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A CHARACTERIZATION OF COMPLEX $\mathcal{AUMD}$ BANACH SPACES VIA TANGENT MARTINGALES

1. Introduction

In this paper we will characterize complex separable Banach spaces $\mathcal{B}$ in which $\mathcal{B}$-valued analytic martingale difference sequences are unconditional (so called $\mathcal{AUMD}$ spaces). In these characterizations we use notion of tangent analytic martingales and tangent analytic Gaussian martingales, defined in this paper. These characterizations are analogical to the well-known theorem [4, p. 285] for $\mathcal{UMD}$ Banach spaces i.e. spaces in which all martingale difference sequences are unconditional.

THEOREM. *Let $\mathcal{B}$ be a Banach space $\mathcal{UMD}$. If p satisfies $1 < p < \infty$, then there exists a constant c_p (depending only on $\mathcal{B}$ and p) such that for every pair $\{d_k\}$ and $\{d_k^*\}$ of $\mathcal{B}$-valued tangent martingales difference sequences such that $E\|d_k\|^p$ and $E\|d_k^*\|^p$ are finite for each k, we have*

$$E\Big\|\sum_{k=0}^{n} d_k^*\Big\|^p \le c_p E\Big\|\sum_{k=0}^{n} d_k\Big\|^p, \quad n = 1, 2, \ldots$$

We shall prove analogical theorems to the above one for $\mathcal{AUMD}$ Banach spaces. We remark that the class of $\mathcal{AUMD}$ Banach spaces is strictly larger than the class of complex $\mathcal{UMD}$ Banach spaces and includes such space as complex $L^1[0,1]$ which is not $\mathcal{UMD}$ space.

2. Main definitions

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{F}_0 = \{\emptyset, \Omega\} \subset \mathcal{F}_1 \subset \ldots \mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \ldots \subset \mathcal{F}$ be a filtration (a nondecreasing sequence of sub-σ-fields), $\mathcal{B}$ be a complex separable Banach space, $\mathcal{B}^*$ be a dual space of $\mathcal{B}$ and $M_n = \sum_{k=0}^{n} d_k$ be a B-valued martingale with respect to $(\mathcal{F}_n)$ (i.e. for each integer k,

d_k are measurable relative to $\mathcal{F}_k$ and Bochner integrable functions with $E(d_{k+1}|\mathcal{F}_k) = 0)$.

DEFINITION 1. A sequence of random variables (v_k) is $(\mathcal{F}_k)$-predictable if v_k is $\mathcal{F}_{k-1}$-measurable for $k = 1, 2, \ldots$

DEFINITION 2. Let $\Omega = [0,1]^{\mathcal{N}}$, suppose that $\mathcal{F}$ is the product σ-field of Borel subsets of $[0,1]$ and P the product measure of the normalized Lebesgue maesures. We take the filtration $\mathcal{F}_n$ on Ω, where $\mathcal{F}_n$ stand for the σ-field generated by the first n coordinates $\theta_1, ..\theta_n$ of $\theta = (\theta_k) \in \Omega$.

An analytic martingale is a sequence $(M_n)_{n=0}^{\infty}$ of $\mathcal{B}$-valued functions of the following form

$$M_n = d_0 + \sum_{k=1}^{n} h_{k-1}(\theta_1, \ldots, \theta_{k-1}) e^{2\pi\imath\theta_k},$$

where $d_0, h_0 \in \mathcal{B}$ and h_k are $\mathcal{B}$-valued functions of k-variables, $\mathcal{F}_k$ measurable (with the convention that $\mathcal{F}_0 = \{\emptyset, \Omega\}$).

DEFINITION 3. If $(\varepsilon_i)_{i=0}^{n}$ is a deterministic sequence of numbers from $\{-1, 1\}$, then the martingale $N_n = \sum_{k=0}^{n} \varepsilon_k d_k$ is called (ε_i)-transform of the martingale M_n.

DEFINITION 4. A complex Banach space $\mathcal{B}$ is called $\mathcal{AUMD}$ (Analytic Unconditional Martingale Differences), if for each $p > 1$ there exists a constant c_p which depends only on p and $\mathcal{B}$ such that for every $\mathcal{B}$-valued analytic martingale M_n and its every (ε_i)-transform N_n we have

$$E\|M_n\|^p \leq c_p E\|N_n\|^p.$$

DEFINITION 5. A real-valued function $\Psi : \mathcal{B} \times \mathcal{B} \to [-\infty, \infty)$ is said to be skew-plurisubharmonic if for every $x, y, u \in \mathcal{B}$ the following two functions (defined on the complex plane) $g_{\pm}(z) = \Psi(x + uz, y \pm uz)$ are subharmonic i.e. $g_{\pm}(z)$ is upper semicontinuous and for all $a \in \mathcal{C}$, $r \in \mathcal{R}$, $\int_0^1 g_{\pm}(a + re^{2\pi\imath\theta}) d\theta \geq g_{\pm}(a)$, where $g_{\pm}(z)$ stands for either $g_+(z) = \Psi(x + uz, y + uz)$ or $g_-(z) = \Psi(x + uz, y - uz)$.

DEFINITION 6. A complex random variable $\xi = \xi_1 + \imath\xi_2$ is called complex Gaussian if ξ_1, ξ_2 are independent real-valued centered Gaussian random variables with equal variances; if additionally variances are equal to 1 then ξ is called standard complex Gaussian.

DEFINITION 7. A random vector G with values in a complex Banach space $\mathcal{B}$ is Gaussian, if $\varphi(G)$ is a complex Gaussian variable for all $\varphi \in \mathcal{B}^*$.

DEFINITION 8. A martingale $M_n = \sum_{k=0}^{n} d_k$ with values in a complex Banach space $\mathcal{B}$ is analytic Gaussian if for each positive integer k, d_k has $\mathcal{F}_{k-1}$-conditinally almost surely the distribution of a Gaussian $\mathcal{B}$-valued vector.

DEFINITION 9. Two $(\mathcal{F}_n)$ - adapted martingales $M_n = \sum_{k=0}^{n} d_k$ and $M'_n = \sum_{k=0}^{n} d'_k$ are said to be tangent relative to $(\mathcal{F}_n)$ if for each $n \in N$ the pair (d_n, d'_n) is conditionally i.i.d. almost surely, given $\mathcal{F}_{n-1}$.

3. Main results

Let us start with the following fact.

PROPOSITION 1 [3, see Proposition 2.1.2 p.70]. *If a real-valued function $\Psi : D \subset \mathcal{B} \to [-\infty, \infty)$ is upper semicontinuous and bounded from above then the sequence of functions $\Psi_k(x) = \sup_{y \in D}\{\Psi(y) - k\|x - y\|\}$ defined on D for $k = 1, 2, 3 \ldots$ fulfills:*

1. *Ψ_k is uniformly continuous on D;*
2. *$\Psi_1 \le \sup_{x \in D} \Psi(x)$;*
3. *$\Psi_{k+1}(x) \le \Psi_k(x)$ and $\lim_{k\to\infty} \Psi_k(x) = \Psi(x)$ for $x \in D$.*

The proof of this fact for $\mathcal{B} = C$ (for the complex plane) from the book of Steven Krantz [3] carry over to the case of arbitrary Banach space $\mathcal{B}$ by substituting the absolute value with the norm.

We shall need also the following simple lemma proved in [5, Lemma 3].

LEMMA 1. *Let $p > 0$, $c_p \in \mathcal{R}$ and let d_k be random vectors with values in a complex Banach space $\mathcal{B}$. If the inequality*

$$E\Big\|\sum_{k=0}^{n} \varepsilon_k d_k\Big\|^p \le c_p E\Big\|\sum_{k=0}^{n} d_k\Big\|^p,$$

holds true for all sequences of predictable random variables (respectively for all non-random sequences) (ε_k) with values in $\{-1, 1\}$, then for each predictable sequence $(x_k), (y_k)$ of real random variables (respectively for each non-random sequence) with $|x_k|, |y_k| \le 1$, we have

$$E\Big\|\sum_{k=0}^{n}(x_k + \imath y_k) d_k\Big\|^p \le d_p E\Big\|\sum_{k=0}^{n} d_k\Big\|^p,$$

where

$$d_p = \begin{cases} \dfrac{2c_p}{2^p - 1} & \textit{for } p \in (0,1] \\ 2^p c_p & \textit{for } p \geq 1. \end{cases}$$

Next theorems characterize $\mathcal{AUMD}$ Banach spaces via analytic tangent martingales.

THEOREM 1. *A complex Banach space $\mathcal{B}$ is $\mathcal{AUMD}$ iff for each $p \in (0,\infty)$ there exists a constant c_p depending only on p and $\mathcal{B}$ such that*

$$E\Big\|d_0 + \sum_{k=1}^{n} h_{k-1}(\theta_1,\ldots,\theta_{k-1},\theta'_1,\ldots,\theta'_{k-1})e^{2\pi\imath\theta_k}\Big\|^p$$
$$\leq c_p E\Big\|d_0 + \sum_{k=1}^{n} h_{k-1}(\theta_1,\ldots,\theta_{k-1},\theta'_1,\ldots,\theta'_{k-1})e^{2\pi\imath\theta'_k}\|^p,$$

where h_k are measurable (with respect to σ-field generated by the first coordinates $\theta_1,..\theta_k,\theta'_1,..\theta'_k$) and bounded functions with values in $\mathcal{B}$ and $h_0, d_0 \in \mathcal{B}$ are constant vectors.

Proof. $\Rightarrow$ We shall use Fubini Theorem and Lemma 1 with constants $x_k + \imath y_k = e^{2\pi\imath\theta'_k}$ for $k = 1,2,\ldots$ and $x_0 + \imath y_0 = 1$. Let us fix $\theta'_1, \theta'_2, \ldots$ and take $d_k = h_{k-1}(\theta_1,\ldots,\theta_{k-1},\theta'_1,\ldots\theta'_{k-1})e^{2\pi\imath\theta_k}$ in Lemma 1. Since $\mathcal{B}$ is $\mathcal{AUMD}$ Banach space and $e^{2\pi\imath\theta'_k}$ is from the unite circle, then the assumptions of Lemma 1 are satisfied. Hence

$$E\Big\|d_0 + \sum_{k=1}^{n} h_{k-1}(\theta_1,\ldots,\theta_{k-1},\theta'_1,\ldots\theta'_{k-1})e^{2\pi\imath\theta_k}\Big\|^p$$
$$= \int_0^1 \ldots \int_0^1 \Big\|d_0 + \sum_{k=1}^{n} h_{k-1}(\theta_1,\ldots,\theta_{k-1},\theta'_1,\ldots\theta'_{k-1})$$
$$\times e^{2\pi\imath\theta_k}\Big\|^p d\theta_1,\ldots,d\theta_k, d\theta'_1,\ldots d\theta'_k$$
$$\leq d_p \int_0^1 \ldots \int_0^1 \Big\|d_0 + \sum_{k=1}^{n} h_{k-1}(\theta_1,\ldots,\theta_{k-1},\theta'_1,\ldots\theta'_{k-1})$$
$$\times e^{2\pi\imath\theta'_k} e^{2\pi\imath\theta_k}\Big\|^p d\theta_1,\ldots,d\theta_k, d\theta'_1,\ldots d\theta'_k$$
$$= d_p \int_0^1 \ldots \int_0^1 \Big\|d_0 + \sum_{k=1}^{n} h_{k-1}(\theta_1,\ldots,\theta_{k-1},\theta'_1,\ldots\theta'_{k-1})$$
$$\times e^{2\pi\imath\theta'_k} e^{2\pi\imath\theta_k}\Big\|^p d\theta'_1,\ldots d\theta'_k, d\theta_1,\ldots,d\theta_k$$

$$\leq d_p^2 E\Big\|d_0 + \sum_{k=1}^{n} h_{k-1}(\theta_1, \ldots, \theta_{k-1}, \theta_1', \ldots \theta_{k-1}') e^{2\pi i \theta_k'}\Big\|^p.$$

$\Leftarrow$ Let (ε_k) be a sequence of numbers in $\{-1, 1\}$. Since $\varepsilon_0 \varepsilon_k e^{2\pi i \theta_k'}$ and $e^{2\pi i \theta_k'}$ have the same distribution we have by the assumptions

$$E\Big\|d_0\varepsilon_0 + \sum_{k=1}^{n} \varepsilon_k h_{k-1}(\theta_1, \ldots, \theta_{k-1}) e^{2\pi i \theta_k}\Big\|^p$$

$$\leq c_p E\Big\|d_0 + \sum_{k=1}^{n} \varepsilon_0 \varepsilon_k h_{k-1}(\theta_1, \ldots, \theta_{k-1}) e^{2\pi i \theta_k'}\Big\|^p$$

$$= c_p E\Big\|d_0 + \sum_{k=1}^{n} h_{k-1}(\theta_1, \ldots, \theta_{k-1}) e^{2\pi i \theta_k'}\Big\|^p$$

$$\leq c_p^2 E\Big\|d_0 + \sum_{k=1}^{n} h_{k-1}(\theta_1, \ldots, \theta_{k-1}) e^{2\pi i \theta_k}\Big\|^p.$$

This completes the proof.

Now we shall give a characterization of $\mathcal{AUMD}$ Banach spaces via analytic Gaussian martingales. We shall start with the following lemma.

LEMMA 2. *Let $\beta \geq 1$ and let $\mathcal{B}$ be a complex Banach space such that*

$$E\|M_n\| \leq E\beta\|N_n\|,$$

for all martingales $M_n = d_0 + \sum_{k=1}^{n} h_k(\xi_1, \ldots, \xi_{k-1})\xi_k$ with respect to the filtration $\mathcal{F}_n = \sigma(\xi_1, \ldots \xi_n)$ and its (ε_i)-transforms N_n, where $h_1 = x \in \mathcal{B}$, $h_k : \mathcal{C}^{k-1} \to \mathcal{B}$ are measurable and bounded functions and $(\xi_k)_{k=1}^{\infty}$ is a sequance of independent, standard complex Gaussian random variables. Let $\Phi(x, y) = \beta\|y\| - \|x\|$ for $x, y \in \mathcal{B}$ and let us define

$$\Psi_n(x, y) = \inf_{M_n, N_n, d_0 = 0, \|h_k\| \leq n} E\Phi(x + M_n, y + N_n),$$

where infimum is taken over all martingales M_n of the above form and its (ε_i)-transforms N_n. Then

$$\Psi(x, y) = \lim_{n \to \infty} \Psi_n(x, y)$$

for $x, y \in \mathcal{B}$ exists and is the maximal function among functions $\overline{\Psi}$ satisfying the following conditions:

1. $\overline{\Psi}$ *is upper semicontinuous and locally bounded*;
2. *for each* $x, y, u \in \mathcal{B}$ $E\overline{\Psi}(x+u\xi, y \pm u\xi)$ *is finite and*

$$E\overline{\Psi}(x+u\xi, y\pm u\xi) \geq \overline{\Psi}(x,y),$$

where ξ *is a standard complex Gaussian variable*;

3. $\overline{\Psi}(x,y) \leq \Phi(x,y)$, $0 = \overline{\Psi}(0,0) \leq \overline{\Psi}(x, \pm x)$.

Proof. Ψ is well defined since $\Psi_{n+1} \leq \Psi_n$. Let us assume that another function $\overline{\Psi}$ enjoys conditions 1,2,3, then we have

$$\begin{aligned} &E\Phi(x+M_n, y+N_n) \geq \\ &\geq E\overline{\Psi}(x+M_n, y+N_n) \\ &= EE\Big[\overline{\Psi}\Big(x+\sum_{k=1}^{n-1} h_k\xi_k + h_n\xi_n, y + \sum_{k=1}^{n-1}\varepsilon_k h_k \xi_k + \varepsilon_n h_n \xi_n\Big) \mid \xi_1, \ldots, \xi_{k-1}\Big] \\ &\geq \ldots \geq \overline{\Psi}(x,y) \end{aligned}$$

(by the induction over n). Hence $\Psi_n \geq \overline{\Psi}$ and $\Psi \geq \overline{\Psi}$. Putting $M_n \equiv 0, N_n \equiv 0$ in the definition of Ψ_n we obtain, that $\Psi_n \leq \Phi$. Clearly, Ψ_n is upper semicontinuous as the infimum of continuous functions. Since $0 \leq \Psi(0,0) \leq \Phi(0,0) = 0$ then $\Psi(0,0) = 0$. Moreover, applying $E\Phi(y+M_n, \pm y + N_n) \geq 0$, we have

$$\begin{aligned} \Psi_n(x,y) &= \inf_{M_n, N_n, d_0 = 0, \|h_k\| \leq n} E\Phi(x+M_n, y+N_n) \\ &= \inf_{M_n, N_n, d_0 = 0, \|h_k\| \leq n} E\beta\|y+N_n\| - E\|x+M_n\| \\ &\geq \inf_{M_n, N_n, d_0 = 0, \|h_k\| \leq n} E\beta\|y+N_n\| - E\|y+M_n\|) - \|x-y\| \\ &\geq -\|x-y\|. \end{aligned}$$

Hence $-\|x-y\| \leq \Psi(x,y) \leq \Psi_n(x,y) \leq \Phi(x,y) = \beta\|y\| - \|x\| \leq \beta\|y\| + \|x\|$. Therefore $\Psi^+(x,y) \leq \beta\|y\| + \|x\|$, $\Psi^-(x,y) \leq \|x-y\|$. So $\|\Psi_n\| \leq (\beta + 1)\|y\| + 2\|x\|$ and $\|\Psi\| \leq (\beta+1)\|y\| + 2\|x\|$. Hence $E\Psi(x+u\xi, y \pm u\xi)$ is finite and Ψ_n, Ψ are locally bounded.

To show that $E\Psi(x+u\xi, y\pm u\xi) \geq \Psi(x,y)$ it suffies to show that for the function Ψ_n and $\|u\| \leq n$ we have $E\Psi_n(x+u\xi, y \pm u\xi) \geq \Psi_{n+1}(x,y)$. By Lebesgue Theorem we have

$$\begin{aligned} E\Psi(x+u\xi, y\pm u\xi) &= \lim_{n\to\infty, n \geq \|u\|} E\Psi_n(x+u\xi, y\pm u\xi) \\ &\geq \lim_{n\to\infty} \Psi_{n+1}(x,y) = \Psi(x,y). \end{aligned}$$

Let us define a norm on $\mathcal{B} \times \mathcal{B}$ as $\|(x,y)\| := \|x\| + \|y\|$. We have

$$\begin{aligned} |\Phi(x,y) - \Phi(x_1,y_1)| &= |\beta\|y\| - \|x\| - \beta\|y_1\| + \|x_1\|| \\ &\le |\beta\|y\| - \beta\|y_1\|| + |\|x_1\| - \|x\|| \\ &\le \beta(\|y - y1\| + \|x - x_1\|) \\ &= \beta\|(x,y) - (x_1,y_1)\|. \end{aligned}$$

Hence $\Phi(x,y)$ is the Lipschitz function with the constant β.

Let us fix $x, y, u \in \mathcal{B}, \|u\| \le n$, $c > 0$. Now let us divide the complex plane $\mathcal{C}$ into countably many disjoint squeres K_i without the top and the right edges such that the dimeters d_i of sets $D_i \subset \mathcal{B} \times \mathcal{B}, D_i = \{(x,y) + (u, \pm u)z,\ z \in K_i\}$ satisfy

$$d_i = \sup_{z_1,z_2 \in K_i} \{(u,\pm u)(z_1 - z_2)\} \le 2\|u\| \sup_{z_1,z_2\in K_i} |z_1 - z_2| \le \frac{c}{\beta}.$$

Now we define for all $m \in \mathcal{N}$ and for $(\overline{x},\overline{y}) \in \{\ (x,y) + (u,\pm u)z,\ z \in \mathcal{C}\}$ the following functions

$$\Psi_n^m(\overline{x},\overline{y}) = \sup_{(x_1,y_1)\in D_i} \{\Psi_n(x_1,y_1) - m\|(\overline{x},\overline{y}) - (x_1,y_1)\|\} \text{ for } (\overline{x},\overline{y}) \in D_i.$$

Since Φ is Lipschitz function we have that Ψ_n is bounded from above on D_i. Hence from Proposition 2 $\lim_{m\to\infty} \Psi_n^m = \Psi_n$ and the functions $\xi \to \Psi_n^m(x + u\xi, y \pm \xi)$ are continuous on K_i. Let us fix $\varepsilon > 0$. Then by the definition of Ψ_n, for each $\xi \in K_i$, we have

$$\begin{aligned} \Psi_n^m(x + u\xi, y \pm u\xi) &\ge \Psi_n(x + u\xi, y \pm u\xi) \\ &\ge E\Phi(x + u\xi + M_n^\xi, y \pm u\xi + N_n^\xi) - \varepsilon, \end{aligned}$$

for some martingale M_n^ξ and its (ε_i)-transform N_n^ξ.

Let us observe that by taking in the definition of the function Ψ_n^m the supremum over $\overline{D_i} = \{(x,y) + (u,\pm u)z\ ,\ z \in \overline{K_i}\}$ we can extend Ψ_n^m continuously on $\overline{D_i}$. Since $\Psi_n \le \sup_{x\in\overline{D_i}} |\Phi(x)|$ and $\overline{D_i}$ is bounded, the extention is bounded from above.

Let us define for fixed $\xi \in \overline{K_i}$ the following functions

$$g_\xi(\eta) = \Psi_n^m(x + u\eta, y \pm u\eta) - E\Phi(x + u\eta + M_n^\xi, y \pm u\eta + N_n^\xi).$$

The Lebesgue Theorem implies that g_ξ is continuous and $g_\xi(\xi) > -\varepsilon$. By the comptactness of $\overline{K_i}$ we have, that there exist finitely many points ξ_i^s in K_i and disjoint sets $P_i^s, \xi_i^s \in P_i^s$ covering K_i and corresponding martingales $M_n^{\xi_i^1}, \ldots M_n^{\xi_i^{n_i}}$ such that $\|h_k^{\xi_i^s}\| \le n$ and $g_{\xi_i^s}(\eta) > -\varepsilon$ for $\eta \in P_i^s, s = 1,2,\ldots,n_i$. From these martingales we build for $\xi \in K_i$ a random vector

$M_n^i(\xi,\xi_1,\ldots,\xi_n) = u\xi + M_n^{\xi_i^s}$ for $\xi \in P_i^s$ such that we have

$$\Psi_n^m(x+u\xi, y\pm u\xi) \geq E\Phi(x+M_n^i, y+N_n^i) - \varepsilon.$$

Since we have countably many sets K_i, we have also countably many martingales. Since $\|u\| \leq n, \|h_k^{\xi_i^s}\| \leq n$ then for these martingales h_k^i are measurable and uniformly bounded by n. Finally, let us define a martingale M_n as follows:

$$M_{n+1}(\xi,\ldots\xi_n) = M_n^i(\xi,\ldots\xi_n) \quad \text{for } \xi \in K_i.$$

For this martingale the functions h_k are measurable and $\|h_k\| \leq n+1$. Moreover the following inequality is true

$$\Psi_n^m(x+u\xi, y\pm u\xi) \geq E\Phi(x+M_{n+1}, y+N_{n+1}) - \varepsilon.$$

If ξ is a standard complex Gaussian random variable independent on $\xi_1,\ldots$ $\ldots,\xi_n$, then integrating both sides and applying Fubini theorem and the definition of function Ψ_{n+1} we obtain

$$E\Psi_n^m(x+u\xi, y\pm u\xi) \geq \Psi_{n+1}(x,y) - \varepsilon.$$

Now passing to infinity with m we obtain

$$E\Psi_n(x+u\xi, y\pm u\xi) \geq \Psi_{n+1}(x,y) - \varepsilon.$$

This completes the proof of lemma because ε was arbitrary small.

To prove Theorem 3 we will need the following results proved in [5].

LEMMA 3. *Let $\Phi : \mathcal{B}\times\mathcal{B} \to [-\infty,\infty)$ be a continuous function locally bounded from above (i.e. a function bounded from above on bounded sets), $\Psi(x,y) = \inf_{M_n,N_n} E\Phi(x+M_n, y+N_n)$, where the infimum is taken over all bounded analytic martingales M_n starting from $x=0$ and their ε_i-transforms N_n.*

Then Ψ is a maximal skew-plurisubharmonic function such that

$$\Psi(x,y) \leq \Phi(x,y) \quad \textit{for} \quad (x,y) \in \mathcal{B}\times\mathcal{B}.$$

THEOREM 2. *A complex Banach space $\mathcal{B}$ is $\mathcal{AUMD}$ iff there exists a function $\Psi : \mathcal{B}\times\mathcal{B} \to [-\infty,\infty)$ such that*

1. *Ψ is skew-plurisubharmonic;*
2. *$\Psi(x,\pm x) \geq \Psi(0,0) > 0$;*
3. *$\Psi(x,y) \leq \Psi(0,0) + \|y\|$ for $x,y \in \mathcal{B}$;*
4. *$\Psi(x,y) \leq \|y\|$ on the set $\{(x,y) : \|x\|+\|y\| \geq 1\}$.*

Now we shall state and prove the main theorem.

THEOREM 3. *A complex separable Banach space $\mathcal{B}$ is $\mathcal{AUMD}$ iff for all tangent analytic Gaussian martingales M_n, M_n' with values in $\mathcal{B}$ and a fixed $p > 0$ there exists a constant $d_p \geq 1$ such that*

$$E\|M_n\|^p \leq d_p E\|M_n'\|^p.$$

Proof. $\Rightarrow$ Let $\mathcal{B}$ be a separable $\mathcal{AUMD}$ Banach space with constant c_p for $p > 0$. Let us take the function Ψ_p from Lemma 3 for the following function

$$\Phi_p(x,y) = c_p\|y\|^p - \|x\|^p \quad \text{if } p \in (0,1],$$
$$\Phi_p(x,y) = 2^{p-1}c_p\|y\|^p - \|x\|^p \quad \text{if } p > 1.$$

First we will check integrability of the function $\Psi_p(x+\xi, y+\eta)$, where ξ and η are $\mathcal{B}$-valued Gaussian vectors. Let us notice that for all $x, y \in \mathcal{B}$ we have

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p \quad \text{if } p \in (0,1]$$

and

$$\|x+y\|^p \leq 2^{p-1}(\|x\|^p + \|y\|^p) \quad \text{if } p > 1.$$

Hence, for $p \in (0,1]$, we have

$$-\|x+M_n\|^p = -\|y+M_n+x-y\|^p \geq -\|y+M_n\|^p - \|x-y\|^p$$

and for $p > 1$, we have

$$-\|x+M_n\|^p \geq -2^{p-1}\|y+M_n\|^p - 2^{p-1}\|x-y\|^p.$$

Since $\mathcal{B}$ is $\mathcal{AUMD}$, we have for $p \in (0,1]$

$$\begin{aligned}\Psi_p(x,y) &= \inf_{M_n,N_n} c_p\|y+N_n\|^p - \|x+M_n\|^p \\ &\geq \inf_{M_n,N_n} c_p\|y+N_n\|^p - \|y+M_n\|^p - \|x-y\|^p \geq -\|x-y\|^p\end{aligned}$$

and for $p > 1$, we have

$$\begin{aligned}\Psi_p(x,y) &= \inf_{M_n,N_n} c_p 2^{p-1}\|y+N_n\|^p - \|x+M_n\|^p \\ &\geq \inf_{M_n,N_n} 2^{p-1}(c_p\|y+N_n\|^p - \|y+M_n\|^p) - 2^{p-1}\|x-y\|^p \\ &\geq -2^{p-1}\|x-y\|^p.\end{aligned}$$

Hence $\Psi_p^+(x+\xi, y+\eta)$ and $\Psi_p^-(x+\xi, y+\eta)$ are integrable. So $\Psi_p(x+\xi, y+\eta)$ is integrable for $p > 0$. Moreover, $\Psi_p(x, \pm x) \geq \Psi_p(0,0) = 0$.

Let $M_n = \sum_{k=0}^{n} d_k$, $M'_n = \sum_{k=0}^{n} d'_k$ be two tangent analytic Gaussian martingales. Then $\frac{d_k+d'_k}{\sqrt{2}}$, $\frac{d_k-d'_k}{\sqrt{2}}$ are $\mathcal{F}_{k-1}$-conditionally independent and they have $\mathcal{F}_{k-1}$-conditionally the same symmetric Gaussian distributions as d_k and d'_k.

Let $d_p = c_p$ for $p \in (0,1]$ and $d_p = 2^{p-1}c_p$ for $p > 1$.

If $\mathcal{B}$ is $\mathcal{AUMD}$ then the function Ψ_p is skew-plurisubharmonic and such that $\Psi_p(0,0) = 0$ and $\Psi_p(x,y) \le d_p\|y\|^p - \|x\|^p$. Hence

$$
\begin{aligned}
&d_pE\|M'_n\|^p - E\|M_n\|^p\\
&\ge E\Psi_p(M_n, M'_n) = EE[\Psi_p(M_{n-1} + d_n, M'_{n-1} + d'_n)|\mathcal{F}_{n-1}]\\
&= EE[\Psi_p(M_{n-1} + \tfrac{d_n+d'_n}{\sqrt{2}}, M'_{n-1} + \tfrac{d_n-d'_n}{\sqrt{2}})|\mathcal{F}_{n-1}]\\
&= \int_0^1\int_0^1 EE[\Psi_p(M_{n-1} + \tfrac{e^{2\pi i\theta}d_n+e^{2\pi i\theta'}d'_n}{\sqrt{2}}, M'_{n-1} + \tfrac{e^{2\pi i\theta}d_n-e^{2\pi i\theta'}d'_n}{\sqrt{2}})|\mathcal{F}_{n-1}]d\theta d\theta'\\
&= \int_0^1 EE\int_0^1[\Psi_p(M_{n-1} + \tfrac{e^{2\pi i\theta}d_n+e^{2\pi i\theta'}d'_n}{\sqrt{2}}, M'_{n-1} + \tfrac{e^{2\pi i\theta}d_n-e^{2\pi i\theta'}d'_n}{\sqrt{2}})|\mathcal{F}_{n-1}]d\theta' d\theta\\
&\ge \int_0^1 EE[\Psi_p(M_{n-1} + \tfrac{e^{2\pi i\theta}d_n}{\sqrt{2}}, M'_{n-1} + \tfrac{e^{2\pi i\theta}d_n}{\sqrt{2}})|\mathcal{F}_{n-1}]d\theta\\
&\ge E\Psi_p(M_{n-1}, M'_{n-1}) \ge \ldots \ge \Psi_p(d_0, \pm d_0) \ge \Psi(0,0) = 0
\end{aligned}
$$

(by the induction over n).

$\Leftarrow$ Let us take the following martingales

$$
M_n = d_0 + \sum_{k=0}^{n} h_k(\xi_1, \ldots, \xi_{k-1})\xi_k,
$$

$$
M'_n = d_0 + \sum_{k=0}^{n} h_k(\xi_1, \ldots, \xi_{k-1})\xi'_k,
$$

where h_k are measurable and bounded $\mathcal{B}$-valued functions, $\xi_1, \ldots, \xi_n$, $\xi'_1, \ldots, \xi'_n$ are two independent copies of independent sequence of standard complex Gaussian random variables.

If we take (ε_i)-transform N_n of the martingale M_n than M_n, M'_n are tangent analytic Gaussian and $M'_n, \varepsilon_0 N_n$ are also tangent analytic Gaussian. Hence

$$
E\|M_n\|^p \le d_pE\|M'_n\|^p \le d_p^2\|N_n\|^p.
$$

Let $\Phi(x,y) = d_1^2\|y\| - \|x\|$. In this case the function Ψ from Lemma 2 is the maximal (the maximality is crucial for the proof) function satisfying:

a) for all $x, u \in \mathcal{B}$ $E\Psi(x + u\xi, y \pm u\xi) \geq \Psi(x, y)$, where ξ is a standard complex Gaussian random variable;

b) $\Psi(x, \pm x) \geq \Psi(0,0) = 0$ $\Psi \leq \Phi$. The proof will be finished if we show that Ψ is skew-plurisubharmonic because of Theorem 2.

Let us define $\Psi_r(x,y) = E\Psi(x+ur\xi_1, y\pm ur\xi_1)$ for $r > 0, u \in \mathcal{B}$, where ξ_1 is a standard complex Gaussian random variable. Clearly, $\Psi_r(x,y) \geq \Psi(x,y)$ and $\Psi_r(x,y)$ is upper semicontinuous. We shall show that $\lim_{r\to 0}\Psi_r(x,y) = \Psi(x,y)$. Let us notice that $\Psi_{r+s}(x,y) \geq \Psi_s(x,y)$ if $r, s > 0$. Namely, if ξ_1^1, ξ_1^2 are two independent copies of ξ_1, then

$$\begin{aligned}\Psi_{r+s}(x,y) &= E\Psi(x + u(r+s)\xi_1, y \pm u(r+s)\xi_1)\\ &= E\Psi(x + u(r\xi_1^1 + \sqrt{s^2+2rs}\,\xi_1^2), y \pm u(r\xi_1^1 + \sqrt{s^2+2rs}\,\xi_1^2))\\ &\geq E\Psi(x + ur\xi_1^1, y + ur\xi_1^1) = \Psi_r(x,y).\end{aligned}$$

Let us assumme that $\widetilde{\Psi}(x,y) = \lim_{r\to 0}\Psi_r(x,y)$. Clearly $\widetilde{\Psi}(x,y) \geq \Psi(x,y)$.

Since $\Psi^+ \leq d_1^2\|y\| + \|x\|$ and $\Psi^- \leq \|x-y\|$, then

$$\begin{aligned}|\Psi_r(x,y)| &\leq E|\Psi(x + ur\xi_1, y \pm ur\xi_1)|\\ &= E(\Psi^+ + \Psi^-)(x + ur\xi_1, y \pm ur\xi_1)\\ &\leq E(d_1^2+1)\|y \pm ur\xi_1\| + E2\|x + ur\xi_1\|\\ &\leq (d_1^2+1)\|y\| + 2\|x\| + (d_1^2+3)\|u\|rE|\xi_1|.\end{aligned}$$

If $r < 1$, then $|\Psi_r(x,y)| \leq (d_1^2+1)\|y\| + 2\|x\| + (d_1^2+3)\|u\|E|\xi_1|$ and $|\Phi(x + ur\xi_1, y\pm ur\xi_1)| = |d_1^2\|y\pm ur\xi_1\| - \|x+ur\xi_1\|| \leq\leq d_1^2\|y\| + \|x\| + (d_1^2+1)\|u\||\xi_1|$.

Hence from Lebesgue Theorem we obtain

$$\begin{aligned}\widetilde{\Psi}(x,y) &= \lim_{r\to 0} E\Psi(x + ur\xi_1, y \pm ur\xi_1)\\ &\leq \lim_{r\to 0} E\Phi(x + ur\xi_1, y \pm ur\xi_1)\\ &= E\lim_{r\to 0} d_1^2\|y \pm ur\xi_1\| - \|x + ur\xi_1\| = d_1^2\|y\| - \|x\| = \Phi(x,y).\end{aligned}$$

Since $r < 1$ then we have

$$|\Psi_r(x + v\xi, y \pm v\xi)| \leq (d_1^2+1)\|y\| + 2\|x\| + (d_1^2+3)\|v\||\xi| + (d_1^2+3)\|u\|E|\xi_1|.$$

Then again from Lebesgue Theorem follows that

$$E\widetilde{\Psi}(x + v\xi, y \pm v\xi) = \lim_{r\to 0} E\Psi_r(x + v\xi, y \pm v\xi) \geq \lim_{r\to 0}\Psi_r(x,y) = \widetilde{\Psi}(x,y).$$

Hence from Fubini Theorem we have

$$E\Psi_r(x + v\xi, y \pm v\xi) = EE\Psi_r(x + u\xi_1 + v\xi, y + u\xi_1 \pm v\xi)$$
$$\overset{\text{for } \xi, \xi_1 \text{indep.}}{=} E\Psi_r(x + u\xi_1 + v\xi, y + u\xi_1 \pm v\xi)$$
$$\geq E\Psi_r(x + u\xi_1, y + u\xi_1) = \Psi_r(x, y).$$

So $\widetilde{\Psi}$ satisfies the following conditions:

1. $\widetilde{\Psi} \leq d_1^2\|y\| - \|x\|$;
2. $E\widetilde{\Psi}(x + v\xi, y \pm v\xi) \geq \widetilde{\Psi}(x, y)$;
3. $\widetilde{\Psi} \geq \Psi$.

Finally, because Ψ is the maximal function satisfying the above conditions, then $\Psi = \lim_{r \to 0} \Psi_r = \Psi$.

To show that $z \to \Psi(x + vz, y \pm vz)$ is subharmonic it suffies to show that $z \to \Psi_r(x + vz, y \pm vz)$ is subharmonic. Applying again the Lebesgue theorem we obtain for $r < 1$ that

$$|\Psi_r(x + ve^{2\pi i\theta}, y \pm ve^{2\pi i\theta})| \leq (d_1^2 + 1)\|y\| + 2\|x\| + (d_1^2 + 3)(\|v\| + \|u\|E|\xi_1|).$$

Let us notice that in the definition of Ψ_r we have used a fix vector $u \in \mathcal{B}$, but we have showed that $\lim_{r \to 0} \Psi_r = \Psi$, independently on vector u . If we take $u = v$, in the definition of Ψ_r, then $\Psi_r(x + vz, y \pm vz) = E\Psi(x + v(z + \xi_1), y + v(\pm z + \xi_1)$ will be a smooth function of complex z. From a general theory of subharmonic functions follows that if $z \to \Psi_r(x + vz, y \pm vz)$ is smooth and $E\Psi_r(x+v\xi, y \pm v\xi) \geq \Psi_r(x, y)$, where ξ is standard 2-dimensional normal random variable, then the function $z \to \Psi_r(x+vz, y \pm vz)$ is subharmonic. Hence we have a skew-plurisubharmonic function such that

$$\Psi(x, \pm x) \geq \Psi(0, 0) = 0, \quad \Psi(x, y) \leq d_1^2\|y\| - \|x\|.$$

Finally let us define the following function

$$\Psi_1(x, y) = \frac{1 + \Psi(x, y)}{1 + d_1^2}.$$

We shall show that Ψ_1 satisfies conditions 1,2,3,4 from Theorem 2. Namely, we have

$$\Psi_1(0, 0) = \frac{1}{d_1^2 + 1} > 0, \quad \Psi_1(x, \pm x) = \Psi_1(0, 0) + \frac{\Psi(x, \pm x)}{d_1^2 + 1} \geq \Psi_1(0, 0).$$

For all $x, y \in \mathcal{B}$ we have

$$\Psi_1(x, y) \leq \frac{1 + d_1^2\|y\| - \|x\|}{d_1^2 + 1} \leq \Psi_1(0, 0) + \|y\|.$$

For $x, y \in \mathcal{B}$ such that $\|x\| + \|y\| \geq 1$ we have

$$\Psi_1(x,y) \leq \frac{1 + d_1^2\|y\| - \|x\|}{d_1^2 + 1} \leq \frac{(d_1^2+1)\|y\|}{d_1^2+1} = \|y\|.$$

Clearly, Ψ_1 is skew-plurisubharmonic as a linear transform of a skew-plurisubharmonic function Ψ. Hence $\mathcal{B}$ is a complex $\mathcal{AUMD}$ Banach space. This completes the proof.

The following theorems easily follow from the proof of Theorem 3.

THEOREM 4. *A complex separable Banach space $\mathcal{B}$ is $\mathcal{AUMD}$ iff there exists $\beta \geq 1$ such that*

$$E\|M_n\| \leq E\beta\|N_n\|$$

for any martingale $M_n = d_0 + \sum_{k=0}^{n} h_k(\xi_1, \ldots, \xi_{k-1})\xi_k$ and its any (ε_i)-transform N_n, where $h_k \in \mathcal{B}$ are measurable and bounded $\mathcal{B}$-valued functions, $h_1, d_0 \in \mathcal{B}$ and $\xi_1, \ldots, \xi_n$ is a sequence of standard, independent complex Gaussian random variables.

THEOREM 5. *A complex separable Banach space $\mathcal{B}$ is $\mathcal{AUMD}$ iff there exists $\beta \geq 1$ such that*

$$E\|M_n\| \leq E\beta\|M'_n\|$$

for all martingales

$$M_n = d_0 + \sum_{k=1}^{n} h_k(\xi_1, \ldots, \xi_{k-1})\xi_k,$$

$$M'_n = d_0 + \sum_{k=1}^{n} h_k(\xi_1, \ldots, \xi_{k-1})\xi'_k,$$

where $h_k \in \mathcal{B}$ are measurable and bounded $\mathcal{B}$-valued functions, $h_1, d_0 \in \mathcal{B}$ and $\xi_1, \ldots, \xi_n, \xi'_1, \ldots, \xi'_n$ are two independent copies of independent sequence of complex standard Gaussian random variables.

References

[1] D. J. H. Garling, *On martingales with values in a complex Banach space*, Math. Proc. Camb. Phil. Soc.(1988) 104–399.

[2] N. J. Kalton, *Plurisubharmonic functions on quasi-Banach spaces*, Studia Math. 84 (1986), 297–324.

[3] Steven G. Kranz, *Functions theory of several complex variables*, PWN, Warszawa 1991.

[4] T. R. McConnell, *Decoupling and stochastic integration in UMD Banach spaces*, Probab. Math. Statist. 10 (2) (1989), 283–295.
[5] M. Piasecki, *A geometrical characterization of AUMD Banach spaces via subharmonic functions*, Demonstratio Math. 30 (1997), 641–654.

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