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ON THE ARONSZAJN PROPERTY FOR AN INTEGRO-DIFFERENTIAL EQUATION IN BANACH SPACES

In this paper we shall present two existence theorems for local solutions of an initial value problem for a nonlinear integro-differential equation in Banach space. Moreover, we also shall prove that the set of all these solutions is an R_δ in the Aronszajn sense [1], i.e. it is homeomorphic to the intersection of a decreasing sequence of compact absolute retracts. Let us recall that in the case of ordinary differential equations this problem was investigated by many authors. For example see to [10], [11] and [6], [9].

1. Consider a Cauchy problem

$$x'(t) = f(t, x(t)) + \int_0^t g(t, s, x(s))\, ds, \tag{1}$$

$$x(0) = 0 \tag{2}$$

in a Banach space E. We assume that $D = \langle 0, d\rangle$, $B = \{x \in E: \|x\| \le b\}$ and $f: D \times B \to E$, $g: D^2 \times B \to E$ are bounded continuous functions. Let

$$m_1 = \sup\{\|f(t,x)\|: t \in D,\ x \in B\},$$
$$m_2 = \sup\{\|g(t,s,x)\|: t, s, \in D,\ x \in B\}.$$

We choose a positive number a such that $a \le d$ and

$$m_1 a + m_2 a^2 \le b. \tag{3}$$

Let $J = \langle 0, a\rangle$. Denote by $C = C(J, E)$ the Banach space of continuous functions $z: J \to E$ with the usual norm $\|z\|_c = \max_{t\in J} \|z(t)\|$. Let $Q = \{x \in C: \|x\|_c \le b\}$. For $t \in J$ and $x \in Q$ put

$$\widetilde{g}(t, x) = \int_0^t g(t, s, x(s))\, ds.$$

Fix $\tau \in J$ and $x \in Q$. As the set $J \times x(J)$ is compact, from the continuity of g it follows that for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|g(t,s,x(s)) - g(\tau,s,x(s))\| < \varepsilon \quad \text{for } t,s \in J \quad \text{with } |t-\tau| < \delta.$$

In view of the inequality

$$\|\tilde{g}(t,x) - \tilde{g}(\tau,x)\| \le m_2|t-\tau| + \int_0^\tau \| g(t,s,x(s)) - g(\tau,s,x(s)) \| \, ds,$$

this implies the continuity of the function $t \to \tilde{g}(t,x)$.

On the other hand, the Lebesgue dominated convergence theorem proves that for each fixed $t \in J$ the function $x \to \tilde{g}(t,x)$ is continuous on Q. Moreover

$$\|\tilde{g}(t,x)\| \le m_2 t \quad \text{for } t \in J \text{ and } x \in Q. \tag{4}$$

2. Assume that h is a Kamke function, i.e. $(t,r) \to h(t,r)$ is a non-negative function defined on $D \times R_+$ which is Lebesgue measurable in t for fixed r, and continuous in r for fixed t, and

(i) for every bounded subset Z of $D \times R_+$ there exists a function ψ_z defined on $(0,d\rangle$ such that $h(t,r) \le \psi_z(t)$ for $(t,r) \in Z$ and ψ_z is Lebesgue integrable on $[c,d]$ for every $c > 0$;

(ii) for each c, $0 < c \le d$, the identically zero function is the only absolutely continuous function on $\langle 0,c\rangle$ which satisfies $z'(t) = h(t,z(t))$ almost everywhere on $\langle 0,c\rangle$ and such that $D_+z(0) = z(0) = 0$.

THEOREM 1. *If*

$$\|f(t,x) - f(t,y)\| \le h(t,\|x-y\|) \quad \textit{for } t \in D \textit{ and } x,y \in B,$$

and the set $g(D^2 \times B)$ *is relatively compact in* E, *then the set* S *of all solutions of the problem* (1)–(2), *defined on* J, *is an* R_δ.

Proof. Let us remark that on J the problem (1)–(2) is equivalent to

$$x'(t) = f(t,x(t)) + \tilde{g}(t,x), \quad x(0) = 0.$$

Let $W = \bigcup_{0 \le \lambda \le a^2} \lambda \overline{\text{conv}}\, g(D^2 \times B)$. By (3) and the Mazur Lemma, W is a compact subset of B. Fix $n \in N$ and $v \in Q$. Put $t_i = ai/n$ for $i = 0,1,\ldots,n$. We define a mapping $u_n(v)\colon J \to B$ by

$$u_n(v)(0) = 0,$$

$$u_n(v)(t) = u_n(v)(t_i) + \int_{t_i}^{t} f(s, u_n(v)(t_i))\, ds + \int_{t_i}^{t} \tilde{g}(s,v)\, ds$$

for $t \in \langle t_i, t_{i+1}\rangle$, $i = 0,1,\ldots,n-1$.

Similarly as in [7] and [8] it can be shown that

(5) $\|u_n(v)(t) - u_n(v)(\tau)\| \le M|t-\tau|$ for $t, \tau \in J$ and $v \in Q$,

where $M = m_1 + m_2 a$;

(6) $u_n(v)(t) \in V_n$ for $t \in J$ and $v \in Q$,

where V_n is a compact subset of B defined by

$$V_0 = \{0\}, \quad V_{k+1} = \bigcup_{0 \le \lambda \le a} \lambda \overline{\operatorname{conv}}\, f(J \times V_k) + W \quad \text{for } k = 0, 1, \ldots, n-1;$$

(7) $D_+\|u_n(v)(t) - u_m(v)(t)\|$

$$\le \min\Big(\mu(t), h\big(t, \|u_n(v)(t) - u_m(v)(t)\|\big) + 2\varepsilon(t, q_n)\Big)$$

for $m \ge n$, $t \in J$ and $v \in Q$, where $\varepsilon(t,p) = \sup_{0 \le r \le p} h(t,r)$,

$\mu(t) = \sup\{\|f(t,x) - f(t,y)\| \colon \|x\| \le Mt, \ \|y\| \le Mt\}$ and $q_n = Ma/n$;

(8) $$\Big\|u_n(v)(t) - \int_0^t f\big(s, u_n(v)(s)\big)\, ds - \int_0^t \widetilde{g}(s,v)\, ds\Big\| \le m(q_n)$$

for $t \in J$ and $v \in Q$, where $m(p) = \int_0^a \min\big(\mu(t), \varepsilon(t,p)\big)\, dt$ for $p \ge 0$; moroever, $\lim_{p \to 0_+} m(p) = 0$;

(9) for any $\varepsilon > 0$ and $v_0 \in Q$ there exists $\delta > 0$ such that

$$\int_0^t \|\widetilde{g}(s,v) - \widetilde{g}(s,v_0)\|\, ds \le \varepsilon \quad \text{for } t \in J, \ v \in Q, \ \|v - v_0\|_c < \delta,$$

and consequently $\|u_n(v)(t) - u_n(v_0)(t)\| \le m_n(\varepsilon)$, where $m_0(p) = 0$, $m_{k+1}(p) = m\big(m_k(p)\big) + p$ for $k = 0, 1, 2, \ldots$ and $p \ge 0$; obviously $\lim_{p \to 0_+} m_n(p) = 0$.

Let u_n denote the mapping $v \to u_n(v)$ for $v \in Q$. From (5) and (6) it follows that $u_n(Q)$ is a relatively compact set in C. Since, by (9), u_n is continuous, it is completely continuous mapping $Q \to Q$. Furthermore, analogously as in [5], the inequality (7) implies

(10) $\|u_n(v)(t) - u_m(v)(t)\| \le w_n(t)$ for $m \ge n$, $t \in J$, $v \in Q$,

where w_n is the maximal solution of $z'(t) = \min\big(\mu(t), h(t, z(t)) + 2\varepsilon(t, q_n)\big)$ issuing from $(0,0)$. Since w_n uniformly converges to 0 as $n \to \infty$, from (10) we conclude that the sequence (u_n) converges uniformly on Q to a limit u. By passing to the limit in (8), we obtain

(11) $$u(v)(t) = \int_0^t f\big(s, u(v)(s)\big)\, ds + \int_0^t \widetilde{g}(s,v)\, ds \qquad (t \in J, \ v \in Q).$$

Since u is the uniform limit of the sequence of completely continuous mappings u_n, u is a completely continuous mapping $Q \to Q$. Remark that

$$u(v)'(t) = f\big(t, u(v)(t)\big) + \widetilde{g}(t, v) \quad \text{for } t \in J \text{ and } v \in Q.$$

Now we shall show that for each $\varepsilon > 0$

$$v \mid \langle 0, \varepsilon\rangle = z \mid \langle 0, \varepsilon\rangle \Rightarrow u(v) \mid \langle 0, \varepsilon\rangle = u(z) \mid \langle 0, \varepsilon\rangle \qquad (v, z \in Q). \tag{12}$$

Indeed, if $v, z \in Q$ and $v(t) = z(t)$ for $t \in \langle 0, \varepsilon\rangle$, then $\widetilde{g}(t, v) = \widetilde{g}(t, z)$ for $t \in \langle 0, \varepsilon\rangle$, and hence

$$\begin{aligned} D_+\|u(v)(t) - u(z)(t)\| &\le \|u(v)'(t) - u(z)'(t)\| \\ &= \big\|f\big(t, u(v)(t)\big) - f\big(t, u(z)(t)\big)\big\| \\ &\le \min\big(\mu(t), h\big(t, \|u(v)(t) - u(z)(t)\|\big)\big) \quad \text{for } t \in \langle 0, \varepsilon\rangle. \end{aligned}$$

Since $u(v)(0) = u(z)(0) = 0$ and h is a Kamke function, by Olech's Lemma [5, Lemma 1] this implies $\|u(v)(t) - u(z)(t)\| = 0$ for $t \in \langle 0, \varepsilon\rangle$. This proves (12). We see that the mapping $v \to u(v)$ satisfies all assumptions of a Vidossich theorem [11; Corollary 1.2]. By applying this theorem, we conclude that the set Fix u is an R_δ. From (11) it is clear that Fix $u \subset S$. Conversely, let $v \in S$. Since f satisfies the Kamke condition, the Cauchy problem

$$z'(t) = f\big(t, z(t)\big) + \widetilde{g}(t, v), \quad z(0) = 0 \tag{13}$$

has a unique solution $z = u(v)$. As v satisfies (13), we get $v = u(v)$, so that $v \in \operatorname{Fix} u$. Thus $S = \operatorname{Fix} u$ which ends our proof.

3. Let α be the Kuratowski measure of noncompactness in E.

THEOREM 2. *If there exist Lebesgue integrable functions* h: $D \to R_+$ *and* k: $D^2 \to R_+$ *such that*

$$\alpha\big(f(t, X)\big) \le h(t)\alpha(X) \quad \textit{and} \quad \alpha\big(g(t, s, X)\big) \le k(t, s)\alpha(X) \tag{14}$$

for $t, s \in D$ *and for each subset* X *of* B, *then the set* S *of all solutions of the problem* (1)–(2), *defined on* J, *is an* R_δ.

Proof. Put

$$r(x) = \begin{cases} x & \text{for } x \in B \\ \frac{bx}{\|x\|} & \text{for } x \in E \setminus B. \end{cases}$$

Then r is a continuous function $E \to B$ and

$$r(X) \subset \bigcup_{0 \le \lambda \le 1} \lambda X \quad \text{for } X \subset E,$$

so that $\alpha\big(r(X)\big) \le \alpha(X)$ for each bounded subset X of E. Consequently, putting

$$\overline{f}(t, x) = f\big(t, r(x)\big),$$

$$\overline{g}(t,s,x) = g\big(t,s,r(x)\big) \quad (t,s \in D,\ x \in E),$$

we obtain bounded continuous functions $\overline{f}$: $D \times E \to E$ and $\overline{g}$: $D^2 \times E \to E$, satisfying (14) for bounded subsets X of E, such that on J the problem (1)–(2) is equivalent to

$$x'(t) = \overline{f}\big(t,x(t)\big) + \int_0^t \overline{g}\big(t,s,x(s)\big)\,ds, \quad x(0) = 0.$$

For simplicity, we shall write f and g instead of $\overline{f}$ and $\overline{g}$, respectively. We define a mapping F by

$$F(x)(t) = \int_0^t f\big(s,x(s)\big)\,ds + \int_0^t \widetilde{g}(s,x)\,ds \quad (t \in J,\ x \in C).$$

By the Lebesgue dominated convergence theorem, from the considerations of Section 1 we deduce that F is a continuous mapping $C \to C$. Moreover, $F(C)$ is an equiuniformly continuous subset of C, $F(x)(0) = 0$ for $x \in C$ and for each $\varepsilon > 0$

$$x \mid \langle 0,\varepsilon\rangle = y \mid \langle 0,\varepsilon\rangle \Rightarrow F(x) \mid \langle 0,\varepsilon\rangle = F(y) \mid \langle 0,\varepsilon\rangle \quad (x,y \in C).$$

From (3) and (4) it follows that

$$\|F(x)(t)\| \le b \quad \text{for } x \in C.$$

Consequently, a function $x \in C$ is a solution of (1)–(2) iff $x = F(x)$. Now we shall show that

(15) Each sequence (x_n) in C such that

$$\lim_{n\to\infty} \|x_n - F(x_n)\|_c = 0 \text{ has a limit point.}$$

Let (x_n) be a sequence in C such that

$$\lim_{n\to\infty} \|x_n - F(x_n)\|_c = 0. \tag{16}$$

Put $V = \{x_n\colon n \in N\}$ and $V(t) = \{x_n(t)\colon n \in N\}$. As $V \subset \{x_n - F(x_n)\colon n \in N\} + F(V)$, from (16) it follows that the set V is equicontinuous. Thus the function $t \to v(t) = \alpha\big(V(t)\big)$ is continuous on J. By (14) and Heinz's theorem [4, Th. 2.1] we have

$$\alpha\big(\{\widetilde{g}(\tau,x_n)\colon x \in N\}\big) = \alpha\Big(\Big\{\int_0^\tau g\big(\tau,s,x_n(s)\big)\,ds\colon n \in N\Big\}\Big)$$

$$\le 2\int_0^\tau \alpha\big(\{g\big(\tau,s,x_n(s)\big)\colon n \in N\}\big)\,ds = 2\int_0^\tau \alpha\big(g\big(\tau,s,V(s)\big)\big)\,ds$$

$$\le 2\int_0^\tau k(\tau,s)\alpha(V(s))\,ds = 2\int_0^\tau k(\tau,s)v(s)\,ds \quad \text{for } \tau\in J,$$

and further

$$\alpha\Big(\Big\{\int_0^t \widetilde{g}(\tau,x_n)\,d\tau\colon n\in N\Big\}\Big) \le 2\int_0^t \alpha(\{\widetilde{g}(\tau,x_n)\colon n\in N\})\,d\tau$$

$$\le 4\int_0^t \Big[\int_0^\tau k(\tau,s)v(s)\,ds\Big]\,d\tau \quad \text{for } t\in J.$$

Similarly

$$\alpha\Big(\Big\{\int_0^t f(s,x_n(s))\,ds\colon n\in N\Big\}\Big) \le 2\int_0^t \alpha(\{f(s,x_n(s))\colon n\in N\})\,ds$$

$$\le 2\int_0^t h(s)v(s)\,ds \quad \text{for } t\in J.$$

Since $V(t)\subset\{x_n(t)-F(x_n)(t)\colon n\in N\}+F(V)(t)$ and $\alpha(\{x_n(t)-F(x_n)(t)\colon n\in N\})=0$, we have $\alpha(V(t))\le\alpha(F(V)(t))$. This implies that

$$v(t)\le\alpha(F(V)(t))$$

$$\le \alpha\Big(\Big\{\int_0^t f(s,x_n(s))\,ds\colon n\in N\Big\}\Big)+\alpha\Big(\Big\{\int_0^t \widetilde{g}(\tau,x_n)d\tau\colon n\in N\Big\}\Big)$$

$$\le 2\int_0^t h(s)v(s)\,ds+4\int_0^t\Big[\int_0^\tau k(\tau,s)v(s)\,ds\Big]\,d\tau \quad \text{for } t\in J.$$

Consequently

$$v(t)\le 2\int_0^t h(s)v(s)\,ds+4\int_0^t q(t,s)v(s)\,ds \quad \text{for } t\in J, \tag{17}$$

where

$$q(t,s)=\int_s^t k(\tau,s)\,d\tau \quad \text{for } 0\le s\le t\le a.$$

The function $t\to q(t,s)$ is continuous and the function $s\to q(t,s)$ is integrable, because

$$\int_0^t q(t,s)\,ds=\int_0^t\Big[\int_s^t k(\tau,s)\,d\tau\Big]\,ds=\int_0^t\Big[\int_0^\tau k(\tau,s)\,ds\Big]\,d\tau.$$

As the function v is continuous, from (17) we deduce that $\alpha(V(t))=v(t)=0$ for $t\in J$. Therefore the set $V(t)$ is relatively compact in E. Hence, by

Ascoli's theorem, V is a relatively compact subset of C. This proves (15). Applying now Th. 5 of [9], we conclude that the set $S = \mathrm{Fix}\,F$ is an R_δ.

4. The continuity of the functions f and g guaranted that a solution of (1)–(2) was of class C^1. Clearly the integrals in (1) make sense for many functions f and g which are not continuous. In this case we must replace classical solutions by solutions in the Caratheodory sense (cf. [12], p. 42). In this section we assume that

1^0 $(t,x) \to f(t,x)$ is a function from $D \times E$ into E which is strongly measurable in t and continuous in x, and there exists an integrable function $m_1\colon D \to R_+$ such that $\|f(t,x)\| \le m_1(t)$ for $t \in D$ and $x \in B$;

2^0 $(t,s,x) \to g(t,s,x)$ is a function from $D^2 \times B$ into E which is strongly measurable in (t,s) and continuous in x, and there exists an integrable function $m\colon D^2 \to R_+$ such that $\|g(t,s,x)\| \le m(t,s)$ for $t,s \in D$ and $x \in B$. It is clear that the function $m_2\colon D \to R_+$ defined by $m_2(t) = \int_0^t m(t,s)\,ds$ is integrable. Choose a positive number a in such a way that $a \le d$ and $\int_0^a \big(m_1(t) + m_2(t)\big)\,dt \le b$. Let $J = \langle 0, a\rangle$.

THEOREM 3. *If the functions f and g satisfy* (14), *then the set S of all Caratheodory solutions of the problem* (1)–(2), *defined on J, is an R_δ.*

Proof. It follows from 2^0 that for each fixed $x \in C$ the function $t \to \widetilde{g}(t,x)$ is strongly measurable and $\|\widetilde{g}(t,x)\| \le m_2(t)$ for $t \in J$, $x \in C$.

Moreover, by the Lebesgue dominated convergence theorem for each fixed $t \in J$ the function $x \to \widetilde{g}(t,x)$ is continuous on C. Hence we may repeat the proof of Theorem 2.

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