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Wawrzyniec Sadkowski

GLOBAL EXACT CONTROLLABILITY FOR GENERALIZED WAVE EQUATION

1. Introduction

In the paper we will investigate a problem of exact controllability on $\langle 0,T\rangle$, $T\in(0,\infty)$, T finite for the generalized wave system

$$\text{(IBVP)}\qquad \begin{cases} u_{tt}+A(x,D)=z, \ x\in\Omega, \ t\in(0,T\rangle \\ u(x,0)=\Phi(x), \ u_t(x,0)=\psi(x), x\in\Omega \\ D^{\beta}u|_{\Gamma}=0 \text{ for } |\beta|\le m-1, \ t\in\langle 0,T\rangle, \end{cases}$$

where $A(x,D)$ is a linear elliptic operator of order $2m$ [4]; $\Omega\subset R^n$, $\Gamma=\partial\Omega, T>0$. The problem is a follows: given T an initial functions ψ,Φ, find a corresponding control function $z(.)$ in a suitable Hilbert space, driving the system to a desired state $u(.,T)=z_1(.)$, $u'(.,T)=z_2(.)$ at a time T. This problem has been studied in [1], [2], [6] for $A(x,D)=-\Delta$ and $A(x,D)=\Delta^2$. The way we approach the problem consists of three stages:

First, this system will be set in an abstract form on the Hilbert space $H_0^m(\Omega)\times L_2(\Omega)$. Next, it will be proved that the following operator $\Lambda=\begin{bmatrix}0 & I\\ -A & 0\end{bmatrix}$ generates a strongly continuous unitary group on $H_0^m(\Omega)\times L_2(\Omega)$ and that this operator is a Riesz-spectral one. Finally, a sufficient condition will be checked out for exact controllability for the state linear system $\Sigma(A,B,-)$.

Assumption 1. Let $\Omega\subset R^n, n\ge 1$ be a bounded domain with smooth boundary Γ and closure $\overline{\Omega}$. Let functions $a_{pq}:\overline{\Omega}\to R$ be given, where $p=(p_1,p_2,\ldots,p_n)$, $q=(q_1,q_2,\ldots q_n)$, are multiindicies with

$$|p|=\sum_{i=1}^{n}p_i, \ |q|=\sum_{i-1}^{n}q_i.$$

Let $H^m(\Omega)$ and $H_0^m(\Omega)$ be Sobolev spaces with the norms $\|.\|_H^s, s\in R$. Consider a linear elliptic operator $A_0(x,D)$ for order $2m, m\in N$, in the divergence form

$$(1) \qquad A_0(x,D) = \sum_{|p|=|q|=0}^{m} (-1)^{|p|} D^p(A_{pq}(x)D^q).$$

ASSUMPTION 2. We have $a_{pq} \in C(\overline{\Omega})$, $a_{pq} = a_{pq}$, $|p| \leq m$, $|q| \leq m$.

ASSUMPTION 3. The operator $A_0(x, D)$ is strongly elliptic in Ω, e.i., there exists a constant $c > 0$ such that

$$\sum_{|p|=|q|=m} a_{pq}(x)\xi^p\xi^q \geq c|\xi|^{2m}, \text{ for all } x \in \overline{\Omega}, \xi \in R^n.$$

Denote by $a(u, v)$ a bilinear form

$$(2) \qquad a(v,w) = \sum_{|p|=|q|=0}^{m} \int_\Omega a_{pq} D^p v(x) D^q w(x) dx, \quad w, v \in H^m\Omega.$$

Assumption 3 implies that the operator $A_0(x, D)$ satisfies Gärding's inequality, i.e., there exist constants $C_1 > 0$ and $C_2 \geq 0$ such that $a(v,v) \geq C_1\|v\|^2_{H^m(\Omega)} - C_2\|v\|^2_{L_2(\Omega)}$ for any $v \in H_0^m(\Omega)$. If $C_2 \neq 0$, the operator $A_0(x, D)$ will be replaced by the operator

$$(3) \qquad A(x,D) = A_0(x,D) + \lambda I,$$

where I is the identity operator and $\lambda > C_2$. Then for any $v \in H_0^m(\Omega)$

$$(4) \qquad a(v,v) \geq C_1\|v\|^2_{H^m(\Omega)}.$$

With the elliptic operator $A(x, D)$, we associate a linear operator A in $L_2(\Omega)$, given by $A_u = A(x, D)u$ for $u \in D(A) = H^{2m}(\Omega) \cap H_0^m(\Omega)$.

LEMMA 1.
i) *A is a positive self-adjoint operator,*
ii) $0 \in \rho(A)$,
iii) *there exists $A^{\frac{1}{2}}$ and it is self-adjoint,*
iv) $0 \in \rho(A^{\frac{1}{2}})$,
v) $a(v, w) = (A^{\frac{1}{2}}v, A^{\frac{1}{2}}w)$,
vi) $D(A^{\frac{1}{2}}) = H_0^m(\Omega)$.

Proof. i) Due to Assumption 2, the operator A given by (3) can be extended to a self-adjoint operator in $L_2(\Omega)$ [4, p. 126]. ii) The inequality (4) implies that $0 \in \rho(a)$. The properties iii)–vi) have been proved in [4, p. 29, p. 109] and in [1, p. 606, p. 609].

We set $(IBVP)$ problem in the form

$$(5) \qquad \frac{dw}{dt} = \Lambda w + Bz,$$

(6) $$w_0 = w(0) = \begin{pmatrix} \Phi \\ \psi \end{pmatrix},$$

on the space $\mathbb{H}_0 = H_0^m(\Omega) \times L_2(\Omega)$ with $D(\Lambda) = (H^{2m}(\Omega) \cap H_0^m(\Omega)) \times H_0^m(\Omega)$, where $\Lambda \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ I \end{bmatrix}$, $w = \begin{bmatrix} u \\ u_t \end{bmatrix}$.

Denote by $(,)$ the usual inner product in $L_2(\Omega)$ and let

(7) $$(w_1, w_2)_{\mathbb{H}_0} = (A^{\frac{1}{2}} v_1, A^{\frac{1}{2}} v_2) + (z_1, z_2),\ w_1 = \begin{pmatrix} v_1 \\ z_1 \end{pmatrix},\ v_2 = \begin{pmatrix} v_2 \\ z_2 \end{pmatrix}.$$

LEMMA 2. $\mathbb{H}_0$ *with the inner product given by* (7) *is a Hilbert space.*

The proof of this Lemma is similar to that one from [1], and it is based on the results of Lemma 1.

2. A C_0 unitary group for the generalized wave equation

Consider a C_0 unitary group U on a Hilbert space H. The well-known Stone's Theorem [3, p. 41] states that A is the generator of the group U on H iff A is skew-adjoint. As a consequence of the above result we shall prove the following:

THEOREM 1. *If the assumption* 1–3 *are satisfied, then the operator* Λ *is an infinitesimal generator of a* C_0 *unitary group on the space* $\mathbb{H}_0$.

Proof. First, it will be shown that $i\Lambda$ is a self-adjoint operator on $\mathbb{H}_0(\Omega)$. To see this let us observe that

$$\left(i\begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right)_{\mathbb{H}_0(\Omega)} = \left(i\begin{bmatrix} z_2 \\ -Az_1 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right)_{\mathbb{H}_0(\Omega)}$$
$$= \left(\begin{bmatrix} iz_2 \\ -iAz_1 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right)_{\mathbb{H}_0(\Omega)} = (iA^{\frac{1}{2}} z_2, A^{\frac{1}{2}} v_1) + (-iAz_1, v_2)$$
$$= i(z_2, Av_1) - i(Az_1, v_2) = (z_2, -iAv_1) + (Az_1, iv_2)$$
$$= i(z_2, Av_1) + (Az_1, iv_2) = (A^{\frac{1}{2}} z_1, iA^{\frac{1}{2}} v_2) + (z_2, -iAv_1)$$
$$= \left(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \begin{bmatrix} iv_2 \\ -iAv_1 \end{bmatrix}\right) = \left(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, i\begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right).$$

It means that $(i\Lambda)^* = i\Lambda$, i.e., $i\Lambda$ is a self-adjoint operator on $\mathbb{H}_0(\Omega)$. But this implies that Λ is a skew-adjoint operator on the space $\mathbb{H}_0(\Omega)$ and therefore, by the Stone's Theorem, Λ generates a C_0 unitary group on $\mathbb{H}_0(\Omega)$. So, Λ has a dense domain in $\mathbb{H}_0(\Omega)$ and Λ is a closed operator.

Let us consider, on the Hilbert space, an abstract nonhomogeneous Cauchy problem

(8) $$\frac{dz(t)}{dt} = Az(t) + f(t),\ t \geq 0,\ z(0) = z_0,$$

where $f : \langle 0, T\rangle \to H$.

DEFINITION 1. Let A be the infinitesimal generator of a C_0 semigroup $T(t)$, $t \geq 0$. For given $z_0 \in H$ and $f \in L^1(\langle 0,T\rangle; H)$ the function $z \in C(\langle 0,T\rangle; H)$ defined by

$$z(t) = T(t)z_0 + \int_0^t T(t-s)f(s)ds, \quad 0 \leq t \leq T, \tag{9}$$

is called the mild solution of the initial value problem (7) on T.

LEMMA 3. ([3], p. 106; [1], Lemma 3.1.5, p. 104). *If $f \in L_p(\langle 0,T\rangle; H)$ for some $p \geq 1$ and $z_0 \in H$ then there exists a unique mild solution z of the equation* (8) *given by the formula* (9).

3. A Riesz-spectral operator for the generalized wave equation

In this section, it will be presented a convenient representation for linear generalized wave equation. A Riesz basis for non-self-adjoint (exactly skew-adjoint) operators will be constructed.

DEFINITION 2. A sequence of vectors $\{\Phi_n, n \geq 1\}$ in a Hilbert space H forms a Riesz basis for H if the following two condition are satisfied:
a) $\overline{\operatorname{span}\{\Phi_n, n \geq 1\}} = H$,
b) there exist positive constants m and M such that for arbitrary $N \in \mathbb{N}$ and arbitrary scalars $\alpha_n, n = 1, \dots N$, we have

$$m \sum_{n=1}^{N} |\alpha_n|^2 \leq \left\| \sum_{n=1}^{n} \alpha_n \Phi_n \right\|^2 \leq M \sum_{n=1}^{N} |\alpha_n|^2.$$

DEFINITION 3. Suppose that A is a linear, closed operator on a Hilbert space H, with simple eigenvalues $\{\mu_n, n \geq 1\}$ and suppose that corresponding eigenvectors $\{\Phi_n, n \geq 1\}$ form a Riesz basis in H. If the closure of $\{\mu_n, n \geq 1\}$ is totally disconnected, then we call A a Riesz-spectral operator. By a totally disconnectedness of a set $\overline{\{\mu_n, n \geq 1\}}$ we mean property that there are no two points $\lambda, \mu \in \overline{\{\mu_n, n \geq 1\}}$ which can be joint by a segment lying entirely in $\overline{\{\mu_n, n \geq 1\}}$.

Remark 1. The above definition covers the case when A has a one accumulation point.

LEMMA 4. [4, p. 369] *Assume that the operator A defined by (3) satisfies Assumptions 1,2,3 and (4). Then the eigenvalue problem*

$$\text{(EPA)} \qquad \begin{cases} A\varphi - \mu\varphi = 0 \text{ on } \Omega, \\ D^\alpha \varphi = 0 \text{ on } \Gamma \text{ for } |\alpha| \leq m-1 \end{cases}$$

has countably many eigenvalues μ which are real. All the eigenvalues have finite multiplicity. If we count the eigenvalues according to their multiplicity, then $0 < \mu_1 < \mu_2 \leq \ldots$ and $\mu_k \to \infty$ as $k \to \infty$. There exists a complete orthonormal system of eigenvectors $\{\varphi_k\}$, $\varphi_k \in H_0^m(\Omega)$, in the space $L_2(\Omega)$:

$$\int_\Omega \varphi_k \varphi_s dx = \begin{cases} 1 & \text{for } k = s \\ 0 & \text{for } k \neq s \end{cases}$$

and $A\varphi_k = \lambda_k^2 \varphi_k$ if we denote $\mu_k = \lambda_k^2$.

Now, the eigenvalue problem for a skew-adjoint operator will be considered:

$$\text{(EP}\Lambda\text{)} \qquad \Lambda U = BU, \quad U = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix},$$

which is equivalent to the system

$$z_1 = \beta^{-1} z_2, \ \beta \neq 0$$
$$A z_2 = -\beta^2 z_2.$$

By Lemma 4, it is clear that $\beta_k = \pm i\lambda_k$, $k \in \mathbb{N}$. So, we obtain eigenvectors in the space $\mathbb{H}_0(\Omega) = H_0^m(\Omega) x L_2(\Omega)$

$$\Phi_{-k} = \left\{ \begin{matrix} -\frac{i}{\lambda_k}\varphi_k \\ \varphi_k \end{matrix} \right\}, \ \Phi_k = \left\{ \begin{matrix} \frac{1}{\lambda_k}\varphi_k \\ \varphi_k \end{matrix} \right\}, \ k \in \mathbb{N}.$$

THEOREM 2. *The eigenvectors $\Phi_{\pm k}$, $k \in \mathbb{N}$ form a Riesz basis and Λ is a Riesz-spectral operator.*

Proof. First we shall prove that $\{\Phi_{\pm k}\}$, $k \in \mathbb{N}$, is complete in $\mathbb{H}_0(\Omega)$. Suppose that $z = \binom{z_1}{z_2}$ is orthogonal to every Φ_k. Then

$$0 = \left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} \frac{-i\varphi_k}{\lambda_k} \\ \varphi_k \end{pmatrix} \right)_{\mathbb{H}_0(\Omega)} = \left(A^{\frac{1}{2}} z_1, A^{\frac{1}{2}} \left(-\frac{i}{\lambda_k} \varphi_k \right) \right)_{L_2(\Omega)} + (z_2, \varphi_k)_{L_2(\Omega)}$$
$$= \left(z_1, \frac{-i}{\lambda_k} A\varphi_k \right)_{L_2(\Omega)} + (z_2, \varphi_k)_{L_2(\Omega)} = (z_1, -i\lambda_k \varphi_k)_{L_2(\Omega)} + (z_2, \varphi_k)_{L_2(\Omega)}$$
$$= \overline{-i\lambda_k}(z_1, \varphi_k)_{L_2(\Omega)} + (z_2, \varphi_k)_{L_2(\Omega} = -(-i)\lambda_k (z_1, \varphi_k)_{L_2(\Omega)} + (z_2, \varphi_k)_{L_2 9\Omega)}$$
$$= i\lambda_k (z_1, \varphi_k)_{L_2(\Omega)} + (z_2, \varphi_k)_{L_2(\Omega)},$$

and similarly

$$0 = \left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} \frac{i\varphi_k}{\lambda_k} \\ \varphi_k \end{pmatrix} \right)_{\mathbb{H}_0(\Omega)} = -i\lambda_k (z_1, \varphi_k)_{L_2(\Omega)} + (z_2, \varphi_k)_{L_2(\Omega)}.$$

Adding and substracting above equalities, we have $0 = (z_2, \varphi_k)$ and $i\lambda_k(z_1, \varphi_k) = 0$. Since the system $\{\varphi_k, k \geq 1\}$ is complete in $L_2(\Omega)$ and

$\lambda_k > 0$ we conclude that $z_1 = 0$ and $z_2 = 0$. Thus the system $\{\Phi_{\pm k}, k \in \mathbb{N}\}$ is complete in $\mathbb{H}_0(\Omega)$. Λ has only one accumulation point.

Now we want to check that the condition b) from Definition 2 holds. To do that, let us observe that

$$\Big\| \sum_{n=-N, n\neq 0}^{N} \alpha_n \Phi_n \Big\|^2_{\mathbb{H}_0(\Omega)} = \Big\| \sum_{n=-N}^{1} \alpha_n \Phi_n + \sum_{n=1}^{N} \alpha_n \Phi_n \Big\|^2_{\mathbb{H}_0(\Omega)}$$

$$= \Big\| \sum_{n=1}^{N} \alpha_{-n} \Phi_{-n} + \sum_{n=1}^{N} \alpha_n \Phi_n \Big\|^2_{\mathbb{H}_0(\Omega)}$$

$$= \Big\| A^{\frac{1}{2}} \Big\{ \sum_{n=1}^{N} \Big[\alpha_{-n} \Big(\frac{-i}{\lambda_n} \varphi_n \Big) + \alpha_n \Big(\frac{i}{\lambda_n} \varphi_n \Big) \Big] \Big\} \Big\|^2_{L_2(\Omega)}$$

$$+ \Big\| \sum_{n=1}^{N} \alpha_{-n} \varphi_n + \sum_{n=1}^{N} \alpha_n \varphi_n \Big\|^2_{L_2(\Omega)}$$

$$= \Big(\sum_{n=1}^{N} \Big[-i \frac{\alpha_{-n}}{\lambda_n} + \frac{i\alpha_n}{\lambda_n} \Big] \varphi_n, \sum_{n=1}^{N} \Big[-1 \frac{\alpha_{-n}}{\lambda_n} + \frac{i\alpha_n}{\lambda_n} \Big] A\varphi_n \Big)$$

$$+ \Big\| \sum_{n=1}^{N} (\alpha_{-n} + \alpha_n) \varphi_n \Big\|^2_{L_2(\Omega)}$$

$$= (-i)\overline{(i)} \sum_{n=1}^{N} \Big(\frac{\alpha_{-n} - \alpha_n}{\lambda_n}, \frac{\alpha_{-n} - \alpha_n}{\lambda_n} \lambda_n^2 \Big) + \sum_{n=1}^{\infty} (\alpha_{-n} - \alpha_n) \overline{(\alpha_{-n} + \alpha_n)}$$

$$= \sum_{n=1}^{N} (\alpha_{-n} - \alpha_n) \overline{(\alpha_{-n} + \alpha_n)} + \sum_{n=1}^{N} |\alpha_{-n} + \alpha_n|^2$$

$$= \sum_{n=1}^{N} |\alpha_{-n} - \alpha_n|^2 + \sum_{n=1}^{N} |\alpha_{-n} + \alpha_n|^2 = 2 \sum_{n=1}^{N} (|\alpha_{-n}|^2 + |\alpha_n|^2).$$

Therefore condition b) is satisfied for $n = M = 2$. By Theorem 1, Λ is a generator of C_0 unitary group on the space $\mathbb{H}_0(\Omega)$, so Λ is a closed operator with dense domain in $\mathbb{H}_0(\Omega)$. Hence Λ is a Riesz-spectral operator.

Remark 2. It can be checked that $\{\Phi_{\pm k}, k \in \mathbb{N}\}$ is an orthogonal basis in $\mathbb{H}_0(\Omega)$ and $\|\Phi_{\pm k}\|^2_{\mathbb{H}_0(\Omega)} = 2$.

Later on, the following orthonormal basis will be used

$$\Phi_{-k} = \left\{ \begin{array}{c} \frac{-i}{\sqrt{2}\lambda_k} \varphi_k \\ \frac{1}{\sqrt{2}} \varphi_k \end{array} \right\}, \; \Phi_k = \left\{ \begin{array}{c} \frac{i}{\sqrt{2}\lambda_k} \varphi_k \\ \frac{1}{\sqrt{2}} \varphi_k \end{array} \right\}, \; k \in N.$$

4. Controllability

Generally, we shall consider the following class of infinite-dimensional systems with input z and output y:

$$\dot{u}(t) = Au(t) + Bz(t), \ t \geq 0, \quad u(0) = u_0, \tag{10}$$

$$y(t) = Cu(t) + Dz(t). \tag{11}$$

DEFINITION 4. Let $\Sigma(A, B, C, D)$ denote the state linear system, where A is the infinitesimal generator of the strongly continuous semigroup $S(t)$ on a Hilbert space H, B is a bounded linear operator from a Hilbert space U into H, C is a bounded linear operator from H into a Hilbert space Y, and D is a bounded operator from U into Y. The system $\Sigma(A, B, C, D)$ will be considered for all initials states $u_0 \in H$ and all inputs $z \in L_2(\langle 0, T\rangle, U)$. The state function $u(.)$ is the mild solution of (10) and it is given by

$$u(t) = S(t)u_0 + \int_0^t S(t-s)Bz(s)ds, \quad 0 \leq t \leq T.$$

If A is a Riesz-spectral operator, we shall call $\Sigma(A, B, C, D)$ a Riesz spectral system. If $C = D = 0$ we use the notation $\Sigma(A, B, C, -)$. For the state linear system $\Sigma(A, B, -)$, we need the following:

DEFINITION 5.

a) The controllability map of $\Sigma(A, B, -)$ on $\langle 0, T\rangle$ (for some finite $T > 0$) is the bounded linear map $B^T : L_2(\langle 0, T\rangle, u) \to H$ defined by $B^T z \equiv \int_0^T S(t-s)Bz(s)ds$.

b) $\Sigma(A, B, -)$ is exactly controllable on $\langle 0, T\rangle$ (for some finite $T > 0$) iff all points in H can be reached from the origin at time T, i.e., iff the range $B^T = H$.

For the state linear system $\Sigma(A, B, -)$, we have the following sufficient and necessary conditions for exact controllability.

THEOREM 3 [1, p. 147]. *The system $\Sigma(A, B, -)$ is exactly controllable on $\langle 0, T\rangle$ if and only if one of the following equivalent conditions hold for some $\gamma > 0$ and all $u \in H$:*
i) $\langle L_B^T u, u\rangle \geq \gamma\|u\|_H^2$,
ii) $\|B^{T^*}u\|_2^2 \equiv \int_0^T \|(B^{T^*}u)(s)\|_U^2 ds \geq \gamma\|u\|_H^2$,
iii) $\int_0^T \|B^* S^*(s)u\|_U^2 ds \geq \gamma\|u\|_H^2$,
iv) $\ker B^{T^*} - \{0\}$ *and* $ranB^{T^*}$ *is closed.*

Remark 3. In our case $H = \mathbb{H}_0(\Omega)$, $B = \begin{bmatrix} 0 \\ I \end{bmatrix}$, $A = \Lambda$, $U = L_2(\Omega)$.

THEOREM 4. [1, p. 41, c]. *Suppose that Λ is a Riesz-spectral operator with simple eigenvalues $\{\mu_n, n \geq 1\}$ and corresponding eigenvectors $\{\Phi_n, n \geq 1\}$. Let $\{\psi_n, n \geq 1\}$ are the eigenvectors of Λ^* such that $\langle \Phi_n, \psi_n \rangle = \delta_n^m$. Then Λ is the infinitesimal generator of C_0 semigroup if and only if $\sup_{n\geq 1} Re\, \mu_n < \infty$. Additionaly, $S(t)$ is given by $S(t) = \sum_{n=1}^{\infty} e^{\mu_n t}\langle ., \psi_n\rangle \Phi_n, t \geq 0$.*

THEOREM 5. *In our case $\Lambda = \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix}$, $\Lambda^* = \begin{bmatrix} 0 & -I \\ A & 0 \end{bmatrix}$. Both operators Λ and Λ^* have the same set of eigenvalues $\mu_n = \pm i\lambda_n$, $n \in \mathbb{N}$ and the eigenvectors $\{\Phi_{\pm n}, n \in \mathbb{N}\}$. It is clear that $Re\, \mu_n$ and $\sup_{n\geq 1} Re\, \mu_n = 0 < \infty$. By Theorem 1 the operator Λ is a Riesz-spectral one. From Theorem 4, we see that $S(t)$ has a representation*

$$S(t) = \sum_{n=1}^{\infty}(e^{-i\lambda_n t}\langle ., \Phi_{-n}\rangle \Phi_{-n} + e^{i\lambda_n t}\langle ., \Phi_n\rangle \Phi_n, \ t \geq 0.$$

After a tedious calculation it can be found that

$$S(t)\begin{pmatrix}\Phi \\ \psi\end{pmatrix} = \begin{cases} \sum_{n=1}^{\infty} \left[(\Phi, \varphi_n)\cos\lambda_n t + \frac{1}{\lambda_n}(\psi, \varphi_n)\right]\varphi_n \\ \sum_{n=1}^{\infty} \left[-\lambda_n(\Phi, \varphi_n)\sin\lambda_n t + \cos\lambda_n t(\psi, \varphi_n)\right]\varphi_n. \end{cases} \tag{12}$$

In the paragraph 2, it has been proved that Λ is a generator of strongly continuous semigroup $S(t)$, $t \geq 0$ on $\mathbb{H}_0(\Omega)$ (exactly a group for $t \in R$). By Theorem 4, $S(t)$ has the representation given by (12). Applying Theorem 3, iii), we see that for $z \in L_2(\langle 0, T\rangle; L_2(\Omega))$, our system will be exactly controllable on $\langle 0, T\rangle$ iff there exists $\gamma > 0$ such that

$$\|B^*S^*(.)v\|^2_{L_2(\langle 0,T\rangle; L_2(\Omega))} \geq \gamma\|v\|^2_{\mathbb{H}_0(\Omega)}, \text{ for any } v \in \mathbb{H}_0(\Omega),\ v = \begin{pmatrix}\Phi \\ \psi\end{pmatrix}. \tag{13}$$

The proof of this inequality is like that one in [1, p. 150], [2,p. 10]. From the group property of $S(t)$ and Remark 1, we conclude that $S^*(t) = S(-t)$ for $t \in R$. It is clear that $B^* = [0, I]$, so

$$\left\|B^*S^*(t)\begin{pmatrix}\Phi \\ \psi\end{pmatrix}\right\|^2_{L_2(\Omega)} = \sum_{n=1}^{\infty}[-\lambda_n(\Phi, \varphi_n)\sin(-\lambda_n t) + (\psi, \varphi_n)\cos(-\lambda_n t)]^2$$

$$= \sum_{n=1}^{\infty}\left[\lambda_n(\Phi, \varphi_n)^2\sin^2\lambda_n t + \lambda_n(\Phi, \varphi_n)(\psi, \varphi_n)\sin 2\lambda_n t + (\psi, \varphi_n)^2\cos^2\lambda_n t\right].$$

Thus

$$\left\| B^* S^* \begin{pmatrix} \Phi \\ \psi \end{pmatrix} \right\|^2_{L_2(\langle 0,T\rangle, L_2(\Omega))}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \left[\lambda_n^2 (\Phi, \psi_n)^2 \left(T - \frac{\sin 2\lambda_n T}{2\lambda_n} \right) + 2\lambda_n (\Phi, \varphi_n)(\psi, \varphi_n) \left(1 - \frac{\cos 2\lambda_n T}{2\lambda_n} \right) \right.$$
$$\left. + (\psi, \varphi_n)^2 \left(T + \frac{\sin 2\lambda_n T}{2\lambda_n} \right) \right].$$

After routine calculation we obtain

$$\|v\|_{\mathbb{H}_0(\Omega)} = \sum_{n=1}^{\infty} \left[\lambda_n^2 (\Phi, \varphi_n)^2 + (\psi, \varphi_n)^2 \right].$$

So, the inequality (13) is equivalent to the following one

$$\frac{1}{2} \sum_{n=1}^{\infty} \left[\lambda_n^2 (\Phi, \varphi_n)^2 \left(T - \frac{\sin 2\lambda_n T}{2\lambda_n} \right) + 2\lambda_n (\Phi, \varphi_n)(\psi, \varphi_n) \cdot \left(\frac{1 - \cos 2\lambda_n t}{2\lambda_n} \right) \right.$$
$$\left. + (\psi, \varphi_n)^2 \left(T + \frac{\sin 2\lambda_n T}{2\lambda_n} \right) \right] \geq \gamma^2 \sum_{n=1}^{\infty} \left[\lambda_n^2 (\Phi, \varphi_n)^2 + (\psi, \varphi_n)^2 \right].$$

Putting $P = T - 2\gamma^2$ we see that the above inequality holds iff for any $n \in N$

$$P - \frac{\sin 2\lambda_n T}{2\lambda_n T} > 0, \quad P + \frac{\sin 2\lambda_n T}{2\lambda_n T} > 0$$

and

$$P^2 - \frac{\sin^2 2\lambda_n T}{2\lambda_n} - \left(\frac{1 - \cos 2\lambda_n T}{2\lambda_n} \right)^2 \geq 0.$$

But the last inequality is equivalent to $T - 2\gamma^2 = P \geq |\frac{\sin \lambda_n T}{\lambda_n}|$, while the first two to $T - 2\gamma^2 = P > |\frac{\sin 2\lambda_n T}{2\lambda_n}| = |\frac{\sin \lambda_n T}{\lambda_n}||\cos \lambda_n T|$. Since $\lambda_n \to +\infty$, therefore $\sup_n |\frac{\sin \lambda_n T}{\lambda_n}| < T$ for any $T > 0$ what, in turn, guarantees the existence of a required $\gamma > 0$.

This results shows that for all $T > 0$ system is exactly controllable.

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INSTITUTE OF MATHEMATICS
WARSAW UNIVERSITY OF TECHNOLOGY
Pl. Politechniki 1
00-661 Warszawa, POLAND

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