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SPECTRAL REPRESENTATION AND CHARACTERIZATIONS OF LOCALLY ALGEBRAIC LINEAR TRANSFORMATIONS

1. Introduction

The purpose of this paper is to generalize a spectral representation theorem of Williamson [7] and to simplify its proof. We show that this theorem is, essentially, a spectral representation of locally algebraic linear transformations which can be deduced by elementary methods of the linear algebra. Due to Kaplansky [3] a linear transformation T on a complex space E is said to be *locally algebraic*, if, for each $x \in E$, there exists a non-zero polynomial f such that $f(T)x = 0$. As main tool for our results serves a decomposition of the underlying space which is in the finite dimensional case well-known from the theory of the Jordan canonical form.

We first introduce some notation. E will always be a complex linear space. $L(E)$ denotes the set of all linear transformations on E and E^*, the algebraic dual of E. $\sigma_p(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not one-to-one}\}$ is the point spectrum of T. If E is a locally convex space, then $\mathcal{L}(E)$ is the set of all continuous linear transformations on E and E' the topological dual of E. $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ has no inverse in } \mathcal{L}(E)\}$ is the spectrum of T. $T \in L(E)$ is called *quasi-nilpotent*, if, for every $x \in E$, there is a non-negative integer n such that $T^n x = 0$. Given $M \subset E$ the linear hull of M is denoted by $[M]$.

In the following we make several times use of the fact that a linear transformation $T \in L(E)$ is locally algebraic if and only if $[x, Tx, T^2x, \ldots]$ is finite dimensional for each $x \in E$.

2. Williamson's spectral representation theorem

If J is an arbitrary index set, then $\mathbb{C}^J = \prod_{j \in J} \mathbb{C}$ and $\mathbb{C}_J = \{(\eta_j) \in \mathbb{C}^J : \eta_j \neq 0 \text{ for at most finitely many } j \in J\}$. Given $x = (\xi_j) \in \mathbb{C}^J$ and

$y = (\eta_j) \in \mathbb{C}_J$ let $\langle x, y \rangle = \sum \xi_j \eta_j$. This bilinear form places $\mathbb{C}^J$ and $\mathbb{C}_J$ in duality. $\mathbb{C}^J$ and $\mathbb{C}_J$ shall be endowed with the respective weak topologies $\sigma(\mathbb{C}^J, \mathbb{C}_J)$ and $\sigma(\mathbb{C}_J, \mathbb{C}^J)$. The dual transformation T' of $T \in \mathcal{L}(\mathbb{C}^J)$ is defined by $\langle Tx, y \rangle = \langle x, T'y \rangle$.

DEFINITION. [7] A continuous linear transformation T on a locally convex space E is called *adequately restricted* (*a.r.*), if

$$\limsup_{n\to\infty} |l(T^n x)|^{1/n} < \infty \quad \text{for all} \quad x \in E,\ l \in E'.$$

Williamson establishes the following spectral representation theorem (see [7, Theorems 4.1, 4.2 and Lemma 4.1]):

THEOREM 1. *Let J be a countable index set. If $T \in \mathcal{L}(\mathbb{C}^J)$ is a.r., then for each $\lambda \in \sigma(T)$ there are unique a.r. linear transformations P_λ and Q_λ such that*

$$P_\lambda P_\mu = \delta_{\lambda\mu} P_\lambda, \quad P_\lambda Q_\mu = \delta_{\lambda\mu} Q_\lambda, \quad Q_\lambda Q_\mu = \delta_{\lambda\mu}(T - \lambda I) Q_\lambda,$$
$$I = \sum_{\lambda\in\sigma(T)} P_\lambda, \quad T = \sum_{\lambda\in\sigma(T)} (\lambda P_\lambda + Q_\lambda).$$

The transformations P_λ and Q_λ commute with T and with each other. The Q_λ are quasi-nilpotent.

We show in §5 that the countability condition can be dropped and specify the transformations P_λ.

It can be shown that $\mathbb{C}_J' = \mathbb{C}_J^*$ and $\mathcal{L}(\mathbb{C}_J) = L(\mathbb{C}_J)$. By dual transformation $\mathcal{L}(\mathbb{C}^J)$ and $\mathcal{L}(\mathbb{C}_J) = L(\mathbb{C}_J)$ are isomorphic. Obviously $T \in \mathcal{L}(\mathbb{C}^J)$ is a.r. if and only if T' is a.r.. Furthermore $\sigma(T) = \sigma(T')$. In consequence of these facts the theorem remains valid, if $\mathcal{L}(\mathbb{C}^J)$ is replaced by $L(\mathbb{C}_J)$ and one can look for a proof with the methods of linear algebra.

Remark. Körber [4, Satz 6] shows that, if J is a countable set, then $T \in \mathcal{L}(\mathbb{C}^J)$ is a.r. if and only if $\sigma(T)$ is countable. If T is not a.r., then $\mathbb{C} \setminus \sigma(T)$ is countable ([4, Satz 2]).

We proceed with deriving our main tool.

3. Decomposition of a linear space relative to a locally algebraic linear transformation

It is well-known from the theory of the Jordan canonical form that a finite dimensional space E decomposes into a direct sum relative to a linear transformation T on E due to

$$(1)\qquad E = \bigoplus_{\lambda\in\sigma_p(T)} H_\lambda(T), \quad H_\lambda(T) = \bigcup_{n\in\mathbb{N}} \ker(T-\lambda I)^n$$

(see, e.g., [1, (13.18)]). In the infinite dimensional case we have

PROPOSITION 1. *Given* $T \in L(E)$, *decomposition* (1) *holds if and only if* T *is locally algebraic.*

Proof. We first assume T to be locally algebraic. Let x be an arbitrary point in E, $E_x = [x, Tx, T^2x, \dots]$ and T_x the restriction of T onto E_x. Since T is locally algebraic, E_x is finite dimensional and, therefore, decomposes due to (1). Taking additionally into account that $\sigma_p(T_x) \subset \sigma_p(T)$ and $H_\lambda(T_x) \subset H_\lambda(T)$, we obtain

$$x \in E_x = \bigoplus_{\lambda\in\sigma_p(T_x)} H_\lambda(T_x) \subset \sum_{\lambda\in\sigma_p(T)} H_\lambda(T).$$

Since x is arbitrary, it follows that $E = \sum_{\lambda\in\sigma_p(T)} H_\lambda(T)$. Suppose the sum is not direct. Then there exist $\lambda_1, \dots, \lambda_n \in \sigma_p(T)$, $\lambda_i \neq \lambda_j$ for $i \neq j$, and vectors $x_k \in H_{\lambda_k}(T) \setminus \{0\}$ such that $\sum_{k=1}^n x_k = 0$. For each x_k there is a $n_k \in \mathbb{N}$ with $(T-\lambda_k I)^{n_k} x_k = 0$. We assume n_1 to be chosen minimal, so that $\tilde{x}_1 := (T-\lambda_1 I)^{n_1-1}x_1 \neq 0$. Now let $g(X) = \prod_{k=2}^n (X-\lambda_k)^{n_k}$. From $(T-\lambda_1 I)^{n_1}x_1 = 0$ it follows that $\tilde{x}_1$ is an eigenvector of T with eigenvalue λ_1. Hence $g(T)\tilde{x}_1 = g(\lambda_1)\tilde{x}_1 \neq 0$. However, this is a contradiction to $0 = \sum_{k=1}^n x_k = (T-\lambda_1 I)^{n_1-1} g(T) \sum_{k=1}^n x_k = (T-\lambda_1 I)^{n_1-1} g(T) x_1 = g(T)\tilde{x}_1$.

Conversely, if (1) applies, then for fixed $x \in E$ there exist $\lambda_1, \dots, \lambda_n \in \sigma_p(T)$ and vectors $x_k \in H_{\lambda_k}(T)$ such that $x = \sum_{k=1}^n x_k$. Since $x_k \in H_{\lambda_k}(T)$, there are non-negative integers n_k with $(T-\lambda_k I)^{n_k} x_k = 0$ which implies that $\prod_{k=1}^n (T-\lambda_k I)^{n_k} x = 0$. Thus T is locally algebraic. ∎

4. A characterization of locally algebraic linear transformations

For the spaces $\mathbb{C}_J$ with countable index set J the following result is also contained in [7, Lemma 3.3], but the proof does not carry over to the general case.

PROPOSITION 2. $T \in L(E)$ *is locally algebraic if and only if*

$$\limsup_{n\to\infty} |l(T^n x)|^{1/n} < \infty \quad \text{for all } x \in E,\ l \in E^*.$$

Proof. Suppose T is locally algebraic. We first consider the case $x \in H_\lambda(T)$ for fixed $\lambda \in \sigma_p(T)$. Let $N \in \mathbb{N}$ be such that $(T-\lambda I)^N x = 0$. If $\lambda = 0$, then, obviously, $\limsup_{n\to\infty} |l(T^n x)|^{1/n} = 0$ for every $l \in E^*$. If $\lambda \neq 0$ and $n \geq N$, then

$$T^n x = [(T-\lambda I) + \lambda I]^n x = \lambda^n \sum_{k=0}^{N-1} \binom{n}{k} \lambda^{-k} (T-\lambda I)^k x.$$

Using $\binom{n}{k} \le n^k \le n^N$ whenever $k \le N$ we obtain

$$|l(T^n x)| \le |\lambda|^n n^N \sum_{k=0}^{N-1} |\lambda|^{-k} |l((T-\lambda I)^k x)| \quad \text{for all } l \in E^*.$$

The sum on the right-hand side is independent of n. Hence

$$\limsup_{n\to\infty} |l(T^n x)|^{1/n} \le |\lambda|.$$

Now let x be arbitrary. Since T is locally algebraic, by Proposition 1, there exist certain $\lambda_k \in \sigma_p(T)$ and $x_k \in H_{\lambda_k}(T)$, $k = 1, \dots, m$, such that $x = \sum x_k$. From the above it follows at once that

$$\limsup_{n\to\infty} |l(T^n x)|^{1/n} \le \sup_{k=1}^{m} |\lambda_k| < \infty, \quad l \in E^*. \tag{2}$$

On the other hand, if T is not locally algebraic, then there exists a vector $x \in E$ such that the set $M = \{x, Tx, T^2x, \dots\}$ is linearly independent. Now let $\tilde{l}$ be a linear form on $[M]$ defined by $\tilde{l}(T^n x) = n!$, $n \in \mathbb{N}_0$, and l an extension of $\tilde{l}$ onto E. Then $\limsup_{n\to\infty} |l(T^n x)|^{1/n} = \infty$. ■

Remarks. 1. For other characterizations of locally algebraic linear transformations see [5].

2. From relation (2) it is clear that locally algebraic linear transformations with bounded point spectrum $\sigma_p(T)$ are characterized by the property

$$\sup_{x\in E,\, l\in E^*} \limsup_{n\to\infty} |l(T^n x)|^{1/n} < \infty.$$

5. Spectral representation of locally algebraic linear transformations

For fixed locally algebraic linear transformation T we denote by $P_\lambda = P_\lambda(T)$ the projection of E onto $H_\lambda(T)$ relative to the direct sum decomposition (1), i.e. the mapping $x = \sum_{\mu\in\sigma_p(T)} x_\mu \mapsto x_\lambda$. Then $I = \sum P_\lambda$ and $P_\lambda P_\mu = \delta_{\lambda\mu} P_\lambda$. There is no question about convergence, since $\sum P_\lambda x$ has at most finitely many non-zero terms for each $x \in E$. Further we define $Q_\lambda = (T - \lambda I)P_\lambda$. We note that P_λ, as a projection, is locally algebraic; Q_λ is locally algebraic, since it turns out to be quasi-nilpotent.

THEOREM 2. *If $T \in L(E)$ is locally algebraic, then*

$$T = \sum_{\lambda\in\sigma_p(T)} (\lambda P_\lambda + Q_\lambda),$$

$$P_\lambda Q_\mu = \delta_{\lambda\mu} Q_\lambda, \quad Q_\lambda Q_\mu = \delta_{\lambda\mu}(T - \lambda I)Q_\lambda.$$

The transformations P_λ and Q_λ commute with T and with each other. The Q_λ are quasi-nilpotent.

Essential for our proof is decomposition (1), the remaining conclusions are largely those of Williamson.

Proof. Since $TP_\lambda = \lambda P_\lambda + Q_\lambda$, it holds that $T = TI = T\sum P_\lambda = \sum TP_\lambda = \sum(\lambda P_\lambda + Q_\lambda)$. Because of the T-invariance of the spaces $H_\lambda(T)$ we have $P_\lambda T P_\mu = \delta_{\lambda\mu} TP_\lambda$; thus $P_\lambda T = P_\lambda T \sum_{\mu\in\sigma_p(T)} P_\mu = \sum_{\mu\in\sigma_p(T)} P_\lambda T P_\mu = \sum_{\mu\in\sigma_p(T)} \delta_{\lambda\mu} TP_\lambda = TP_\lambda$. From this it follows that P_λ and Q_λ commute with T and with each other. The other relations are easy to see. Given $x \in E$ and $\lambda \in \sigma_p(T)$ we have $(T - \lambda I)^n P_\lambda x = 0$ for some $n \in \mathbb{N}$. Since $(T - \lambda I)^n P_\lambda = Q_\lambda^n$, the transformations Q_λ are quasi-nilpotent. ∎

PROPOSITION 3. *If $T \in L(E)$ is locally algebraic, then $\sigma_p(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ has no inverse in $L(E)\}$.*

Proof. It is sufficient to show that every one-to-one locally algebraic linear transformation is onto as well. If $T \in L(E)$ is locally algebraic and one-to-one, then the restrictions $T|_{E_x}$ of T onto $E_x = [x, Tx, T^2x, \ldots]$ are one-to-one and, since the spaces E_x are finite dimensional, onto as well. Hence $TE_x = E_x$ and $TE = T\bigcup E_x = \bigcup TE_x = \bigcup E_x = E$. ∎

We are now able to show that the countabilty condition in Williamson's theorem is dispensable: Let J be an arbitrary index set and $T \in \mathcal{L}(\mathbb{C}^J)$ a.r.; then $T' \in \mathcal{L}(\mathbb{C}_J)$ is a.r. and therefore, by Proposition 2, locally algebraic (recall that $\mathbb{C}'_J = \mathbb{C}^*_J$ and $\mathcal{L}(\mathbb{C}_J) = L(\mathbb{C}_J)$). Thus Theorem 2 is applicable on T' and, if we take into account that $\sigma_p(T') = \sigma(T')$ due to Proposition 3 and that $\sigma(T') = \sigma(T)$, the assertions of Theorem 1 follow by dual transformation. This consideration reveals the P_λ in Theorem 1 to be the dual transformations of the projections of $\mathbb{C}_J$ onto $H_\lambda(T')$.

Remark. In finite dimensional spaces E a suitable decomposition of the spaces $H_\lambda(T)$ into T-invariant subspaces leads to the Jordan canonical form. We did not succeed in generalizing this decomposition to the infinite dimensional case. For the spaces $E = \mathbb{C}_\mathbb{N}$ such a generalization can be found in a work of Körber [4, Satz 4 and 5]. But his proof makes use of the following non-valid assertion [4, Lemma 5]: *Let $T \in L(\mathbb{C}_\mathbb{N})$ be locally algebraic and $M_x = [x, Tx, T^2x, \ldots]$, $x \in \mathbb{C}_\mathbb{N}$. If M_x and M_y are not decomposable into T-invariant subspaces, then either $M_x \subset M_y$ or $M_y \subset M_x$ or $M_x \bigcap M_y = \{0\}$.* For a counterexample we define the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

which can be regarded as a locally algebraic linear transformation on $\mathbb{C}_\mathbb{N}$.

Let $e_k = (\delta_{jk})_{j\in\mathbb{N}}$, $k \in \mathbb{N}$ (δ_{jk} is the Kronecker symbol). Then the spaces $M_{e_1+e_3} = [e_1 + e_3, e_2]$ and $M_{e_1} = [e_1, e_2]$ are not decomposable into A-invariant subspaces, but none of the relations asserted above hold.

Herzog [2] gives a normal form for the non-locally-algebraic linear transformations on $\mathbb{C}_{\mathbb{N}}$. His result goes back to a work of Ulm [6].

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