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Mateusz Wiśniewski

EXTREMES IN MULTIVARIATE MIXING SEQUENCES

1. Introduction

The classical results concerning a limit behaviour of extreme order statistics and point processes of exceedances in one-dimensional mixing stationary sequences can be found in Leadbetter et al. [6]. The multivariante i.i.d. case was discussed by a lot of authors (see for instance Wiśniewski [11]). Hsing [4] and Hüsler [5] investigated weak convergence of the maximum vectors under stationarity and mixing conditions. Extremes and exceedances of high boundaries by stationary and nonstationary multivariate random sequences in the rare events context were presented by Falk et al. [2]. The purpose of this paper is to extend results of Wiśniewski [11] and some of results of Hsing [4] and Hüsler [5] to the stationary case of multidimensional extreme order statistics and multivariate point processes of exceedances. Identical with Hüsler's [5] assumptions are used, apart from the condition D_d^Ω which is a slightly stronger version of the long range dependence condition D_d (see Section 3). Especially, the condition D' is assumed which guarantees that the clustering of extreme values does not occur. In the second Section we set up notation and compiles some basic facts of the theory of Point Processes. In Section 3 we provide a detailed exposition of assumptions. Section 4 is devoted to the study of the weak convergence of point processes. The obtained limit distributions are just the same as in the i.i.d. case. Hence the extremal index function (see Nandagopalan [9] or Perfekt [10]) is constant and equal to 1. It is worth pointing out that, as opposed to the univariate case, the limit distributions do not have to be Poisson type (compare with Falk et al. [2], Section 10.6). This effect result only from an allowable dependence between the various components of the multivariate observations. On account of Nandagopalan et al. [8] it is obvious that the

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obtained limit point processes have Compound Poisson distributions. The dependence structure among the components of the multivariate distributions is an important aspect in the multivariate analysis. In this Section we will look more closely at independence and complete dependence of the components of the limit processes. Also the relation between max-infinite divisibility and infinite divisibility of the limit point processes of exceedances are established. In Section 5 we deal with the limit extreme order statistics and their dependence structure. The last Section contains lemmas.

2. Notation

Let $\aleph(p)$ denote the set $\{1,2,\ldots,p\}$ for an arbitrary $p \in \aleph$. Fix $d \in \aleph$. Let us assume that $\overline{\xi} = \{\overline{X}_n : n \in \aleph\}$ is a stationary sequence of d-dimensional random vectors $\overline{X}_n = (X_{ni} : i \in \aleph(d))$ with a common distribution function F. For any $G \subset \aleph(d)$ we use the symbol F_G to denote the distribution function of the vector $\overline{X}_{nG} = (X_{ni} : i \in G)$. Let $\overline{M}_n^k$ denote for $k \in \aleph(n)$, $n \in \aleph$ the vector of k-th extreme order statistic (defined componentwise). We also define by $\overline{M}_{nG}^k(A)$ the k-th extreme order statistic in the sequence $\{\overline{X}_{jG} : j \in \aleph, j/n \in A\}$, where $A \subset (0,1]$, $G \subset \aleph(d)$. The main question deals with the convergence of the distribution of $\overline{M}_n^k$ under a suitable normalization

$$P(\overline{M}_n^k \leq \overline{u}_n) \xrightarrow[n\to\infty]{w} H^k(\overline{x}) \quad \text{for } k \in \aleph \tag{1}$$

where $\overline{u}_n = \overline{u}_n(\overline{x}) = \overline{a}_n\overline{x}+\overline{b}_n, \overline{x} \in \Re^d$ and $\overline{a}_n, \overline{b}_n$ are sequences in $\Re^d$, $\overline{a}_n$ with positive components. (Arithmetical operations are meant componentwise).

Let $\{S_n : n \in \aleph\}$ denote the sequence of the d-variate point processes of exceedances obtained on the base $\overline{\xi}$ and previously considered normalization (see Wiśniewski [11, 12] and the one-dimensional case in Leadbetter et al. [6]). This means that

$$S_n(\Delta, \overline{x}) = \sum_{i=1}^{d} \sum_{j:\frac{j}{n}\in\delta_i} \overline{1}_{\{X_{ji}>u_{ni}\}}, \quad \text{for } \overline{x} \in \Re^d,\ \Delta = \bigcup_{i=1}^{d}(\delta_i \times \{i\}),$$

where δ_i are Borel subsets of $(0,1]$ and $\overline{1}_A$ denotes the characteristic function of the set A. The marginal point processes of the process S_n are defined in the following way

$$S_{ni}(\delta_i) = S_n(\delta_i \times \{i\}), \quad \text{for } i \in \aleph(d).$$

For any point process $\overline{\eta}$ we can define the avoidance function $F_{\overline{\eta}}$ (see Daley et al. [1]). We have

$$F_{S_n}(\Delta) = P[S_{ni}(\delta_i) = 0 : i \in \aleph(d)].$$

The distribution of a simple point process defined on $(0,1]$ is determined by values of its avoidance function on the sets Δ, where δ_i are finite sums of subintervals of $(0,1]$ (see Leadbetter et al. [6]). According to the above property, we use avoidance functions for characterization of limit distributions. For example, in the i.i.d. case the avoidance function of the limit point process of exceedances S has the form

$$F_S(\Delta) = \prod_{G\in\Gamma} H^1_G(\overline{x}_G)^{m(\delta^*_G)}, \tag{2}$$

where m denotes Lebesgue measure on $(0,1]$,

$$\Gamma = \{\{i_1,\ldots,i_k\} : 1 \le i_1 < \ldots < i_k \le d,\ k \in \aleph(d)\}$$

and

$$\delta^*_G = \bigcap_{i\in G} \delta_i \setminus \bigcup_{i\notin G} \delta_i \quad \text{(see Wiśniewski [11]).}$$

We also need a multivariate version of the Kallenberg theorem on a weak convergence of point processes (see Wiśniewski [11]). For point processes defined on space $(0,1]$ this result can be expressed in the following way.

PROPOSITION 2.1. *If a sequence of d-variate point processes $\overline{\eta}_n$ satisfies the conditions:*

(i) $\eta_{ni} \xrightarrow[n\to\infty]{w} \eta_i$, $i \in \aleph(d)$, *where η_i are simple point processes,*

(ii) *there exist limits*

$$\lim_{n\to\infty} F_{\overline{\eta}_n}(\Delta),$$

for all $\Delta = \cup_{i=1}^d \delta_i \times \{i\}$, δ_i are finite sums of subintervals of $(0,1]$, then

$$\overline{\eta}_n \xrightarrow[n\to\infty]{w} \overline{\eta}$$

where $\overline{\eta}$ is simple and possesses the marginals η_i.

3. Assumptions

It is well-known that in the i.i.d. case (1) is equivalent to

$$n\{1 - F[\overline{u}_n(\overline{x})]\} \xrightarrow[n\to\infty]{} -\ln H^1(\overline{x}), \quad \text{for all } \overline{x} \in D_H, \tag{3}$$

where

$$D_H = \{\overline{x} \in \Re^d : H^1(\overline{x}) > 0\}.$$

From now on we make Assumption (3).

Subsets I and J of $\aleph$ are called m-separated if $\min(J) - \max(I) \ge m$ or $\min(I) - \max(J) \ge m$. Let us set $B_n(I) = \cap_{k\in I} \cap_{i\in\aleph(d)} B^n_{ki}$, where

$$B^n_{ki} = \{X_{ki} \le u_{ni}\}. \tag{4}$$

Additionally we need the following assumptions : (the notation is adapted from Hüsler [5]).

Condition $D_d(\overline{u}_n)$ holds if a sequence $\alpha_{n,m}$ exists such that for all n and m

$$\left|P[B_n(I \cup J)] - P[B_n(I)]P[B_n(J)]\right| \leq \alpha_{n,m},$$

for all $I, J \subset \aleph(n)$ which are m-separated and such that $\alpha_{n,m_n^*} \to 0$, for some sequence $m_n^* = o(n)$.

Condition $D_d'(\overline{u}_n)$ holds if

$$\limsup_{n\to\infty} n \sum_{1 \leq i \leq \frac{n}{r}} P[\overline{X}_1 \not\leq \overline{u}_n, \overline{X}_i \not\leq \overline{u}_n] \underset{r\to\infty}{\longrightarrow} 0.$$

Condition $D_d^*(\overline{u}_n)$ holds if

$$n \sum_{1 \leq j \leq j' \leq d} P[X_{1j} > u_{nj}, X_{1j'} > u_{nj'}] \underset{n\to\infty}{\longrightarrow} 0.$$

Condition $D_d^{**}(\overline{u}_n(\overline{x}))$ holds if

$$n\ \{P[X_{1p} > u_{np}(x_p), X_{1q} > u_{nq}(x_q)] - P[X_{1p} > u_{np}(x_p)]\} \underset{n\to\infty}{\longrightarrow} 0,$$

for all $1 \leq p \neq q \leq d$ and $\overline{x}$ such that $H_p^1(x_p) \geq H_q^1(x_q)$.

In order to get asymptotic results, it is necessary to strengthen the condition D_d in the following way:

Condition $D_d^\Omega(\overline{u}_n)$ holds if condition $D_d(\overline{u}_n)$ holds with (3) replaced by

$$B_{ki}^n = \{X_{ki} \leq u_{ni}\} \ \text{ or } \ B_{ki}^n = \Omega,$$

where Ω denotes the sure event.

Although D_d^Ω is stronger than D_d it is easily seen that strong mixing condition implies D_d^Ω. Furthermore D_d^Ω is still weaker than mixing conditions defined as the Δ-condition (see Hsing et al. [3]).

4. On weak convergence of multivariate point processes of exceedances

THEOREM 4.1. *Let a stationary sequence $\overline{\xi}$ satisfy (3) and $D_d^\Omega(\overline{u}_n(\overline{x}))$, $D_d'(\overline{u}_n(\overline{x}))$, for all $\overline{x} \in D_H$. Then*

$$S_n(\overline{x}) \underset{n\to\infty}{\overset{w}{\longrightarrow}} S(\overline{x}), \quad \text{for all } \overline{x} \in D_H,$$

$S(\overline{x})$ is a simple point process with Poisson marginals and the avoidance function of the form (2).

Proof. The main idea of the proof is adopted from the proof of Theorem 5.2.1 in Leadbetter et al. [6]. This theorem shows that

$$S_{ni}(x_i) \xrightarrow[n\to\infty]{w} S_i(x_i) \quad \text{for all } x_i,$$

where $S_i(x_i)$ are simple Poisson point processes. Thanks to the above and Proposition 2.1 it is sufficient to show that

$$\lim_{n\to\infty} F_{S_n(\overline{x})}(\Delta) \quad \text{exist for all } \Delta = \bigcup_{i=1}^{d} \delta_i \times \{i\},$$

where δ_i are finite sums of subintervals of $(0,1]$. Since

$$\bigcup_{G\in\Gamma} \delta_G^* \times G = \Delta \quad \text{and} \quad \delta_{G_1}^* \cap \delta_{G_2}^* = \emptyset \text{ for } G_1 \neq G_2$$

we see that

$$F_{S_n(\overline{x})}(\Delta) = P\left[\bigcap_{G\in\Gamma} \{S_{ni}(\delta_G^*, x_i) = 0 : i \in G\}\right]$$

It is easy to check that for any $A \subset (0,1]$, $i \in \aleph(d)$, $k \in \aleph$

$$\{S_{ni}(A) < k\} = \{M_{ni}^k(A) \leq u_{ni}\}. \tag{5}$$

Thus we get

$$F_{S_n(\overline{x})}(\Delta) = P\left[\bigcap_{G\in\Gamma} \{\overline{M}_{nG}^1(\delta_G^*) \leq \overline{u}_{nG}(\overline{x}_G)\}\right].$$

The sets δ_G^* are disjoint and consist of finite sums of disjoint subintervals of $(0,1]$. Therefore, we can use Lemma 6.2. (see Section 6) to obtain the convergence:

$$\lim_{n\to\infty} F_{S_n(\overline{x})}(\Delta) = \prod_{G\in\Gamma} H_G^1(\overline{x}_G)^{m(\delta_G^*)},$$

which proves the theorem.

COROLLARY 4.2. *Under the assumptions of Theorem* 4.1, *if moreover* $D_d^*(\overline{u}_n(\overline{z}))$ *holds for some* $\overline{z}$ *with* $0 < H_i^1(z_i) < 1$ *for each* $i \in \aleph(d)$, *then for all* $\overline{x} \in D_H$ $S(\overline{x})$ *have independent components* $S_i(x_i)$.

The proof is completed by using Corollaries 2 and 6 of Wiśniewski [11] and Theorem 2.4 of Hüsler [5].

COROLLARY 4.3. *Under the assumptions of Theorem* 4.1, *if moreover* $D_d^{**}(\overline{u}_n(\overline{z}))$ *holds for some* $\overline{z}$, *then for all* $\overline{x} \in D_H$ $S(\overline{x})$ *have almost surely equal components* (*i.e.* $S_1(z) = \ldots = S_d(z)$ *a.s. for all* $z \in \Re$).

It is sufficient to use Corollaries 3 and 6 of Wiśniewski [11] together with Theorem 3.6 of Hüsler [5].

Since a multivariate simple point process is Poisson process if and only if its marginals are independent Poisson point processes (see Corollary 1. of Wiśniewski [11]), Corollary 5.3 asserts that limit point processes of exceedances are not always those of Poisson.

COROLLARY 4.4. *Under the assumptions of Theorem* 4.1, *if moreover* F *is max-infinitly divisible* (*i.e. for all* $n \in \aleph$ $F^{1/n}$ *is a distribution function – see Falk et al. [2]*), *then for all* $\overline{x} \in D_H$ *the distributions of* $S(\overline{x})$ *are infinitely divisible* (*see the definition in Matthes* [7]).

It results from Corollaries 5 and 6 of Wiśniewski [11].

5. On weak convergence of multivariate extreme order statistics

A weak convergence of point processes implies a weak convergence of their finite dimensional distributions (see Lemma 9.1.IV in Daley et al. [1]). According to the above remark and (5) we can use the results of the last section to obtain the similar ones for extreme order statistics.

THEOREM 5.1. *Under the assumptions of Theorem* 4.1,

$$P[\overline{M}_n^k \leq \overline{u}_n(\overline{x})] \underset{n\to\infty}{\overset{w}{\longrightarrow}} H^k(\overline{x}), \quad \textit{for all } k \in \aleph.$$

It is important point to note here that H^k are the same as suitable limit distributions in the i.i.d. case. Indeed, the obtained avoidance function of the limit point process of exceedances is identical to the suitable one in the i.i.d. case.

COROLLARY 5.2. *Under the assumptions of Corollary* 4.2, *for all* $k \in \aleph$, H^k *have independent components.*

COROLLARY 5.3. *Under the assumptions of Corollary* 4.3, *for all* $k \in \aleph$, H^k *have almost surely equal components.*

6. Lemmas

LEMMA 6.1. *Let* A *be a subinterval of* $(0,1]$ *and* $G \subset \aleph(d)$. *If a stationary sequence* $\overline{\xi}$ *satisfies the conditions* (3) *and* $D_d(\overline{u}_n(\overline{x}))$, $D'_d(\overline{u}_n(\overline{x}))$ *for all* $\overline{x} \in D_H$, *then*

$$P[\overline{M}^1_{nG}(A) \leq \overline{u}_{nG}(\overline{x}_G)] \underset{n\to\infty}{\longrightarrow} H^1_G(\overline{x}_G)^{m(A)}, \quad \textit{for all } \overline{x} \in \Re^d.$$

The proof is slightly different from the proof of Corollary 3.6.4 of Leadbetter et al. [6].

LEMMA 6.2. *Let* $A_1, \ldots, A_p$ *be disjoint subintervals of* $(0,1]$ *and* $G_1, \ldots$ $\ldots, G_p \subset \aleph(d)$ (*not necessarily different*). *If a stationary sequence* $\overline{\xi}$ *satisfies* *the conditions (3) and $D_d^{\Omega}(\overline{u}_n(\overline{x}))$, $D_d'(\overline{u}_n(\overline{x}))$, for all $\overline{x} \in D_H$, then*

$$P[\overline{M}^1_{nG_1}(A_1) \le \overline{u}_{nG_1}(\overline{x}_{G_1}), \ldots, \overline{M}^1_{nG_p}(A_p) \le \overline{u}_{nG_p}(\overline{x}_{G_p})] \underset{n\to\infty}{\longrightarrow} \prod_{i=1}^{p} H^1_{G_i}(\overline{x}_{G_i})^{m(A_i)}, \quad \text{for all } \overline{x} \in \Re^d.$$

Proof. To shorten notation we write

$$g_n(B_1, \ldots, B_p) = P[\overline{M}^1_{nG_1}(B_1) \le \overline{u}_{nG_1}(\overline{x}_{G_1}), \ldots, \overline{M}^1_{nG_p}(B_p) \le \overline{u}_{nG_p}(\overline{x}_{G_p})].$$

Without loss of generality we can assume that $a_1 < \ldots < a_p$ for $a_i \in A_i$. For $i \in \aleph(p)$, $n \in \aleph$, let the intervals $\overline{A}_i^n$, A_i^n be given by:

$$\overline{A}_i^n \cup A_i^n = A_i,\ \overline{A}_i^n \cap A_i^n = \emptyset,\ a < b \ \text{ for } \ a \in \overline{A}_i^n,\ b \in A_i^n \ \text{ and } \ \#\{j \in \aleph : \frac{j}{n} \in A_i^n\} = m_n^*,$$

where m_n^* originates from D_d^{Ω}. We can write

$$\left| g_n(A_1, \ldots, A_p) - \prod_{i=1}^{p} g_n(A_i) \right|$$

$$\le \left| g_n(A_1, \ldots, A_p) - g_n(\overline{A}_1^n, \ldots, \overline{A}_p^n) \right| + \left| g_n(\overline{A}_1^n, \ldots, \overline{A}_p^n) - \prod_{i=1}^{p} g_n(\overline{A}_i^n) \right| + \left| \prod_{i=1}^{p} g_n(\overline{A}_i^n) - \prod_{i=1}^{p} g_n(A_i) \right| = S_1(n) + S_2(n) + S_3(n).$$

Thanks to Lemma 6.1 it is sufficient to show that $S_1(n) = o(n)$, $S_2(n) = o(n)$, $S_3(n) = o(n)$ as $n \to \infty$. The sets

$$\{j \in \aleph : \frac{j}{n} \in \overline{A}_1^n\}, \ldots, \{j \in \aleph : \frac{j}{n} \in \overline{A}_p^n\}$$

are m_n^* – separated. Thus we can use D_d^{Ω} and deduce that

$$S_2(n) \le p\alpha_{n,m_n^*}$$

(compare to Lemma 3.2.2 of Leadbetter et al. [6]), and so $S_2(n) = o(n)$ as $n \to \infty$. For all sufficiently large n the sets

$$\left\{ j \in \aleph : \frac{j}{n} \in A_1^n \right\}, \ldots, \left\{ j \in \aleph : \frac{j}{n} \in A_p^n \right\}$$

are also m_n^* - separated. Hence

$$\left| g_n(A_1^n, \ldots, A_p^n) - \prod_{i=1}^{p} g_n(A_i^n) \right| \leq p\alpha_{n,m_n^*}.$$

According to the above remark and from the inequality

$$S_1(n) \leq 1 - g_n(A_1^n, \ldots, A_p^n)$$

it follows that the condition

$$S_1(n) = o(n) \quad \text{as} \quad n \to \infty \tag{6}$$

holds if for all $i \in \aleph(p)$ $g_n(A_1^n) \to 1$ as $n \to \infty$. But for any $\varepsilon > 0$ and $i \in \aleph(p)$ we have

$$H_{G_i}^1(\overline{x}_{G_i})^\varepsilon \leq \liminf_{n\to\infty} g_n(A_i^n) \leq \limsup_{n\to\infty} g_n(A_i^n) \leq 1$$

and so (6) holds. According to

$$S_3(n) \leq \sum_{i=1}^{p} \left| g_n(A_i^n) - g_n(A_i) \right|$$

the analysis similar to that in the proof of (6) shows that $S_3(n) = o(n)$ as $n \to \infty$, which completes the proof.

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DEPARTMENT OF MATHEMATICS
TECHNICAL UNIVERSITY OF KIELCE
Al. Tysiąclecia PP 7
25-314 KIELCE, POLAND
E-mail: ztpmw@eden.tu.kielce.pl

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