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STABILITY AND ASYMPTOTIC PROPERTIES OF A $2n$-DIMENSIONAL SYSTEM OF DIFFERENTIAL EQUATIONS

Abstract. The stability and asymptotic properties of a real $2n$-dimensional system $x' = A(t)x + h(t,x)$ are studied. Here $A(t)$ is a square block-diagonal matrix with blocks of order two and $h(t,x)$ is a vector function. The method is based on the combination of the technique of complexification and that of vector Lyapunov functions.

1. Introduction

In [6] the stability and asymptotic behaviour of a two-dimensional system

$$x' = A(t)x + h(t,x)$$

were studied by means of the method of complexification and the method of Lyapunov functions. The system was converted to one equation $z' = a(t)z + b(t)\bar{z}$ with complex-valued coefficients a, b. The results were obtained on the assumption $\liminf_{t\to\infty}(|a(t)| - |b(t)|) > 0$ (or $\liminf_{t\to\infty}(|\operatorname{Im} a(t)| - |b(t)|) > 0$) and extend several results of K. Tatarkiewicz [8] and Z. Artstein and E. F. Infante [1]. The case $|b(t)| > |\operatorname{Im} a(t)|$ was investigated in [2] and the achieved results generalize those of J. Radzikowski [7]. The existence of bounded solutions in unstable cases was studied in [3].

In the present paper we attempt to weaken the requirement

$$\liminf_{t\to\infty}(|a(t)| - |b(t)|) > 0$$

from [6]. Instead of a scalar Lyapunov function we shall use a suitable vector Lyapunov function. The advantage of the use of the vector Lyapunov function V consists in the fact that each component of V satisfies less rigid requirements than the usual scalar vector Lyapunov function. Since the use of vector Lyapunov functions is natural and typical for multidimensional

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systems, we shall formulate our results for the $2n$-dimensional system of coupled differential equations. An example of $\mathbb{R}$-linear equation of the form

$$z' = a(t)z + b(t)\bar{z}$$

with complex coefficients a, b is given such that the assumptions for the asymptotic stability of this equation are fulfilled but the condition $|a(t)| > |b(t)|$ is violated on any interval of the form $[T, \infty)$.

2. Preliminaries

Consider a real $2n$-dimensional system

$$x' = A(t)x + h(t, x), \tag{1}$$

where $A(t) = (a_{jk}(t))$, $j, k = 1, \dots, 2n$ is a square matrix, $x = (x_1, \dots, x_{2n})$ and $h(t, x) = (h_1(t, x_1, \dots, x_{2n}), \dots, h_{2n}(t, x_1, \dots, x_{2n}))$ is a vector function. We suppose that $a_{jk}(t)$ are continuous on $J = [t_0, \infty)$, $a_{2l-1,j} = 0$ and $a_{2l,j} = 0$ for $j \geq 2l + 1$ or $j \leq 2l - 2$, $(l = 1, \dots, n;\ j = 1, \dots, 2n)$, i.e. the matrix $A(t)$ is of a block-diagonal form

$$A(t) = \begin{pmatrix} A_1(t) & 0 & 0 & \dots & 0 \\ 0 & A_2(t) & 0 & \dots & 0 \\ 0 & 0 & A_3(t) & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \dots & A_n(t) \end{pmatrix},$$

where

$$A_j(t) = \begin{pmatrix} a_{2j-1,2j-1}(t) & a_{2j-1,2j}(t) \\ a_{2j,2j-1}(t) & a_{2j,2j}(t) \end{pmatrix}.$$

Further, $h(t, x)$ is assumed to be continuous on

$$\Gamma_r = J \times \{(x_1, \dots, x_{2n}) \in \mathbb{R}^{2n} : \max_{j=1,\dots,n} (x_{2j-1}^2 + x_{2j}^2) < r^2 \leq \infty\}. \tag{2}$$

First, we shall perform the complexification of (1). For this purpose put

$$z_j = x_{2j-1} + i x_{2j} \quad (j = 1, \dots, n).$$

We have

$$\begin{aligned} \begin{bmatrix} x'_{2j-1} \\ x'_{2j} \end{bmatrix} &= \begin{pmatrix} a_{2j-1,2j-1}(t) & a_{2j-1,2j}(t) \\ a_{2j,2j-1}(t) & a_{2j,2j}(t) \end{pmatrix} \begin{bmatrix} x_{2j-1} \\ x_{2j} \end{bmatrix} + \begin{bmatrix} h_{2j-1}(t, x_1, \dots, x_n) \\ h_{2j}(t, x_1, \dots, x_n) \end{bmatrix} \\ &= \begin{bmatrix} a_{2j-1,2j-1}(t)x_{2j-1} + a_{2j-1,2j}(t)x_{2j} \\ a_{2j,2j-1}(t)x_{2j-1} + a_{2j,2j}(t)x_{2j} \end{bmatrix} + \begin{bmatrix} h_{2j-1}(t, x_1, \dots, x_n) \\ h_{2j}(t, x_1, \dots, x_n) \end{bmatrix}. \end{aligned}$$

Hence

$$\begin{aligned}
z_j' &= (a_{2j-1,2j-1}x_{2j-1} + a_{2j-1,2j}x_{2j} + h_{2j-1}) + \\
&\quad + i(a_{2j,2j-1}x_{2j-1} + a_{2j,2j}x_{2j} + h_{2j}) = \\
&= (a_{2j-1,2j-1} + ia_{2j,2j-1})x_{2j-1} + (a_{2j-1,2j} + ia_{2j,2j})x_{2j} + f_j = \\
&= (a_{2j-1,2j-1} + ia_{2j,2j-1})\frac{z_j + \bar{z}_j}{2} + (a_{2j-1,2j} + ia_{2j,2j})\frac{z_j - \bar{z}_j}{2i} + f_j = \\
&= \frac{1}{2}[(a_{2j-1,2j-1} + a_{2j,2j}) + i(a_{2j,2j-1} - a_{2j-1,2j})]z_j + \\
&\quad + \frac{1}{2}[(a_{2j-1,2j-1} - a_{2j,2j}) + i(a_{2j,2j-1} + a_{2j-1,2j})]\bar{z}_j + f_j
\end{aligned}$$

for $j = 1, \dots, n$, where

$$h_k = h_k\left(t, \frac{z_1 + \bar{z}_1}{2}, \frac{z_1 - \bar{z}_1}{2i}, \dots, \frac{z_n + \bar{z}_n}{2}, \frac{z_n - \bar{z}_n}{2i}\right) \quad (k = 1, \dots, 2n),$$

$$f_j = f_j(t, z_1, \dots, z_n) = h_{2j-1} + ih_{2j}.$$

Thus, denoting

$$a_j(t) = \frac{1}{2}[(a_{2j-1,2j-1}(t) + a_{2j,2j}(t)) + i(a_{2j,2j-1}(t) - a_{2j-1,2j}(t))],$$

$$b_j(t) = \frac{1}{2}[(a_{2j-1,2j-1}(t) - a_{2j,2j}(t)) + i(a_{2j,2j-1}(t) + a_{2j-1,2j}(t))],$$

we have a system

$$z_j' = a_j(t)z_j + b_j(t)\bar{z}_j + f_j(t, z_1, \dots, z_n) \quad (j = 1, \dots, n), \tag{3}$$

where

$$t \in J,\ z_j \in \Omega_r = \{z \in \mathbb{C} : |z| < r\} \quad (j = 1, \dots, n). \tag{4}$$

Throughout the paper $\mathbb{C}$ denotes the set of all complex numbers, $\mathbb{R}$ the set of real numbers. Let $\mathbb{N}_n = \{1, \dots, n\}$. By Re z, Im z and $\bar{z}$ we mean the real part, the imaginary part and the conjugate of a complex number z, respectively. For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ we put $|z| = \sum_{j=1}^n |z_j|$, $\|z\| = (\sum_{j=1}^n |z_j|^2)^{\frac{1}{2}}$. $C(\tilde{\Gamma}, \Gamma)$ will denote the class of all continuous functions $\tilde{\Gamma} \to \Gamma$, $C^1(I, \Gamma)$ where I is a subinterval of J will denote the class of all continuously differentiable functions $I \to \Gamma$. For $g \in C(I, \mathbb{C})$ where I is a subinterval of J we define $\sqrt[\mathbb{C}]{g(t)}$ as any fixed function $q \in C(I, \mathbb{C})$ such that $q^2(t) = g(t)$.

Assume $a_j, b_j \in C(J, \mathbb{C})$, $f_j \in C(J \times \Omega_r^n, \mathbb{C})$ $(j = 1, \dots, n)$. Suppose the uniqueness of any initial value problem for (3). In the future, the following hypotheses will play a fundamental role:

(H) There are $g_{kj} \in C^1(J, \mathbb{C})$ $(k = 1, 2; j = 1, \ldots, n)$ such that

$$\sup_{t \in J} [2|g_{1j}(t) + g_{2j}(t)| - (|g_{1j}(t)|^2 + |g_{2j}(t)|^2)] < 2 \tag{5}$$

and

$$\left[i \operatorname{Im} a_j(t) + (-1)^{k-1} \sqrt[\mathbb{C}]{|b_j(t)|^2 - (\operatorname{Im} a_j(t))^2}\right] g_{kj}(t) = b_j(t), \qquad t \in J \tag{6}$$

hold for $k = 1, 2$ and $j = 1, \ldots, n$.

(H1) There are $\alpha_{kj}, \lambda_{kj} \in C(J, \mathbb{R})$ $(k = 1, 2; j = 1, \ldots, n)$ such that

$$\begin{aligned} &\operatorname{Re}[(f_j + g_{kj}\bar{f}_j + g'_{kj}\bar{z}_j)(\bar{z}_j + \bar{g}_{kj}z_j)] \leq \\ &\qquad \leq \alpha_{kj}(t)|z_j + g_{kj}\bar{z}_j|^2 + \lambda_{kj}(t)|z_j + g_{kj}\bar{z}_j| \end{aligned} \tag{7}$$

is satisfied for $(t, z_1, \ldots, z_n) \in J \times \Omega_r^n$, $k = 1, 2; j = 1, 2, \ldots, n$, where $f_j = f_j(t, z_1, \ldots, z_n)$, $g_{kj} = g_{kj}(t)$.

If the hypothesis (H1) is satisfied with $\lambda_{kj}(t) \equiv 0$ $(k = 1, 2; j = 1, \ldots, n)$, i.e. if (7) is replaced by

$$\operatorname{Re}[(f_j + g_{kj}\bar{f}_j + g'_{kj}\bar{z}_j)(\bar{z}_j + \bar{g}_{kj}z_j)] \leq \alpha_{kj}(t)|z_j + g_{kj}\bar{z}_j|^2 \tag{7'}$$

in (H1), we shall write (H0) instead of (H1).

Remark 1. 1. If $b_j(t) \neq 0$ then the condition (6) can be written in the form

$$i \operatorname{Im} a_j(t) + (-1)^k \sqrt[\mathbb{C}]{|b_j(t)|^2 - (\operatorname{Im} a_j(t))^2} = -\overline{b_j(t)}\, g_{kj}(t). \tag{6'}$$

2. If $b_j(t) \neq 0$ for $t \in J$, $a_j, b_j \in C^1(J, \mathbb{C})$ and

$$\sup_{t \in J} \left| \frac{\operatorname{Im} a_j(t)}{b_j(t)} \right| < 1 \quad \text{or} \quad \inf_{t \in J} \left| \frac{\operatorname{Im} a_j(t)}{b_j(t)} \right| > 1 \tag{8}$$

for $j = 1, \ldots, n$, then the hypothesis (H) is fulfilled.

Remark 2. 1. If $|g_{kj}(t)| \neq 1$, then the condition (7) in (H1) may be replaced by

$$\begin{aligned} &\operatorname{Re}[f_j(t, z_1, \ldots, z_n) + g_{kj}(t)\overline{f_j(t, z_1, \ldots, z_n)})(\bar{z}_j + \overline{g_{kj}(t)} z_j)] \leq \\ &\qquad \leq \beta_{kj}(t)|z_j + g_{kj}(t)\bar{z}_j|^2 + \lambda_{kj}(t)|z_j + g_{kj}(t)\bar{z}_j|, \end{aligned} \tag{9}$$

$$\begin{aligned} &\alpha_{kj}(t) = \beta_{kj}(t) + \\ &\qquad + (1 - |g_{kj}(t)|^2)^{-1}[-\operatorname{Re}(\overline{g_{kj}(t)} g'_{kj}(t)) + |g'_{kj}(t)| \operatorname{sgn}(1 - |g_{kj}(t)|)], \end{aligned} \tag{10}$$

where $\beta_{kj}, \lambda_{kj} \in C(J, \mathbb{R})$ $(k = 1, 2; j = 1, \ldots, n)$.

Really, if (9), (10) hold then

$$\begin{aligned}
&\operatorname{Re}[(f_j + g_{kj}\bar{f}_j + g'_{kj}\bar{z}_j)(\bar{z}_j + \bar{g}_{kj}z_j)] \le \\
&\quad \le \operatorname{Re}[(f_j + g_{kj}\bar{f}_j)(\bar{z}_j + \bar{g}_{kj}z_j)] + \operatorname{Re}[g'_{kj}\bar{z}_j(\bar{z}_j + \bar{g}_{kj}z_j)] \le \\
&\quad \le \left(\beta_{kj} + \operatorname{Re}\frac{g'_{kj}\bar{z}_j}{z_j + g_{kj}\bar{z}_j}\right)|z_j + g_{kj}(t)\bar{z}_j|^2 + \lambda_{kj}|z_j + g_{kj}(t)\bar{z}_j| \le \\
&\quad \le \left[\beta_{kj} + (1 - |g_{kj}(t)|^2)^{-1}\operatorname{Re}\frac{-g'_{kj}\bar{g}_{kj}(z_j + g_{kj}\bar{z}_j) + g'_{kj}\overline{(z_j + g_{kj}\bar{z}_j)}}{z_j + g_{kj}\bar{z}_j}\right] \\
&\quad \times |z_j + g_{kj}(t)\bar{z}_j|^2 + \lambda_{kj}|z_j + g_{kj}\bar{z}_j|
\end{aligned}$$

for all $(t, z_1, \ldots, z_n) \in J \times \Omega_r$.

2. Since

$$\begin{aligned}
&\frac{-\operatorname{Re}(\overline{g_{kj}(t)}g'_{kj}(t)) + |g'_{kj}(t)|\operatorname{sgn}(1 - |g_{kj}(t)|)}{1 - |g_{kj}(t)|^2} \le \\
&\quad \le \frac{(|g_{kj}(t)| + 1)|g'_{kj}(t)|}{|1 - |g_{kj}(t)|^2|} \le \frac{|g'_{kj}(t)|}{|1 - |g_{kj}(t)||},
\end{aligned}$$

$\alpha_{kj}(t)$ in (10) may be replaced by

$$\alpha_{kj}(t) = \beta_{kj}(t) + \frac{|g'_{kj}(t)|}{|1 - |g_{kj}(t)||}.$$

If the hypothesis (H) is satisfied we define scalar-valued functions

(11) $$V_{kj}(t, z_j) = |z_j + g_{kj}(t)\bar{z}_j|^2 \quad (k = 1, 2;\ j = 1, \ldots, n),$$

(12) $$V_0(t, z_1, \ldots, z_n) = \sum_{k=1}^{2}\sum_{j=1}^{n}|z_j + g_{kj}(t)\bar{z}_j|^2$$

and a vector-valued function

(13) $$V(t, z_1, \ldots, z_n) = (V_{11}(t, z_1,), \ldots, V_{1n}(t, z_n); V_{21}(t, z_1), \ldots, V_{2n}(t, z_n)).$$

Notice that, in general, the functions V_{kj} are not positive definite. However, the condition (5) implies the positive definiteness of V_0:

$$\begin{aligned}
V_0(t, z_1, \ldots, z_n) &= \sum_{k=1}^{2}\sum_{j=1}^{n}|z_j + g_{kj}(t)\bar{z}_j|^2 = \\
&= \sum_{j=1}^{n}\sum_{k=1}^{2}(z_j + g_{kj}(t)\bar{z}_j)(\bar{z}_j + \overline{g_{kj}(t)}z_j) =
\end{aligned}$$

$$= \sum_{j=1}^{n} [(2 + |g_{1j}(t)|^2 + |g_{2j}(t)|^2)|z_j|^2 + 2\,\mathrm{Re}((\overline{g_{1j}(t)} + \overline{g_{2j}(t)})z_j^2)] \geq$$

$$\geq \sum_{j=1}^{n} (2 + |g_{1j}(t)|^2 + |g_{2j}(t)|^2 - 2|g_{1j}(t) + g_{2j}(t)|)|z_j|^2;$$

$$V_0(t, z_1, \dots, z_n) \geq \sum_{j=1}^{n} \varkappa_j |z_j|^2, \tag{14}$$

where

$$\varkappa_j = 2 - \sup_{t \in J} [2|g_{1j}(t) + g_{2j}(t)| - (|g_{1j}(t)|^2 + |g_{2j}(t)|^2)] > 0. \tag{15}$$

On the other hand, as we shall see, there are no requirements on the derivatives of V_0 with respect to (3); we estimate only the derivatives of the functions V_{kj} (i.e. only the derivatives of the components of the vector-valued function V).

3. Main results

THEOREM 1. *Let the hypothesis* (H) *and the hypothesis* (H0) *be fulfilled. Assume that*

$$f_j(t, 0 \dots 0) = 0 \quad \text{for } t \in J. \tag{16}$$

If

$$\limsup_{t \to \infty} \int_{t_0}^{t} \mathrm{Re}[a_j(s) + g_{kj}(s)\overline{b_j(s)} + \alpha_{kj}(s)]\, ds < \infty \tag{17}$$

$$(k = 1, 2;\ j = 1, \dots, n),$$

then the trivial solution of (3) *is stable. If*

$$\int_{t_0}^{\infty} \mathrm{Re}[a_j(s) + g_{kj}(s)\overline{b_j(s)} + \alpha_{kj}(s)]\, ds = -\infty \quad (k = 1, 2;\ j = 1, \dots, n), \tag{18}$$

then the trivial solution of (3) is asymptotically stable.

Proof. From (6) it follows that

$$[b_j(t) = 0 \Rightarrow g_{kj}(t)\,\mathrm{Im}\, a_j(t) = 0] \quad \text{for } t \in J,\ k = 1, 2;\ j = 1, \dots, n. \tag{19}$$

Let $\varepsilon \in (0, r)$ and $t_1 \geq t_0$ be arbitrary. Suppose that $z(t) = (z_1(t), \dots, z_n(t))$ is any solution of (3) defined on $[t_1, t_2)$, where $t_2 > t_1$. Put

$$\Theta_{kj}(t) = V_{kj}(t, z_j(t)). \tag{20}$$

Differentiating (20) with respect to t yields

$$\Theta'_{kj}(t) = 2\,\mathrm{Re}[(\bar{z}_j + \bar{g}_{kj}z_j)(z'_j + g_{kj}\bar{z}'_j + g'_{kj}\bar{z}_j)] =$$
$$2\,\mathrm{Re}[(\bar{z}_j + \bar{g}_{kj}z_j)(a_jz_j + b_j\bar{z}_j + f_j + g_{kj}(\bar{a}_j\bar{z}_j + \bar{b}_jz_j + \bar{f}_j) + g'_{kj}\bar{z}_j)] =$$
$$2\,\mathrm{Re}\{(\bar{z}_j + \bar{g}_{kj}z_j)[(a_j + g_{kj}\bar{b}_j)z_j + (g_{kj}\bar{a}_j + b_j)\bar{z}_j]\} + \varphi_{kj},$$

where $\varphi_{kj} = 2\,\mathrm{Re}[(\bar{z}_j + \bar{g}_{kj}z_j)(f_j + g_{kj}\bar{f}_j + g'_{kj}\bar{z}_j)]$, $a_j = a_j(t)$, $b_j = b_j(t)$, $z_j = z_j(t)$, $g_{kj} = g_{kj}(t)$, $f_j = f_j(t, z_1(t), \dots, z_n(t))$. Using (6) we obtain

$$\Theta'_{kj}(t) = 2\,\mathrm{Re}\{(\bar{z}_j + \bar{g}_{kj}z_j)[(a_j + g_{kj}\bar{b}_j)z_j +$$
$$+ (\mathrm{Re}\,a_j + (-1)^{k-1}\sqrt[\mathbb{C}]{|b_j|^2 - (\mathrm{Im}\,a_j)^2})g_{kj}\bar{z}_j]\} + \varphi_{kj}.$$

With respect to (6') and (19) we have

$$\Theta'_{kj}(t) = 2\,\mathrm{Re}[(\bar{z}_j + \bar{g}_{kj}z_j)(a_j + g_{kj}\bar{b}_j)(z_j + g_{kj}\bar{z}_j)] + \varphi_{kj} =$$
$$= 2\Theta_{kj}(t)\,\mathrm{Re}(a_j + g_{kj}\bar{b}_j) + \varphi_{kj}.$$

In view of (H0) the relation

$$\Theta'_{kj}(t) \le 2\Theta_{kj}(t)\,\mathrm{Re}[a_j(t) + g_{kj}(t)\overline{b_j(t)} + \alpha_{kj}(t)]$$

holds for all $t \in [t_1, t_2)$. Thus

$$\Big(\Theta_{kj}(t)\exp\Big\{-2\int_{t_1}^{t}\mathrm{Re}[a_j(s) + g_{kj}(s)\overline{b_j(s)} + \alpha_{kj}(s)]\,ds\Big\}\Big)' \le 0.$$

By the integration over $[t_1, t]$ we get

$$\Theta_{kj}(t) \le \Theta_{kj}(t_1)\exp\Big\{2\int_{t_1}^{t}\mathrm{Re}[a_j(s) + g_{kj}(s)\overline{b_j(s)} + \alpha_{kj}(s)]\,ds\Big\}$$

for $t \in [t_1, t_2)$. Hence

$$\Phi(t) \le \Phi(t_1)\sum_{k=1}^{2}\sum_{j=1}^{n}\exp\Big\{2\int_{t_1}^{t}\mathrm{Re}\,\big[a_j(s) + g_{kj}(s)\overline{b_j(s)} + \alpha_{kj}(s)]\,ds\Big\}, \tag{21}$$

where $\Phi(t) = V_0(t, z_1(t), \dots, z_n(t))$.

Let (16) hold and

$$L = \sum_{k=1}^{2}\sum_{j=1}^{n}\sup_{t_1\le t<\infty}\exp\Big\{2\int_{t_1}^{t}\mathrm{Re}[a_j(s) + g_{kj}(s)\overline{b_j(s)} + \alpha_{kj}(s)]\,ds\Big\}.$$

Suppose that $\delta > 0$ is such that

$$(22)\qquad \delta \le [\min_{j=1,\dots,n} \varkappa_j]^{\frac{1}{2}}[\max_{\substack{k=1,2\\ j=1,\dots,n}} (1+|g_{kj}(t_1)|)^2]^{-\frac{1}{2}}(2L)^{-\frac{1}{2}}\frac{\varepsilon}{\sqrt{n}},$$

where $\varkappa_j$ is defined by (15). In view of (14), (21) for $z(t)$ satisfying $\|z(t_1)\| < \delta$ we have

$$\min_{j=1,\dots,n} \varkappa_j \|z(t)\|^2 \le \Phi(t) \le L\Phi(t_1) \le L\sum_{k=1}^{2}\sum_{j=1}^{n}(1+|g_{kj}(t_1)|)^2 \sum_{j=1}^{n}|z_j(t_1)|^2.$$

Using $\|z(t_1)\| < \delta$ and (22) we obtain

$$\min_{j=1,\dots,n} \varkappa_j \|z(t)\|^2 < 2nL\delta^2 \max_{\substack{k=1,2\\ j=1,\dots,n}} (1+|g_{kj}(t_1)|)^2$$

and

$$(23)\qquad \|z(t)\| < \varepsilon$$

for $t \in [t_1, t_2)$. Since $\varepsilon < r$ the solution $z(t)$ exists for all $t \ge t_1$ and satisfies (23) for all $t \ge t_1$. Hence the trivial solution of (3) is stable.

Let (17) be fulfilled. With respect to stability, there is a $\delta^* > 0$ such that $\|z(t_1)\| < \delta^*$ implies $\|z(t)\| < r$. Therefore

$$\|z(t)\|^2 \le [\min_{j=1,\dots,n} \varkappa_j]^{-1}\Phi(t_1) \times \sum_{k=1}^{2}\sum_{j=1}^{n}\exp\left\{2\int_{t_1}^{t} \operatorname{Re}[a_j(s)+g_{kj}(s)\overline{b_j(s)}+\alpha_{kj}(s)]\,ds\right\}$$

for $t \ge t_1$. Thus the asymptotic stability is proved. ∎

In the following statement we shall consider the system

$$(24)\qquad z_j' = a_j z_j + b_j \bar{z}_j + f_j(t, z_1, \dots, z_n) \quad (j = 1, \dots, n),$$

where $a_j \in \mathbb{C}$, $b_j \in \mathbb{C}$ are constants. We suppose the uniqueness of any initial-value problem for (24).

COROLLARY 1. *Let* $a_j(t) \equiv a_j \in \mathbb{C}$, $b_j(t) \equiv b_j \in \mathbb{C}$ *and* $\mathbb{N}_n = N_0 \cup N_1 \cup N_2$. *Assume* $b_j = \operatorname{Im} a_j = 0$ *for* $j \in N_0$, $|\operatorname{Im} a_j| < |b_j|$ *for* $j \in N_1$ *and* $|\operatorname{Im} a_j| > |b_j|$ *for* $j \in N_2$. *Suppose there exist functions* $\varrho_{kj} \in C(J, \mathbb{R})$ $(k = 1, 2;\ j \in N_1)$ *and* $\varrho_j \in C(J, \mathbb{R})$ $(j \in N_2)$ *such that*

$$(25)\qquad |f_j(t, z_1, \dots, z_n)| \le \varrho_{kj}(t)|z_j + \omega_{kj}\bar{z}_j| \quad \text{for } (t, z_1, \dots, z_n) \in J \times \Omega_r^n \quad (k = 1, 2;\ j \in N_1),$$

$$(26)\qquad |f_j(t,z_1,\dots,z_n)| \le \varrho_j(t)|z_j| \quad \text{for } (t,z_1,\dots,z_n)\in J\times\Omega_r^n \quad (j\in N_0\cup N_2),$$

where

$$\omega_{kj} = \frac{-i\operatorname{Im} a_j + (-1)^{k-1}\sqrt{|b_j|^2-(\operatorname{Im} a_j)^2}}{\bar b_j}.$$

If the conditions

$$(27)\qquad \limsup_{t\to\infty}\int_{t_0}^{t}\Big[\operatorname{Re} a_j + (-1)^{k-1}\sqrt{|b_j|^2-(\operatorname{Im} a_j)^2} + 2\varrho_{kj}(s)\Big]\,ds < \infty \quad (k=1,2;\ j\in N_1),$$

$$(28)\qquad \limsup_{t\to\infty}\int_{t_0}^{t}[\operatorname{Re} a_j + \varrho_j(s)]\,ds < \infty \quad (j\in N_0)$$

and

$$(29)\qquad \limsup_{t\to\infty}\int_{t_0}^{t}\left[\operatorname{Re} a_j\sqrt{\frac{|\operatorname{Im} a_j|-|b_j|}{|\operatorname{Im} a_j|+|b_j|}} + \varrho_j(s)\right]ds < \infty \quad (j\in N_2)$$

are fulfilled then the trivial solution of the system (24) *is stable. If*

$$(30)\qquad \int_{t_0}^{\infty}\Big[\operatorname{Re} a_j + (-1)^{k-1}\sqrt{|b_j|^2-(\operatorname{Im} a_j)^2} + 2\varrho_{kj}(s)\Big]\,ds = -\infty \quad (k=1,2;\ j\in N_1),$$

$$(31)\qquad \int_{t_0}^{\infty}[\operatorname{Re} a_j + \varrho_j(s)]\,ds = -\infty \quad (j\in N_0)$$

and

$$(32)\qquad \int_{t_0}^{\infty}\left[\operatorname{Re} a_j\sqrt{\frac{|\operatorname{Im} a_j|-|b_j|}{|\operatorname{Im} a_j|+|b_j|}} + \varrho_j(s)\right]ds = -\infty \quad (j\in N_2),$$

then the trivial solution of (24) *is asymptotically stable.*

Proof. Put

$$g_{kj}(t)\equiv g_{kj} = \begin{cases} \omega_{kj} & \text{for } k=1,2,\ j\in N_1\\ 0 & \text{for } k=1,2,\ j\in N_0\cup N_2,\ b_j=0\\ \dfrac{b_j}{i(\operatorname{Im} a_j + (-1)^{k-1}\sqrt{(\operatorname{Im} a_j)^2-|b_j|^2})} & \text{for } k=1,2,\ j\in N_2, b_j\neq 0.\end{cases}$$

In view of Remark 1 the hypothesis (H) is fulfilled with the function $g_{kj}(t)$. Clearly $g'_{kj}(t) \equiv 0$. The conditions (25), (26) imply that (24) has the trivial solution. It is easy to verify that

$$\operatorname{Re}[a_j + g_{kj}\bar{b}_j] =$$
$$= \begin{cases} \operatorname{Re} a_j & \text{whenever } j \in N_0 \cup N_2, \\ \operatorname{Re} a_j + (-1)^{k-1}\sqrt{|b_j|^2 - (\operatorname{Im} a_j)^2} & \text{whenever } j \in N_1. \end{cases}$$

For $j \in N_0 \cup N_1$ we have

$$\operatorname{Re}[(f_j(t,z_1,\dots,z_n) + g_{kj}\overline{f_j(t,z_1,\dots,z_n)})(\bar{z}_j + \bar{g}_{kj}z_j)] \le$$
$$\le (1+|g_{kj}|)|f_j(t,z_1,\dots,z_n)||z_j + g_{kj}\bar{z}_j| \le$$
$$\le (1+\operatorname{sgn}|b_j|)\varrho^*_{kj}(t)|z_j + g_{kj}\bar{z}_j|^2$$

for $(t,z_1,\dots,z_n) \in J \times \Omega^n_r$, where

$$\varrho^*_{kj}(t) = \begin{cases} \varrho_{kj}(t) & \text{if } j \in N_1, \\ \varrho_j(t) & \text{if } j \in N_0. \end{cases}$$

Hence, for $j \in N_0$, the condition (7') is satisfied with $\alpha_{kj}(t) = \varrho_j(t)$, and, for $j \in N_1$, with $\alpha_{kj}(t) = 2\varrho_{kj}(t)$.

For $j \in N_2$ we get $|g_{kj}| \ne 1$ and

$$\operatorname{Re}[(f_j(t,z_1,\dots,z_n) + g_{kj}\overline{f_j(t,z_1,\dots,z_n)})(z_j + g_{kj}\bar{z}_j)] \le$$
$$\le (1+|g_{kj}|)|f_j(t,z_1,\dots,z_n)||z_j + g_{kj}\bar{z}_j| \le$$
$$\le (1+|g_{kj}|)\varrho_j(t)|z_j||z_j + g_{kj}\bar{z}_j|.$$

Since

$$V_{kj}(t,z_j) = |z_j + g_{kj}\bar{z}_j|^2 \ge (1-|g_{kj}|)^2|z_j|^2,$$

we have

$$\operatorname{Re}[(f_j(t,z_1,\dots,z_n) + g_{kj}\overline{f_j(t,z_1,\dots,z_n)})(z_j + g_{kj}\bar{z}_j)] \le$$
$$\le \frac{1+|g_{kj}|}{|1-|g_{kj}||}\varrho_j(t)|z_j + g_{kj}\bar{z}_j|^2.$$

Further,

$$\frac{1+|g_{kj}|}{|1-|g_{kj}||} = \frac{|\operatorname{Im} a_j| + (-1)^{k-1}\operatorname{sgn}(\operatorname{Im} a_j)\sqrt{(\operatorname{Im} a_j)^2 - |b_j|^2} + |b_j|}{||\operatorname{Im} a_j| + (-1)^{k-1}\operatorname{sgn}(\operatorname{Im} a_j)\sqrt{(\operatorname{Im} a_j)^2 - |b_j|^2} - |b_j||} =$$

$$= \sqrt{\frac{|\operatorname{Im} a_j| + |b_j|}{|\operatorname{Im} a_j| - |b_j|}} \cdot \frac{\sqrt{|\operatorname{Im} a_j| + |b_j|} + (-1)^{k-1}\operatorname{sgn}(\operatorname{Im} a_j)\sqrt{|\operatorname{Im} a_j| - |b_j|}}{|\sqrt{|\operatorname{Im} a_j| - |b_j|} + (-1)^{k-1}\operatorname{sgn}(\operatorname{Im} a_j)\sqrt{|\operatorname{Im} a_j| + |b_j|}|} =$$

$$= \sqrt{\frac{|\operatorname{Im} a_j| + |b_j|}{|\operatorname{Im} a_j| - |b_j|}} \qquad \text{whenever } b_j \ne 0.$$

However, for $b_j = 0$, we have $g_{kj} = 0$ and

$$\frac{1+|g_{kj}|}{1-|g_{kj}|} = 1 = \sqrt{\frac{|\operatorname{Im} a_j| + |b_j|}{|\operatorname{Im} a_j| - |b_j|}}$$

too. Hence (7') is, for $j \in N_2$, satisfied with

$$\alpha_{kj}(t) = \sqrt{\frac{|\operatorname{Im} a_j| + |b_j|}{|\operatorname{Im} a_j| - |b_j|}}\varrho_j(t).$$

As

$$\operatorname{Re}[a_j + g_{kj}\bar{b}_j + \alpha_{kj}(t)] = \sqrt{\frac{|\operatorname{Im} a_j| + |b_j|}{|\operatorname{Im} a_j| - |b_j|}}\left[\operatorname{Re} a_j \sqrt{\frac{|\operatorname{Im} a_j| - |b_j|}{|\operatorname{Im} a_j| + |b_j|}} + \varrho_j(t)\right]$$

for $j \in N_2$, the statement follows from Theorem 1. ∎

Remark 3. 1. Corollary 1 generalizes Corollary 6 of [6].

2. Corollary 1 is easily applicable to the system

$$z_j' = (a_j + p_j(t))z_j + (b_j + q_j(t))\bar{z}_j \quad (j = 1, \ldots, n),$$

where $a_j, b_j \in \mathbb{C}$, Re $a_j \le 0$, $|\operatorname{Im} a_j| > |b_j|$, $p, q \in C(J, \mathbb{C})$. Here $f_j = p_j z_j + q_j \bar{z}_j$ and (26) is satisfied with $\varrho_j(t) = |p_j(t)| + |q_j(t)|$.

Remark 4. The autonomous equation $z' = az + b\bar{z}$, where $a, b \in \mathbb{C}$ are constants such that $|a| \ne |b|$, has the unique equilibrium $z = 0$, which is a focus ($|b| < |\operatorname{Im} a|, \operatorname{Re} a \ne 0$), centre ($|b| < |\operatorname{Im} a|, \operatorname{Re} a = 0$), node ($|\operatorname{Im} a| \le |b| < |a|$) or saddle point ($|b| > |a|$). If $|b| = |\operatorname{Im} a| = 0$, we have a dicritical node (proper node), if $|b| = |\operatorname{Im} a| \ne 0$, we have a degenerated node. For $|a| > |b|$ and $\operatorname{Re} a < 0$ the equilibrium is stable, for $|a| > |b|$ and $\operatorname{Re} a > 0$ unstable. (The eigenvalues of the considered equation are $\lambda_{1,2} = \operatorname{Re} a \pm \sqrt[\mathbb{C}]{|b|^2 - (\operatorname{Im} a)^2}$.) The following example shows an $\mathbb{R}$-linear equation $z' = a(t)z + b(t)\bar{z}$, where $a, b \in C(J, \mathbb{C})$, such that the equation is stable or asymptotically stable, however the condition $|a(t)| > |b(t)|$ may be violated on any interval of the type $[T, \infty)$.

EXAMPLE 1. Consider a real system

$$\begin{aligned} x_1' &= (\alpha(t) + \gamma\beta(t))x_1 + (\delta - \vartheta)\beta(t)x_2, \\ x_2' &= (\delta + \vartheta)\beta(t)x_1 + (\alpha(t) - \gamma\beta(t))x_2, \end{aligned} \tag{33}$$

where $\alpha, \beta \in C(J, \mathbb{R})$, $\gamma, \delta, \vartheta \in \mathbb{R}$, $\vartheta^2 < \gamma^2 + \delta^2$. The complexification yields

$$z' = a(t)z + b(t)\bar{z}, \tag{34}$$

where $a(t) = \alpha(t) + i\vartheta\beta(t)$, $b(t) = c\beta(t)$, $c = \gamma + i\delta$, $|\vartheta| < |c|$. The hypothesis (H) is satisfied with $g_{11} = (-i\vartheta + \sqrt{|c|^2 - \vartheta^2})/\bar{c}$, $g_{21} = (-i\vartheta - \sqrt{|c|^2 - \vartheta^2})/\bar{c}$. Indeed, $|g_{11}| = |g_{21}| = 1$, $|g_{11} + g_{21}| = 2|\vartheta|/|c| < 2$ and hence (5) holds. Since

$$i \operatorname{Im} a(t) + (-1)^{k-1} \sqrt[\mathbb{C}]{|b(t)|^2 - (\operatorname{Im} a(t))^2} = \beta(t)(i\vartheta + (-1)^{k-1}\sqrt{|c|^2 - \vartheta^2}),$$

the condition (6) is satisfied too. The hypothesis (H0) is fulfilled obviously with $\alpha_{11} = \alpha_{21} \equiv 0$. Moreover, $b(t) = 0$ implies Im $a(t) = 0$. Now, using Theorem 1 we have the stability of (34) and (33) if

$$\limsup_{t\to\infty} \int_{t_0}^{t} [\alpha(s) + (-1)^{k-1}\sqrt{|c|^2 - \vartheta^2}\beta(s)]ds < \infty \qquad (k = 1, 2), \tag{35}$$

and the asymptotic stability if

$$\int_{t_0}^{\infty} [\alpha(s) + (-1)^{k-1}\sqrt{|c|^2 - \vartheta^2}\beta(s)]ds = -\infty \qquad (k = 1, 2). \tag{36}$$

Notice that the euclidean measure (logarithmic norm) μ_2 of a matrix of the system (33) is $\mu_2(t) = \operatorname{Re} a(t) + |b(t)| = \alpha(t) + |c||\beta(t)|$ (the equality $\mu_2(t) = \operatorname{Re} a(t) + |b(t)|$ holds generally for any equation (34)). The well-known conditions for stability and asymptotic stability are

$$\limsup_{t\to\infty} \int_{t_0}^{t} \mu_2(s)\, ds < \infty \tag{37}$$

and

$$\int_{t_0}^{\infty} \mu_2(s)\, ds = -\infty, \tag{38}$$

respectively. In our case, the conditions (35), (36) are clearly better. Namely, taking $t_0 = \pi$, $\alpha(t) \equiv 0$, $\beta(t) = t^{-\nu} \sin t$, where $0 \le \nu \le 1$, we have the stability in view of (35), however (37) is not satisfied. Taking $t_0 = \pi$, $\alpha(t) \equiv -|c|/\pi$, $\beta(t) = \sin t$, we have the asymptotic stability in view of (36), however (38) is not fulfilled. Notice that also logarithmic norms $\mu_1(t) = \max[\operatorname{Re}(a(t) + b(t)) + |\operatorname{Im}(a(t) + b(t))|, \operatorname{Re}(a(t) - b(t)) + |\operatorname{Im}(a(t) - b(t))|]$, $\mu_\infty(t) = \max[\operatorname{Re}(a(t) + b(t)) + |\operatorname{Im}(a(t) - b(t))|, \operatorname{Re}(a(t) - b(t)) + |\operatorname{Im}(a(t) + b(t))|]$ do not satisfy (37), (38), in general, on our assumptions.

THEOREM 2. *Assume that the hypotheses (H) and (H1) are fulfilled. Let* $z(t) = (z_1(t), \ldots\ldots, z_n(t))$ *be any solution of* (3) *defined for* $t \ge t_1$ $(\ge t_0)$.

Then there exist $\mu_j > 0$ *such that*

$$\mu_j|z_j(t)| \leq \sum_{k=1}^{2} V_{kj}(t_1, z_j(t_1)) \exp\Big[\int_{t_1}^{t} \varphi_{kj}(s)\,ds\Big] + \sum_{k=1}^{2} \int_{t_1}^{t} \lambda_{kj}(\tau) \exp\Big[\int_{\tau}^{t} \varphi_{kj}(s)\,ds\Big] d\tau \tag{39}$$

for $t \geq t_1$, $j = 1, \ldots, n$, *where*

$$\varphi_{kj}(t) = \mathrm{Re}[a_j(t) + g_{kj}(t)\overline{b_j(t)} + \alpha_{kj}(t)]. \tag{40}$$

Proof. Following the proof of Theorem 1 we get

$$\Theta'_{kj}(t) \leq 2\varphi_{kj}(t)\Theta_{kj}(t) + 2\Theta^{\frac{1}{2}}_{kj}(t)\lambda_{kj}(t),$$

where

$$\Theta_{kj}(t) = V_{kj}(t, z_j(t)) = |z_j(t) + g_{kj}(t)\overline{z_j(t)}|^2.$$

Putting

$$\Psi_{kj}(t) = \Theta^{\frac{1}{2}}_{kj}(t),$$

we obtain

$$\Psi'_{kj}(t) \leq \varphi_{kj}(t).\Psi_{kj}(t) + \lambda_{kj}(t)$$

for all $t \geq t_1$ for which $\Psi_{kj}(t) \neq 0$. If $t^* \geq t_1$ is such that $\Psi_{kj}(t^*) = 0$, then

$$\begin{aligned}\Psi'_{kj-}(t^*) &= \lim_{t\to t^*-} \frac{\Psi_{kj}(t) - \Psi_{kj}(t^*)}{t - t^*} = \\ &= \lim_{t\to t^*-} \frac{|z_j(t) + g_{kj}(t)\overline{z_j(t)}| - |z_j(t^*) + g_{kj}(t^*)\overline{z_j(t^*)}|}{t - t^*} = \\ &= -\lim_{t\to t^*-} \frac{|z_j(t) - z_j(t^*) + g_{kj}(t)\overline{z_j(t)} - g_{kj}(t^*)\overline{z_j(t^*)}|}{|t - t^*|} \leq \\ &\leq -|z'_j(t^*) + [g_{kj}(t)\overline{z_j(t)}]'_{t=t^*}| \leq 0 \leq \lambda_{kj}(t^*).\end{aligned}$$

Hence

$$\Psi'_{kj-}(t) \leq \varphi_{kj}(t)\Psi_{kj}(t) + \lambda_{kj}(t)$$

for all $t \geq t_1$ and

$$\Psi_{kj}(t) \leq \Psi_{kj}(t_1)e^{\int_{t_1}^{t} \varphi_{kj}(s)\,ds} + \int_{t_1}^{t} \lambda_{kj}(\tau)e^{\int_{\tau}^{t} \varphi_{kj}(s)ds}\,d\tau, \tag{41}$$

where the right-hand side of the last inequality is the maximal solution of

$$u' = \varphi_{kj}(t)u + \lambda_{kj}(t), \qquad u(t_1) = \Psi_{kj}(t_1).$$

From (41) we obtain

$$\sum_{k=1}^{2} \Psi_{kj}(t) = \sum_{k=1}^{2} \Psi_{kj}(t_1) e^{\int_{t_1}^{t} \varphi_{kj}(s)ds} + \sum_{k=1}^{2} \int_{t_1}^{t} \lambda_{kj}(\tau) e^{\int_{\tau}^{t} \varphi_{kj}(s)ds}\, d\tau$$

for $t \geq t_1$. As

$$\sum_{k=1}^{2} \Psi_{kj}(t) \geq \Big[\sum_{k=1}^{2} |z_j(t) + g_{kj}(t)\overline{z_j(t)}|^2 \Big]^{\frac{1}{2}} \geq [\varkappa_j |z_j(t)|^2]^{\frac{1}{2}} \geq \varkappa_j^{\frac{1}{2}} |z_j(t)|,$$

it is clear that there are $\mu_j > 0$ such that

$$\sum_{k=1}^{2} \Psi_{kj}(t) \geq \mu_j |z_j(t)|$$

for $t \geq t_1$. The proof is complete. ∎

COROLLARY 2. *Let the assumptions of Theorem 2 be fulfilled. Let*

$$\limsup_{t\to\infty} \sum_{k=1}^{2} \int_{t_1}^{t} \lambda_{kj}(\tau) \exp\Big[\int_{\tau}^{t_1} \varphi_{kj}(s)\, ds \Big] d\tau < \infty, \tag{42}$$

φ_{kj} *being defined by* (40). *If* $z(t) = (z_1(t), \dots, z_n(t))$ *is any solution of* (3) *defined for* $t \geq t_1$ $(\geq t_0)$ *then*

$$z_j(t) = O\Big(\sum_{k=1}^{2} \exp \int_{t_1}^{t} \varphi_{kj}(s)\, ds \Big) \quad \text{as } t \to \infty. \tag{43}$$

The statement follows immediately from (39).

COROLLARY 3. *Let the assumptions of Theorem 2 be fulfilled and let*

$$\limsup_{t\to\infty} \varphi_{kj}(t) < \eta_j < \infty, \tag{44}$$

$$\lambda_{kj}(t) = O(e^{\eta_j t}) \quad \text{as } t \to \infty \tag{45}$$

for $k = 1, 2$, *where* φ_{kj} *is defined by* (40). *If* $z(t) = (z_1(t), \dots, z_n(t))$ *is any solution of* (3) *defined for* $t \geq t_1$ $(\geq t_0)$ *then* $z_j(t) = O(e^{\eta_j t})$ *as* $t \to \infty$.

Proof. In view of (44) and (45) there exist $L > 0$, $\eta_j^* < \eta_j$ and $T > t_1$ such that $\varphi_{kj}(t) < \eta_j^*$ and $\lambda_{kj}(t) e^{-\eta_j t} \leq L$ for $t \geq T$, $k = 1, 2$. From (39) it follows that

$$\mu_j |z_j(t)| \leq V_0(T, z_1(T), \dots, z_n(T)) e^{\eta_j^*(t-T)} + \sum_{k=1}^{2} \int_{T}^{t} L e^{\eta_j \tau} e^{\eta_j^*(t-\tau)} d\tau \leq \tag{46}$$

$$\leq V_0(T, z_1(T), \dots, z_n(T)) e^{\eta_j^*(t-T)} + 2L e^{\eta_j^* t} (\eta_j - \eta_j^*)^{-1} [e^{(\eta_j - \eta_j^*)t} - e^{(\eta_j - \eta_j^*)T}] \leq$$

$$\leq V_0(T, z_1(T), \dots, z_n(T))e^{\eta_j^*(t-T)} + L_j^* e^{\eta_j t} \leq$$
$$\leq \left[V_0(T, z_1(T), \dots, z_n(T))e^{-\eta_j^* T} + L_j^*\right] e^{\eta_j t} = O(e^{\eta_j t}),$$

where $L_j^* = 2L(\eta_j - \eta_j^*)^{-1}$. ∎

Remark 5. If $\lambda_{kj}(t) \equiv 0$ for $k = 1, 2$, then we can take $L = L_j^* = 0$ in the proof of Corollary 3 and the inequalities (46) yield the following statement: there exists an $\tilde{\eta}_j < \eta_j$ such that $z_j(t) = o(e^{\tilde{\eta}_j t})$ as $t \to \infty$.

THEOREM 3. *Let the assumptions of Theorem 2 be fulfilled. Let*

$$\varphi_{kj}(t) \leq 0 \text{ for } t \geq T\ (\geq t_0), \tag{47}$$

$$\lim_{t \to \infty} \int_{t_1}^{\infty} \varphi_{kj}(t)\, dt = -\infty \tag{48}$$

and

$$\lambda_{kj}(t) = o(\varphi_{kj}(t)) \text{ as } t \to \infty \tag{49}$$

for $k = 1, 2$, where φ_{kj} are defined by (40). *Then for any solution $z(t) = (z_1(t), \dots, z_n(t))$ of* (3) *the relation*

$$\lim_{t \to \infty} z_j(t) = 0$$

holds.

Proof. Let $\varepsilon > 0$. In view of (49) and (47) there exists a $\sigma \geq T$ such that $\lambda_{kj}(t) \leq -\varepsilon \varphi_{kj}(t)$ for $t \geq \sigma$, $k = 1, 2$ and

$$\sum_{k=1}^{2} \int_{\sigma}^{t} \lambda_{kj}(\tau) \exp\Big(\int_{\tau}^{L} \varphi_{kj}(s)\, ds\Big) d\tau \leq$$
$$\leq \varepsilon \sum_{k=1}^{2} \int_{\sigma}^{t} \Big[-\varphi_{kj}(\tau) \exp\Big(\int_{\tau}^{t} \varphi_{kj}(s)\, ds\Big)\Big] d\tau \leq$$
$$\leq \varepsilon \sum_{j=1}^{2} \Big[1 - \exp\Big(\int_{\sigma}^{t} \varphi_{kj}(\tau) d\tau\Big)\Big] < 2\varepsilon$$

for $t \geq \sigma$. Since

$$\exp\Big[\int_{t_1}^{t} \varphi_{kj}(s)\, ds\Big] \to 0 \quad \text{as } t \to \infty,$$

we get from (39)

$$\mu_j|z_j(t)| < 3\varepsilon$$

for large t. Thus $\lim_{t\to\infty}|z_j(t)| = 0$. ∎

EXAMPLE 2. Consider a real system

$$(50)\quad \begin{aligned} x_1' &= (\alpha(t)+\gamma\beta(t))x_1 + (\delta-\vartheta)\beta(t)x_2 + \varrho(t,x_1,x_2)x_1 + h_1(t,x_1,x_2),\\ x_2' &= (\delta+\vartheta)\beta(t)x_1 + (\alpha(t)-\gamma\beta(t))x_2 + \varrho(t,x_1,x_2)x_2 + h_2(t,x_1,x_2),\end{aligned}$$

where $\alpha,\beta \in C(J,\mathbb{R}), \varrho, h_j \in C(J\times\mathbb{R}^2,\mathbb{R}), \gamma,\delta,\vartheta \in \mathbb{R}, \vartheta^2 < \gamma^2+\delta^2$. Suppose there is a $\sigma \in C(J,\mathbb{R})$ such that $\varrho(t,x_1,x_2) \le \sigma(t)$ for $(t,x_1,x_2) \in J\times\mathbb{R}^2$. The complexification yields

$$(51)\qquad z' = a(t)z + b(t)\bar z + f(t,z),$$

where $a(t) = \alpha(t) + i\vartheta\beta(t)$, $b(t) = c\beta(t)$, $c = \gamma + i\delta$, $|\vartheta| < |c|$ and

$$f(t,z) = \psi(t,z) + h_1(t,\frac{z+\bar z}{2},\frac{z-\bar z}{2i}) + ih_2(t,\frac{z+\bar z}{2},\frac{z-\bar z}{2i}),$$
$$\psi(t,z) = \varrho(t,\frac{z+\bar z}{2},\frac{z-\bar z}{2i}).$$

Put

$$g_1 = (-i\vartheta + \sqrt{|c|^2-\vartheta^2})/\bar c, \qquad g_2 = (-i\vartheta - \sqrt{|c|^2-\vartheta^2})/\bar c.$$

Let $\lambda_k \in C(J,\mathbb{R})$ $(k=1,2)$ be such that

$$|f(t,z) + g_k\overline{f(t,z)}| \le \lambda_k(t) \quad (k=1,2).$$

The hypothesis (H) is fulfilled with $g_{11} = g_1$, $g_{21} = g_2$, similarly as in Example 1. Since

$$\begin{aligned}\operatorname{Re}\{[(\psi z + f) + g_k\overline{(\psi z + f)}](\bar z + \bar g_k z)\} &=\\ = \psi|z+g_k\bar z|^2 + |f+g_k\bar f||z+g_k\bar z| &\le\\ \le \psi|z+g_k\bar z|^2 + \lambda_k|z+g_k\bar z| &\le\\ \le \sigma|z+g_k\bar z|^2 + \lambda_k|z+g_k\bar z|&,\end{aligned}$$

the hypothesis (H1) is satisfied with $\alpha_{k1}(t) \equiv \sigma(t)$, $\lambda_{k1}(t) = \lambda_k(t)$ $(k=1,2)$. Assuming that $(x_1(t),x_2(t))$ is a solution of (50) defined for $t \ge t_1$ $(\ge t_0)$ and applying Corollary 2, Corollary 3 and Theorem 3, we obtain following statements:

1° If

$$\limsup_{t\to\infty}\sum_{k=1}^{2}\int_{t_1}^{t}\lambda_k(\tau)\exp\Big\{\int_{\tau}^{t_1}[\alpha(s)+\sigma(s)+(-1)^{k-1}\sqrt{|c|^2-\vartheta^2}\beta(s)]\,ds\Big\}d\tau<\infty,$$

then

$$x_j(t) = O\Big(\sum_{k=1}^{2} \exp\Big\{\int_{t_1}^{t} [\alpha(s) + \sigma(s) + (-1)^{k-1}\sqrt{|c|^2 - \vartheta^2}\beta(s)]\, ds\Big\}\Big)$$

as $t \to \infty$ for $j = 1, 2$.

2° If

$$\limsup_{t\to\infty} [\alpha(t) + \sigma(t) + (-1)^{k-1}\sqrt{|c|^2 - \vartheta^2}\beta(t)] = \varkappa < \infty$$

and

$$\lambda_k(t) = O(e^{\varkappa t}) \quad \text{as } t \to \infty$$

for $k = 1, 2$, then

$$x_j(t) = O(e^{\varkappa t}) \text{ as } t \to \infty \quad (j = 1, 2).$$

3° If

$$\alpha(t) + \sigma(t) + (-1)^{k-1}\sqrt{|c|^2 - \vartheta^2}\beta(t) \le 0 \quad \text{for } t \ge T \ (\ge t_0),$$

$$\lim_{t\to\infty} \int_{t_1}^{t} [\alpha(s) + \sigma(s) + (-1)^{k-1}\sqrt{|c|^2 - \vartheta^2}\beta(s)]\, ds = -\infty$$

and

$$\lambda_k(t) = o\Big(\alpha(t) + \sigma(t) + (-1)^{k-1}\sqrt{|c|^2 - \vartheta^2}\beta(t)\Big) \quad \text{as } t \to \infty$$

for $k = 1, 2$, then

$$(x_1(t), x_2(t)) \to 0 \quad \text{as } t \to \infty.$$

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