1995), 5–12.

[2] V. A. Chechyk, *Investigations of systems of ordinary differential equations with singularities*, Trudy Moskov. Mat. Obshch. 8 (1959), 155–198 (in Russian).

[3] J. Diblík, *The singular Cauchy-Nicoletti problem for system of two ordinary differential equations*, Math. Bohem. 117, No. 1 (1992), 55–67.

[4] G. V. Gaprindashvili, *On solvability of boundary-value Dirichlet problems for systems of nonlinear differential equations with singularities*, Differentsial'nye Uravneniya 27, No 9 (1991), 1521–1525 (in Russian).

[5] I. T. Kiguradze, *Some singular boundary-value problems for ordinary differential equations*, Izd. Tbilis. Univ., Tbilisi, 1975 (in Russian).

[6] Ju. A. Klokov, *On one boundary-value problem with singularities in the endpoints of interval*, Latv. Mat. Ezhegodnik 17 (1976), 179–186 (in Russian).

[7] N. B. Konyuchova, *Singular Cauchy problem for systems of ordinary differential equations*, Zh. Vychisl. Mat. Fiz. 23, No 3 (1983), 629–645 (in Russian).

[8] A. G. Lomtatidze, *On one boundary-value problem for nonlinear ordinary differential equations of the second order with singularities*, Differentsial'nye Uravneniya 22, No 3 (1986), 416–426 (in Russian).

[9] B. Půža, *On a singular boundary-value problem for system of ordinary differential equations*, Arch. Math. (Brno) 13, No 4 (1977), 207–226 (in Russian).

[10] Z. Šmarda, *On the uniqueness of solutions of the singular problem for certain class of integro-differential equations*, Demonstratio Math., 25, No 4 (1992), 834–841.

[11] T. Werner, *Singular Cauchy-Nicoletti problem with singularities of type of poles*, Georgian Math. J., 2 (No 2) (1995), 211–224.

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