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AN EXISTENCE AND UNIQUENESS THEOREM FOR ORDINARY DIFFERENTIAL EQUATIONS IN ORDERED BANACH SPACES

Abstract. We prove that the initial value problem $x'(t) = g(t,x(t)) + h(t,x(t))$, $t \in [0,T]$, $x(0) = a$, is uniquely solvable in certain partially ordered Banach spaces if, with respect to x, g is one-sided Lipschitz continuous, h is montonic decreasing and $g + h$ is quasimonotonic increasing.

1. Introduction

Let $(E, \|\cdot\|)$ be a real Banach space. We consider a partial ordering $\leq$ on E induced by a cone K. A cone K is a closed convex subset of E with $\lambda K \subseteq K$, $\lambda \geq 0$, and $K \cap (-K) = \{0\}$. Then $x \leq y \iff y - x \in K$, and we use the notation $x \ll y$ for $y - x \in \operatorname{Int} K$ and K^* for the dual cone, i.e., the set of all continuous linear functionals φ on E with $\varphi(x) \geq 0$, $x \geq 0$. The cone K is called normal if there is a $\gamma \geq 1$ with $0 \leq x \leq y \Longrightarrow \|x\| \leq \gamma \|y\|$. Furthermore, we will use $m_{\pm}[x,y] = \lim_{h \to 0\pm} h^{-1}(\|x + hy\| - \|x\|)$, $x, y \in E$. For basic properties of a partial ordering of E by a cone as well as of $m_{\pm}$ see [5].

Now let $f : [0,T] \times E \to E$ be continuous and bounded, and let $a \in E$. We consider the initial value problem

$$(1)\qquad \begin{cases} x'(t) = f(t, x(t)), & t \in [0,T], \\ x(0) = a. \end{cases}$$

A function $f : [0,T] \times E \to E$ is called

a) monotone increasing with respect to x if

$$(2)\qquad x, y \in E,\ t \in [0,T],\ x \leq y \Longrightarrow f(t,x) \leq f(t,y),$$

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b) monotone decreasing with respect to x if

$$x, y \in E,\ t \in [0,T],\ x \leq y \Longrightarrow f(t,x) \geq f(t,y), \tag{3}$$

c) quasimonotone increasing with respect to x if

$$x, y \in E,\ t \in [0,T],\ x \leq y, \varphi \in K^*,\ \varphi(x) = \varphi(y) \Longrightarrow \varphi(f(t,x)) \leq \varphi(f(t,y)), \tag{4}$$

d) one-sided Lipschitz continuous with respect to x if there is an $L \in \mathbb{R}$ with

$$x, y \in E,\ t \in [0,T] \Longrightarrow m_-[x - y, f(t,x) - f(t,y)] \leq L\|x - y\|. \tag{5}$$

Various existence and uniqueness theorems for problem (1) are known, demanding and combining conditions (2), (4) and (5); see, e.g., [1], [2], [3], [4], [5], and [7]. In general, condition (5) with $L \leq 0$ is considered as generalization of decreasing functions in one dimension. Less seems to be known for functions satisfying (3). In this paper, we will prove the following assertion.

THEOREM 1. *Let $K \subseteq E$ be a normal cone with $\gamma = 1$ and* $\operatorname{Int} K \neq \emptyset$. *Let $g, h : [0,T] \times E \to E$ be continuous and bounded with the following properties*:

a) *g satifies* (5),
b) *h satisfies* (3),
c) *$f := g + h$ satisfies* (4).

Then the initial value problem (1) *is uniquely solvable on* $[0,T]$.

Remarks. 1) If K is a regular cone then (1) is already solvable on $[0,T]$, if f is satisfying (4), see [4].

2) Setting $g = 0$, Theorem 1 implies that (1) is uniquely solvable if f satisfies (3) and (4). See Example 2 for such a function f for which (5) does not hold. Therefore Theorem 1 is no consequence of Martin's theorem; see [5], p. 232.

3) According to [6], p. 215, a Banach space with a normal cone can be renormed with an equivalent norm for which $\gamma = 1$. Therefore, if $g = 0$, Theorem 1 holds without the condition $\gamma = 1$.

4) See Example 1 for a function f satisfying (3), but for which (1) is not solvable.

2. Proof of Theorem 1

To prove Theorem 1 we will need the following property of m_+. Let the cone K be as in Theorem 1, and let $x \gg 0$ and $y \leq 0$. Then $0 \ll x + hy \leq x$ for $h > 0$ sufficiently small which implies $\|x + hy\| \leq \|x\|$. Therefore we have

$$x, y \in E,\ x \gg 0,\ y \leq 0 \Longrightarrow m_+[x, y] \leq 0. \tag{6}$$

Remark that $x \ll 0$, $y \geq 0 \Longrightarrow m_+[x,y] \leq 0$ since $m_+[x,y] = m_+[-x,-y]$.

P r o o f o f T h e o r e m 1. Since K is normal and $\operatorname{Int} K \neq \emptyset$ we have, according to a construction of Lemmert, Schmidt and Volkmann [4], that there are sequences $(u_n)_{n=1}^\infty$ and $(v_n)_{n=1}^\infty$ in $C^1([0,T],E)$ with the following properties: Setting $r_n = u_n' - f(\cdot, u_n)$ and $s_n = v_n' - f(\cdot, v_n)$, $n \in \mathbb{N}$, we have

$$r_n(t) \ll r_{n+1}(t) \ll 0 \ll s_{n+1}(t) \ll s_n(t), \quad t \in [0,T], \quad n \in \mathbb{N}, \tag{7}$$
$$\text{and } u_n(0) \ll u_{n+1}(0) \ll a \ll v_{n+1}(0) \ll v_n(0), \quad n \in \mathbb{N},$$

$$\lim_{n\to\infty} \max_{t\in[0,T]} \|r_n(t)\| = \lim_{n\to\infty} \max_{t\in[0,T]} \|s_n(t)\| = 0, \tag{8}$$
$$\text{and } \lim_{n\to\infty} u_n(0) = \lim_{n\to\infty} v_n(0) = a.$$

Since f satisfies (4) we have, according to (7) (cf. [4], [8]),

$$u_n(t) \ll u_{n+1}(t) \text{ and } v_{n+1}(t) \ll v_n(t), \quad n \in \mathbb{N}, \quad t \in [0,T], \tag{9}$$

and

$$u_n(t) \ll x(t) \ll v_n(t), \quad n \in \mathbb{N}, \quad t \in [0,T], \tag{10}$$

for every solution x of (1).

Now, for $m, n \in \mathbb{N}$ and $t \in (0,T]$, we have

$$\begin{aligned}
\|u_m - u_n\|'_-(t) =& \\
&= m_-[u_m(t) - u_n(t), u_m'(t) - u_n'(t)] \\
&= m_-[u_m(t) - u_n(t), r_m(t) - r_n(t) + (f(t,u_m(t)) - f(t,u_n(t)))] \\
&\leq m_-[u_m(t) - u_n(t), g(t,u_m(t)) - g(t,u_n(t))] \\
&\quad + m_+[u_m(t) - u_n(t), h(t,u_m(t)) - h(t,u_n(t))] \\
&\quad + \|r_m(t)\| + \|r_n(t)\|.
\end{aligned}$$

Now the properties of g and h together with (9) and (6) imply

$$\|u_m - u_n\|'_-(t) \leq L\|u_m(t) - u_n(t)\| + \|r_m(t)\| + \|r_n(t)\|, \; m,n \in \mathbb{N}, \; t \in (0,T].$$

This together with (8) implies the uniform convergence of $(u_n)_{n=1}^\infty$ on $[0,T]$ to a solution $\underline{u}$ of (1). Analogously, $(v_n)_{n=1}^\infty$ converges uniformly on $[0,T]$ to a solution $\bar{u}$ of (1). According to (10), we have $\underline{u}(t) \leq \bar{u}(t)$, $t \in [0,T]$.

Now for $n \in \mathbb{N}$ and $t \in (0,T]$ we have

$$\begin{aligned}
\|v_n - u_n\|'_-(t) =& \\
&= m_-[v_n(t) - u_n(t), s_n(t) - r_n(t) + (f(t,v_n(t)) - f(t,u_n(t)))] \\
&\leq m_-[v_n(t) - u_n(t), g(t,v_n(t)) - g(t,u_n(t))] \\
&\quad + m_+[v_n(t) - u_n(t), h(t,v_n(t)) - h(t,u_n(t))] + \|s_n(t)\| + \|r_n(t)\|,
\end{aligned}$$

which implies

$$\|v_n - u_n\|'_-(t) \leq L\|v_n(t) - u_n(t)\| + \|s_n(t)\| + \|r_n(t)\|, \quad t \in (0,T], \; n \in \mathbb{N}.$$

Since $\lim_{n\to\infty}(v_n(0) - u_n(0)) = 0$ and according to (8) we have $\lim_{n\to\infty} v_n(t) - u_n(t) = 0$, $t \in [0,T]$, and therefore $\bar{u} = \underline{u}$. According to (10), $\underline{u} = \bar{u}$ is the unique solution of (1). ∎

3. Examples

The first example shows that the initial value problem (1) can be unsolvable if f satisfies (3).

Let $(c, \|\cdot\|)$ be the Banach space of all real convergent sequences $x = (x_n)_{n=1}^{\infty}$, $\|x\| = \sup_{n\in\mathbb{N}} |x_n|$, and let $f : [0,1] \times c \to c$ be defined as

$$f(t,x) = \left(2t, 2\left(t - \sqrt{k(x_1)}\right), 2\left(t - \sqrt{k(x_2)}\right), \ldots\right)$$

with

$$k(s) = \begin{cases} 0 & \text{if } s \le 0, \\ s & \text{if } s \in [0,1], \\ 1 & \text{if } s \ge 1. \end{cases}$$

The function f is continuous, bounded and satisfies (3) for the partial ordering induced by the normal cone $K = \{(x_n)_{n=1}^{\infty} \in c : x_n \ge 0,\ n \in \mathbb{N}\}$.

Now the coordinatewise solution of (1) with $a = 0$ is $x(t) = (t^2, 0, t^2, 0, \ldots)$, $t \in [0,1]$, and therefore (1) is locally unsolvable in c.

The second example shows that even in two dimensional space there are functions $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ with the properties (3) and (4) but without property (5). Let $\|(x_1,x_2)\| = |x_1| + |x_2|$, $(x_1,x_2) \in \mathbb{R}^2$. We consider the cone $K = \{(x_1,x_2) \in \mathbb{R}^2 : 0 \le x_1 \le x_2\}$. Observe that $0 \le x \le y$, $x,y \in \mathbb{R}^2 \Longrightarrow \|x\| \le \|y\|$ and

$$x \le y \iff 0 \le y_1 - x_1 \le y_2 - x_2 \Longrightarrow y_1 - y_2 \le x_1 - x_2.$$

Let $q \in C([0,T],\mathbb{R})$ and $k \in C(\mathbb{R},\mathbb{R})$. Now let $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(t,x) = (q(t), k(x_1 - x_2)).$$

If k is monotone increasing we have

$$x \le y \Longrightarrow 0 = q(t) - q(t) \le k(x_1 - x_2) - k(y_1 - y_2)$$

and therefore $f(t,x) \ge f(t,y)$, $t \in [0,T]$.

Furthermore, we have $K^* = \{(\alpha_1,\alpha_2) \in \mathbb{R}^2 : \alpha_2 \ge 0,\ \alpha_1 + \alpha_2 \ge 0\}$. Now let $x \le y$, $\varphi = (\alpha_1,\alpha_2) \in K^*$ with $\varphi(y-x) = \alpha_1(y_1 - x_1) + \alpha_2(y_2 - x_2) = 0$. Then $(\alpha_1 + \alpha_2)(y_1 - x_1) \le 0$ which implies $(\alpha_1 + \alpha_2)(y_1 - x_1) = 0$ and therefore

$$y_1 = x_1 \Longrightarrow \alpha_2 = 0 \vee y_2 = x_2$$

or

$$\alpha_1 + \alpha_2 = 0 \Longrightarrow \alpha_1 = \alpha_2 = 0 \vee y_1 - y_2 = x_1 - x_2.$$

Therefore $\varphi(f(t,y) - f(t,x)) = \alpha_2(k(y_1 - y_2) - k(x_1 - x_2)) = 0$.

Finally, we consider

$$\begin{aligned} m_-[x-y, f(t,x) - f(t,y)] &= \lim_{h\to 0-} \frac{\|x - y + h(f(t,x) - f(t,y))\| - \|x-y\|}{h} \\ &= \lim_{h\to 0-} \frac{|x_2 - y_2 + h(k(x_1 - x_2) - k(y_1 - y_2))| - |x_2 - y_2|}{h} \\ &= \begin{cases} k(y_1 - y_2) - k(x_1 - x_2) & \text{if } y_2 > x_2, \\ -|k(x_1 - x_2) - k(y_1 - y_2)| & \text{if } y_2 = x_2, \\ k(x_1 - x_2) - k(y_1 - y_2) & \text{if } y_2 < x_2. \end{cases} \end{aligned}$$

Take, for example, $k(s) = \arctan \sqrt[3]{s}$, $s \in \mathbb{R}$, and assume that f is satisfying (5). Then there is an $L \in \mathbb{R}$ such that, for $x, y \in \mathbb{R}^2$ and $y_2 > x_2 = 0$,

$$\arctan \sqrt[3]{y_1 - y_2} - \arctan \sqrt[3]{x_1} \leq L(|x_1 - y_1| + y_2),$$

which implies $\arctan \sqrt[3]{y_1} - \arctan \sqrt[3]{x_1} \leq L|x_1 - y_1|$, $x_1, y_1 \in \mathbb{R}$. This of course does not hold. Altogether,

$$f(t,x) = (q(t), \arctan \sqrt[3]{x_1 - x_2})$$

is satisfying (3) and (4) but not (5).

References

[1] A. Chaljub-Simon, R. Lemmert, S. Schmidt and P. Volkmann, *Gewöhnliche Differentialgleichungen mit quasimonoton wachsenden rechten Seiten in geordneten Banachräumen*, General Inequalities 6 (Oberwolfach 1990), pp. 307–320, International Series of Numerical Mathematics 103, Birkhäuser, Basel, 1992.

[2] R. Lemmert, *Existenzsätze für gewöhnliche Differentialgleichungen in geordneten Banachräumen*, Funkcial. Ekvac., Ser. Internac. 32 (1989), 243–249.

[3] R. Lemmert, R. M. Redheffer and P. Volkmann, *Ein Existenzsatz für gewöhnliche Differentialgleichungen in geordneten Banachräumen*, General Inequalities 5 (Oberwolfach 1986), pp. 381–390, International Series of Numerical Mathematics 80, Birkhäuser, Basel 1987.

[4] R. Lemmert, S. Schmidt and P. Volkmann, *Ein Existenzsatz für gewöhnliche Differentialgleichungen mit quasimonoton wachsender rechter Seite*, Math. Nachr. 153 (1991), 349–352.

[5] R. H. Martin, *Nonlinear operators and differential equations in Banach spaces*, Robert E. Krieger Publishing Company, 1987.

[6] H. H. Schaefer, *Topological Vector Spaces*, The MacMillan Company, New York (1966).

[7] V. Ja. Stecenko, *K-pravil'nye konusy*. Doklady Acad. Nauk SSSR 136 (1961), 1038–1040.

[8] P. Volkmann, *Gewöhnliche Differentialungleichungen mit quasimonoton wachsenden Funktionen in topologischen Vektorräumen*, Math. Z. 127 (1972), 157–164.

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