DEMONSTRATIO MATHEMATICA
Vol. XXX No 4 1997

Henryk Oryszczyszyn

ON A GENERAL THEORY OF ORDERED TRAPEZIUM SPACES

Introduction

There exist many various axiom systems of Euclidean geometry, based on various primitive notions. Usually two primitive notions are consider, one — describes metric properties, second is responsible for order.

The approach which we propose allows to investigate in the natural way both the metric structure and the geometrical order with the help of a single notion.

In [1] K. Prażmowski has shown that the relation of isosceles trapezium $\mathbf{T}$, may be used to describe the Euclidean geometry. In this paper we consider the relation of directed isosceles trapezium, denoted by T. This relation may be interpreted in the Euclidean space as follows: $ab\mathrm{T}cd$ means that the vectors $\vec{ab}$ and $\vec{cd}$ have the same direction and the common bisector hyperplane. Clearly this relation induces the metric structure of the underlying Euclidean space, i.e. the equidistance relation. As the analogue of geometrical order we choose an ordering function introduced by E. Sperner in [3].

1. Relation of directed trapezium

Let X be a nonempty set. A relation $\mathbf{T} \subseteq X^2 \times X^2$ is *trapezium relation* if $\mathbf{T}$ satisfies the following axioms (comp. [1]):

T1. $xy\mathbf{T}yx$,

T2. $xy\mathbf{T}zt \wedge xy\mathbf{T}uw \Rightarrow x = y \vee zt\mathbf{T}uw$,

T3. $xy\mathbf{T}zt \Rightarrow zt\mathbf{T}xy$,

T4. $(\exists t)xy\mathbf{T}zt$,

T5. $xy\mathbf{T}uz \wedge xy\mathbf{T}ut \Rightarrow x = y \vee z = t$,

T6. $(\exists z,t)[xx\mathbf{T}yy \Rightarrow z \neq t \wedge zt\mathbf{T}xx \wedge zt\mathbf{T}yy]$.

Now we say that $\mathsf{T} \subseteq X^2 \times X^2$ is a *relation of directed trapezium* if T satisfies the following axioms:

DT1. $xy\mathsf{T}yz \Rightarrow x = y = z$,
DT2. $xy\mathsf{T}zt \wedge xy\mathsf{T}uw \Rightarrow x = y \vee zt\mathsf{T}uw$,
DT3. $xy\mathsf{T}zt \Rightarrow zt\mathsf{T}xy \wedge yx\mathsf{T}tz$,
DT4. $(\exists t)[xy\mathsf{T}zt \vee xy\mathsf{T}tz]$,
DT5. $xy\mathsf{T}uz \wedge xy\mathsf{T}ut \Rightarrow x = y \vee z = t$,
DT6. $(\exists z, t)[xx\mathsf{T}yy \Rightarrow z \neq t \wedge zt\mathsf{T}xx \wedge zt\mathsf{T}yy]$.

Any structure $< X, \mathsf{T} >$, where T is a relation of directed trapezium will be called *ordered trapezium space*. Immediately from DT1–DT6 we obtain

PROPOSITION 1. *T is an equivalence relation in the set $\{(a,b) \in X^2 : a \neq b\}$.* ■

One can see that the axioms DT1–DT6 are modifications of T1–T6.

Given any relation $\mathbf{R} \subseteq X^2 \times X^2$, we define the relation $\Lambda(\mathbf{R})$ as follows:

$$ab\Lambda(\mathbf{R})cd :\Longleftrightarrow ab\mathbf{R}cd \vee ab\mathbf{R}dc.$$

The following theorem holds:

THEOREM 2. *If T is a relation of directed trapezium then $\Lambda(\mathsf{T})$ is a trapezium relation.*

Proof. T1 follows from Proposition 1. The axioms T2, T4 and T6 follow from DT2, DT3, DT4 and DT6 respectively.

To prove T3 we assume $xy\Lambda(\mathsf{T})zt$; then $xy\mathsf{T}zt$ or $xy\mathsf{T}tz$. If $xy\mathsf{T}zt$ then $zt\mathsf{T}xy$ by DT3, hence $zt(\Lambda\mathsf{T})xy$. Now assume $xy\mathsf{T}tz$; then by DT3, $tz\mathsf{T}xy$ and $zt\mathsf{T}yx$, hence $zt\Lambda(\mathsf{T})xy$.

To prove T5 we assume $xy\Lambda(\mathsf{T})uz$ and $xy\Lambda(\mathsf{T})ut$; then ($xy\mathsf{T}uz$ or $xy\mathsf{T}zu$) and ($xy\mathsf{T}ut$ or $xy\mathsf{T}tu$). For example we assume $xy\mathsf{T}uz$ and $xy\mathsf{T}tu$. By DT2 we have $x = y$ or $uz\mathsf{T}tu$. If $uz\mathsf{T}tu$ then $zu\mathsf{T}ut$ by DT3, hence $z = u = t$. ■

A family Σ of involutions of a nonempty set X is called *perfect family of involutions* provided it satisfies the following conditions:

S1. $(\forall a, b \in X)(\forall f, g \in \Sigma)[f(a) = b = g(a) \Rightarrow a = b \vee f = g]$,
S2. $(\forall a, b \in X)(\exists f \in \Sigma)[f(a) = b]$. (cf. [1] and [2]).

From Theorem 2 and Proposition 1 we get:

THEOREM 3. *Let $\mathsf{T} \subseteq X^2 \times X^2$ be a relation of directed trapezium and let $\sigma_b^a = \{\langle x, y\rangle : ab\Lambda(\mathsf{T})xy\}$ for all $a, b \in X$; then the set $\Sigma(\mathsf{T}) = \{\sigma_b^a : a \neq b\}$ is a perfect family of involutions of X.* ■

COMMENTARY. If T is a relation of directed trapezium defined in ordered Euclidean space then σ_b^a is a reflection in the bisector hyperplane of a, b and $\Sigma(\mathsf{T})$ is the set of all reflections in hyperplanes. ∎

Let now $\langle X; \mathsf{T}\rangle$ be a fixed ordered trapezium space. Let $\mathrm{Fix}(\sigma_b^a)$ be the set of all points fixed by σ_a^b; clearly

$$\mathrm{Fix}(\sigma_b^a) = \{x : \sigma_b^a(x) = x\} = \{x : ab\mathsf{T}xx\}.$$

For every $\sigma \in \Sigma(\mathsf{T})$ we define the relation $\sim_\sigma \subseteq X \times X$ as follows:

$$p \sim_\sigma q :\Leftrightarrow p\sigma(p)\mathsf{T}q\sigma(q).$$

PROPOSITION 4. *For every $\sigma \in \Sigma(\mathsf{T})$ the following conditions are satisfied*:

(i) *the relation $\sim_\sigma$ is an equivalence relation in the set $X \setminus \mathrm{Fix}(\sigma)$*;

(ii) *$\sim_\sigma$ divides $X \setminus \mathrm{Fix}(\sigma)$ into exactly two equivalence classes.*

Proof. (i) follows immediately from Proposition 1.

(ii) Let $a \neq \sigma(a) = a'$. If not $a \sim_\sigma a'$ then, by the definition, $aa'\mathsf{T}a'a$, so $a = a'$ by DT1; thus $a \sim_\sigma a'$. Now let x be any element of X such that $x \neq \sigma(x)$. By DT4 we have $aa'\mathsf{T}xx'$ or $aa'\mathsf{T}x'x$ for some $x' \in X$. Hence $a\sigma(a)\mathsf{T}x\sigma(x)$ or $a'\sigma(a')\mathsf{T}x\sigma(x)$ i.e. $a \sim_\sigma x$ or $a' \sim_\sigma x$. ∎

Note that if $\Sigma = \sigma(\mathsf{T})$ is a family of reflections in hyperplanes of an ordered Euclidean space and $\sigma \in \Sigma$ then the hyperplane $\mathrm{Fix}(\sigma)$ divides the space into two (open) halfspaces which are exactly the two equivalence classes of $\sim_\sigma$.

For a given family Σ of transformations of X we call a map $\omega : \Sigma \mapsto X$ a *selecting map* for Σ if $\sigma(\omega(\sigma)) \neq \omega(\sigma)$ for every $\sigma \in \Sigma$.

LEMMA 5. *Every family Σ of transformations of a nonempty set X, wich does not cotain the identity on X has a selecting map.*

Proof. Cleary $X \setminus \mathrm{Fix}(\sigma) \neq \emptyset$ for every $\sigma \in \Sigma$. By the axiom of choice there exists a choice function $\omega : \Sigma \mapsto X$ such that $\omega(\sigma) \in X \setminus \mathrm{Fix}(\sigma)$ for $\sigma \in \Sigma$; of course $\sigma(\omega(\sigma)) \neq \omega(\sigma)$. ∎

For every selecting map ω for $\Sigma(\mathbf{T})$ (thus associated with the relation $\mathbf{T}$) we consider the function $\nu = \nu(\mathbf{T}, \omega) : \Sigma \times X \mapsto \{-1, 0, 1\}$ defined by

$$\nu(\sigma, x) = \begin{cases} 0 & \text{if } \sigma(x) = x \\ 1 & \text{if } \omega(\sigma) \sim_\sigma (x) \\ -1 & \text{otherwise.} \end{cases}$$

LEMMA 6. *If* T *is an ordered trapezium relation and* ω *is any selecting map for the set* $\Sigma = \Sigma(\mathsf{T})$ *then the map* $\nu = \nu(\mathsf{T}, \omega)$ *has the following properties*:

N1. $\nu(\sigma, x) = 0 \Leftrightarrow \sigma(x) = x$,

N2. $\nu(\sigma, x) \cdot \nu(\sigma, \sigma(x)) \neq 1$,

N3. *If* $\sigma(x) \neq x$ *and* $\sigma(y) \neq y$ *then* $x \sim_\sigma y \Leftrightarrow \nu(\sigma, x) \cdot \nu(\sigma, y) = 1$. ∎

Note that if we consider (classical) example of directed trapezium T in an ordered Euclidean space and we identify every reflection $\sigma \in \Sigma = \Sigma(T)$ with its axis $\mathrm{Fix}(\sigma)$ then every function $\nu = \nu(\mathbf{T}, \omega)$ defined as above with ω being a selecting map for Σ corresponds to an ordering function in the sence of Sperner (cf. [3]) suitable for Euclidean space.

2. Family of involutions and ordering function ν

Let Σ be an arbitrary family of bijections of a set X and let ν be a map $\nu : \Sigma \times X \mapsto \{-1,0,1\}$. We define the relation $\Pi = \Pi(\Sigma, \nu)$ as follows:

$$ab\Pi(\Sigma, \nu)cd :\Longleftrightarrow (\exists f \in \Sigma)[f(a) = b \wedge f(c) = d \wedge \nu(f, a) \cdot \nu(f, c) \neq -1].$$

THEOREM 7. *If* Σ *is a perfect family of involutions of* X *and* $\nu : \Sigma \times X \mapsto \{-1,0,1\}$ *has the properties* **N1** *and* **N2** *from Lemma* 5, *then the relation* $\Pi = \Pi(\Sigma, \nu)$ *is a relation of directed trapezium.*

Proof. To prove DT1 we assume that $f(x) = y$ and $f(y) = z$ for some $f \in \Sigma$; then $x = z$. If $x \neq y$ then $\nu(f, x) \cdot \nu(f, y) = -1$ by **N2** but $\nu(f, x) \cdot \nu(f, y) \neq -1$ from definition of Π. Hence $x = y$.

To prove DT2 we consider x, y, z, t, u, w such that $xy\Pi zt$ and $xy\Pi uw$.

Then $f(x) = y$, $f(z) = t$, $g(x) = y$, $g(u) = w$ for some $f, g \in \Sigma$. Assume $x \neq y$; then $f = g$ by S1. Moreover $\nu(f, x) \neq 0$, $\nu(f, x) \cdot \nu(f, z) \neq -1$ and $\nu(f, x) \cdot \nu(f, u) \neq -1$. Hence $[\nu(f, x)]^2 \cdot \nu(f, z) \cdot \nu(f, u) = \nu(f, z) \cdot \nu(f, u) \neq -1$ and thus $zt\Pi uw$.

To prove DT3 we assume that $xy\Pi zt$; then $f(x) = y, f(z) = t$ and $\nu(f, x) \cdot \nu(f, z) \neq -1$ for some $f \in \Sigma$. Immediately we obtain $zt\Pi xy$. To obtain $yx\Pi tz$ it is enough to prove $\nu(f, y) \cdot \nu(f, t) \neq -1$. We can assume that $y \neq f(y)$ and $t \neq f(t)$; $\nu(f, y) \cdot \nu(f, x) = -1$ and $\nu(f, t) \cdot \nu(f, z) = -1$. Hence $\nu(f, x) \cdot \nu(f, y) \cdot \nu(f, t) \cdot \nu(f, z) = 1$. Since $\nu(f, x) \cdot \nu(f, z) \neq -1$ we have $\nu(f, y) \cdot \nu(f, t) = 1 \neq -1$.

To prove DT4, let x, y, z be arbitrary elements of X. By S2 we consider $f \in \Sigma$ such that $f(x) = y$ and we put $t = f(z)$. We have $\nu(f, x) \cdot \nu(f, y) \neq -1$. If $x = y$ then, by **N1**, $\nu(f, x) = 0$ so $\nu(f, x) \cdot \nu(f, z) = 0 \neq -1$; hence $xy\Pi zt$. Analogously, $z = t$ yields $xy\Pi zt$.

Now assume $x \neq y$ and $z \neq t$. By **N2** we get $\nu(f, x) \cdot \nu(f, y) = -1$ and $\nu(f, z) \cdot \nu(f, t) = -1$. If $\nu(f, x) \cdot \nu(f, z) = 1$ then $xy\Pi zt$. Let

$\nu(f,x)\cdot\nu(f,z) = -1$ then we have $-\nu(f,t) = \nu(f,t)\cdot(\nu(f,x)\cdot\nu(f,z)) = \nu(f,x)\cdot(\nu(f,t)\cdot\nu(f,z)) = -\nu(f,x)$. Hence $xy\Pi tz$.

DT5 and DT6 are obvious. ∎

Finally we obtain

THEOREM 8. *A structure $\langle X;\mathsf{T}\rangle$ is an ordered trapezium space iff there exists Σ - a perfect family of involutions of X and a function $\nu:\Sigma\times X\mapsto\{-1,0,1\}$ satisfying* **N1** *and* **N2** *from Lemma* 6 *such that* $\mathsf{T} = \Pi(\Sigma,\nu)$.

Proof. The implication ($\Leftarrow$) follows by Theorem 7. Now let $\langle X;\mathsf{T}\rangle$ be an ordered trapezium space and let $\Sigma = \Sigma(\mathsf{T})$. By Theorem 3, Σ is a perfect family of involutions of X. By Lemma 5, there exists a selectiong map ω for Σ; let $\nu = \nu(\mathsf{T},\omega)$. By Lemma 6, ν satisfies the conditions **N1** and **N2**.

Now we prove that $\mathsf{T} = \Pi(\Sigma,\nu)$. Let $\mathsf{T}' = \Pi(\Sigma,\nu)$. Assume that $xy\mathsf{T}x't'$. If $x = y$ and $x' = y'$ then by Theorem 7 $xy\mathsf{T}'x'y'$. Assume that $x\neq y$, then with $\sigma = \sigma^x_y \in \Sigma$ we have $\sigma(x') = y'$; thus $x\sigma(x)\Pi x'\sigma(x')$ hence $x\sim_\sigma x'$ and $\nu(\sigma,x)\cdot\nu(\sigma,x') = 1\neq -1$; thus $xy\mathsf{T}'x'y'$.

On the other hand, if $xy\mathsf{T}'x'y'$ then there exists $\sigma\in\Sigma$ such that $\sigma(x) = y$ and $\sigma(x') = y'$ and $\nu(\sigma,x)\cdot\nu(\sigma,x')\neq -1$. If $\nu(\sigma,x)\neq 0$ and $\nu(\sigma,x')\neq 0$ then $\nu(\sigma,x)\cdot\nu(\sigma,x') = 1$ hence, by Lemma 6, $x\sim_\sigma x'$ and $xy\mathsf{T}x'y'$. Notice that $\nu(\sigma,x) = 0$ and $\nu(\sigma,x') = 0$ gives us $xy\mathsf{T}x'y'$ as well. ∎

All the above considerations we may recapitulate in

THEOREM 9 (representation theorem).

$$T = \Pi(\Sigma(T),\nu(T,\omega)).\ \blacksquare$$

In general, given any perfect family of involutions Σ of a nonempty set X and arbitrary functions $\nu_1,\nu_2:\Sigma\times X\mapsto$\{-1,0,1\} satisfying **N1** and **N2**; one asks when $\Pi(\Sigma,\nu_1) = \Pi(\Sigma,\nu_2)$. We have

THEOREM 10. *If σ is a perfect family of involutions of X, $\nu_1,\nu_2:\Sigma\times X\mapsto\{\text{-}1,0,1\}$ are functions satisfying* **N1** *and* **N2** *then the following conditions are equivalent:*

(i) $\Pi(\Sigma,\nu_1) = \Pi(\Sigma,\nu_2)$,

(ii) *for every $\sigma\in\Sigma$ there exists a constant $c = \pm 1$ such that $\nu_1(\sigma,x)\cdot\nu_2(\sigma,x) = c$, for every $x\in X\setminus\mathrm{Fix}(\sigma)$.*

Proof. (i⇒ii) Let $\sigma\in\Sigma$, $a\in X\setminus\mathrm{Fix}(\sigma)$. Assume e.g. $\nu_1(\sigma,a)\cdot\nu_2(\sigma,a) = 1$. Let $y\in X\setminus\mathrm{Fix}(\sigma)$ with $\nu_1(\sigma,y)\cdot\nu_2(\sigma,y) = -1$. If $\nu_1(\sigma,a) = \nu_1(\sigma,y) = 1$, then $\nu_2(\sigma,a) = 1$ and $\nu_2(\sigma,y) = \nu_1(\sigma,\sigma(a)) = 1 = \nu_1(\sigma,\sigma(y)) = \nu_2(\sigma,\sigma(a)) = -1$. Hence $a\sigma(a)\Pi(\Sigma,\nu_1)y\sigma(y)$ but not $a\sigma(a)\Pi(\Sigma,\nu_2)y\sigma(y)$.

(ii$\Rightarrow$i) Let $\sigma \in \Sigma$ and e.g. $\nu_1(\sigma, x) \cdot \nu_2(\sigma, x) = 1$, for every $x \in X \setminus \mathrm{Fix}(\sigma)$. Assume $a\sigma(a)\Pi(\Sigma, \nu_1)b\sigma(b)$ for some $a, b \in X \setminus \mathrm{Fix}(\sigma)$; then $\nu_1(\sigma, a) \cdot \nu_2(\sigma, b) = 1$. If $\nu_1(\sigma, a) = 1$ then $\nu_1(\sigma, b) = 1$ and by assumption $\nu_2(\sigma, b) = 1$. To prove $a\sigma(a)\Pi(\Sigma, \nu_2)b\sigma(b)$ it suffices to show that $\nu_2(\sigma, a) = 1$. Assume $\nu_2(\sigma, b) = -1$ then $\nu_2(\sigma, a) \cdot \nu_2(\sigma, b) = -1$. Hence $\nu_1(\sigma, a) \cdot \nu_2(\sigma, a) \cdot \nu_1(\sigma, b) \cdot \nu_2(\sigma, b) = -1$ but it is not true. ■

References

[1] K. Prażmowski, *On the isosceles trapezium configuration in an abstract way*, Bull. Polish Acad. Sci., Math., Vol. 32, No 3-4 (1985).

[2] K. Prażmowski, *Geometry over groups with central symetries as the only involutions*, Mitteeilungen aus der Math. Seminar, Giessen, 1990.

[3] E. Sperner, *Die Ordungsfunctionen einer Geometrie*, Math. Ann., Bd. 121, 1149.

INSTITUTE OF MATHEMATICS
UNIVERSITY OF BIAŁYSTOK
ul. Akademicka 2
15-267 BIAŁYSTOK, POLAND

Received May 6, 1996.