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THE ORDER OF STARLIKENESS OF THE p-VALENT ALPHA-CONVEX FUNCTIONS AND NUNOKAWA'S CONJECTURE

Let $M_p(\alpha,\beta)$, $p \in N$, $\alpha,\beta \in R$, denote the class of functions of the form $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ such that $\frac{f(z)f'(z)}{z^{2p-1}} \neq 0$ and $Re\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f(z)}\right)\right] > \beta$ for $z \in D = \{z : |z| < 1\}$. In the present paper some relations between the class $M_p(\alpha,\beta)$ and the class $M_1(\frac{\alpha}{p},\frac{\beta}{p})$ are given. Thus we determine a relation between the class $M_1(\frac{1}{p},0)$ and the class $S_p^c = M_p(1,0)$ of p-valent convex functions. Moreover we determine the order of starlikeness for classes $M_p(\alpha,0)$ and S_p^c and show that Nunokawa's conjecture about the order of starlikeness for the class S_p^c is false.

1. Introduction

Let p be a positive integer, α,β be real numbers, $0 \leq \beta < p$. Suppose that a function f of the form

$$(1) \qquad f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n,$$

is analytic in the unit disc $K = \{z : |z| < 1\}$ with $\frac{f(z)f'(z)}{z^{2p-1}} \neq 0$ for $z \in K$. If

$$(2) \qquad J(f) = Re\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f(z)}\right)\right] > \beta$$

for $z \in K$, then f is said to be a p-valent alpha-convex function of order β. We denote the class of such functions by $M_p(\alpha,\beta)$.

Put $M_p(\alpha) = M_p(\alpha,0)$, $M(\alpha) = M_1(\alpha)$.

For $\alpha = 0$ and $\alpha = 1$ we have $M_p(0,\beta) = S_p^*(\beta)$, $M_p(1,\beta) = S_p^c(\beta)$, where $S_p^*(\beta)$ is the well-known class of p-valent starlike functions of order β and $S_p^c(\beta)$ is the well-known class of p-valent convex functions of order β. Let us denote: $S_p^* = S_p^*(0), S^* = S_1^*, S_p^c = S_p^c(0), S^c = S_1^c$.

The class $M(\alpha)$ of alpha-convex functions was introduced by P. T. Mocanu [5] and it has been intensively investigated by S. S. Miller, P. T. Mocanu and M. O. Reade [3], [4] and others. The class $M_p(\alpha,\beta)$ was studied by M. Nunokawa, S. Owa, H. Saitoh, T. Yaguchi, S. K. Lee, T. Sekine, S. Fukai [7], [8], [9] and others. In paper [4] S. S. Miller, P. T. Mocanu and M. O. Reade determined the order of starlikeness for the class $M(\alpha)$.

In the present paper some relation between the class $M_p(\alpha,\beta)$ and the class $M_1(\frac{\alpha}{p},\frac{\beta}{p})$ are given. Thus we determine a relation between the class $M_1(\frac{1}{p},0)$ and the class $S_p^c = M_p(1,0)$. Moreover we determine the order of starlikeness for classes $M_p(\alpha)$ and S_p^c.

In paper [8] there is an information about Nunokawa's conjecture, that the order of starlikeness for the class S_p^c is equal to $\frac{\Gamma(p+\frac{1}{2})}{\sqrt{\pi}\Gamma(p)}$, where $\Gamma(.)$ denotes the Gamma function. In paper [6] M. Nunokawa proved this conjecture.

In the present paper we show that Nunokawa's conjecture is false and determine the correct order of starlikeness for the class S_p^c.

2. Preliminaries

We start from listing some lemmas, which will be useful later on.

LEMMA 1 [1]. *Let p be a positive integer, α,β be real numbers, $\alpha \neq 0, 0 \leq \beta < p$. A function f is in the class $M_p(\alpha,\beta)$ if and only if there exists a function $g \in S_p^*$ such that*

$$f(z) = \left\{ \frac{p}{\alpha} \int_0^z \frac{[g(w)]^{\frac{1}{\alpha}}}{w} dw \right\}^{\alpha}, \ z \in D. \tag{3}$$

Putting $\beta = 0$ and $g(z) = \frac{z^p}{(1+z)^{2p}}$ in Lemma 1 we obtain the function

$$k_{p,\alpha}(z) = \left\{ \frac{1}{\alpha} \int_0^z \frac{w^{\frac{p-\alpha}{\alpha}}}{(1+w)^{\frac{2p}{\alpha}}} dw \right\}^{\alpha}. \tag{4}$$

Denote $k_\alpha = k_{1,\alpha}$. Thus for $\alpha \geq 0$ we have

$$k_{p,\alpha} \in M_p(\alpha,\beta) \text{ and } k_\alpha \in M(\alpha). \tag{5}$$

It is also known that

$$k(z) = \frac{z}{(1+z)^2} \in M(\alpha) \text{ for } \alpha \in [-2,0]. \tag{6}$$

LEMMA 2 [3]. *$M(\alpha) \subset S^*$ for all $\alpha \in R$.*

LEMMA 3 [2]. *If $\alpha \in (0,1)$, then k_α itself is starlike of order zero. The result is sharp.*

LEMMA 4 [4]. *If* $f \in M(\alpha), \alpha \geq 1$, *then* $f \in S^*(\beta(\alpha))$, *where*

$$\beta(\alpha)) = \frac{\Gamma\left(\frac{1}{2} + \frac{1}{\alpha}\right)}{\sqrt{\pi}\Gamma\left(1 + \frac{1}{\alpha}\right)}.$$

The result is sharp.

LEMMA 5. *A function* f *is in the class* $S_p^*(\alpha)$ *if and only if there exists a function* $g \in S_1^*\left(\frac{\alpha}{p}\right)$, *such that* $f = g^p$.

Proof. Let $f \in S_p^*(\alpha)$. Denote by g that branch of the n-th order root of the function f for which $g'(0) = 1$. Then we have

$$Re\frac{zg'(z)}{g(z)} = Re\frac{zf'(z)}{pf(z)} \geq \frac{\alpha}{p},$$

what means that $g \in S_1^*\left(\frac{\alpha}{p}\right)$.

Now, let $f = g^p$ and $g \in S_1^*\left(\frac{\alpha}{p}\right)$. Then f is p-valent and

$$Re\frac{zf'(z)}{f(z)} = Re\ p\frac{zg'(z)}{g(z)} > \alpha,$$

that is $f \in S_p^*(\alpha)$.

3. Main results

THEOREM 1. *A function* f *is in the class* $M_p(\alpha,\beta)$ *if and only if there exists a function* $h \in M_1\left(\frac{\alpha}{p}, \frac{\beta}{p}\right)$, *such that* $f = h^p$.

Proof. By Lemma 1 the function f is in class $M_p(\alpha,\beta)$ if and only if there exists a function $g \in S_p^*(\beta)$, such that the function f satisfies the condition (3). Since by Lemma 5 we have $g = \rho^p$ for some $\rho \in S_1^*\left(\frac{\beta}{p}\right)$, then instead of the condition (3) we can equivalently write:

$$f(z) = \left\{\frac{p}{\alpha}\int_0^z \frac{[\rho(w)]^{\frac{p}{\alpha}}}{w} dw\right\}^{\alpha}$$

or

$$f(z) = [h(z)]^p,$$

where

$$h(z) = \left\{\frac{p}{\alpha}\int_0^z \frac{[\rho(w)]^{\frac{p}{\alpha}}}{w} dw\right\}^{\frac{\alpha}{p}}.$$

It is enough to prove the condition that $h \in M_1\left(\frac{\alpha}{p}, \frac{\beta}{p}\right)$. Since $\rho \in S_1^*\left(\frac{\beta}{p}\right)$, it is an immediate consequence of the definition of h and Lemma 1.

Using Theorem 1 we can write

$$M_p(\alpha,\beta) = \left\{ f^p : f \in M_1\left(\frac{\alpha}{p}, \frac{\beta}{p}\right) \right\}.$$

Putting $\beta = 0$ in Theorem 1 we have:

COROLLARY 1. *A function f is in the class $M_p(\alpha)$ if and only if there exists a function $h \in M\left(\frac{\alpha}{p}\right)$ such that $f = h^p$.*

Assuming $\alpha = 1$ in Corollary 1 we obtain

COROLLARY 2. *A function f is in the class S_p^c if and only if there exists a function $h \in M\left(\frac{1}{p}\right)$, such that $f = h^p$.*

Thus we can write

$$S_p^c = \left\{ f^p : f \in M\left(\frac{1}{p}\right) \right\}.$$

THEOREM 2. *$M_p(\alpha) \subset S_p^*$ for all $p \in N$, $\alpha \in R$.*

Proof. Let $f \in M_p(\alpha)$. From Theorem 1 we get $f = h^p$, where $h \in M\left(\frac{\alpha}{p}\right)$. From Lemma 2 we obtain that $h \in S^*$. Thus, using Lemma 5, we have $f \in S_p^*$.

THEOREM 3. *Let p be a positive integer, α be a real number, $\alpha \geq -2p$. If $f \in M_p(\alpha)$, then $f \in S_p^*(\beta(p,\alpha))$, where*

$$\beta(p,\alpha) = \begin{cases} 0 & \text{for } -2p \leq \alpha < p \\ \dfrac{p\Gamma\left(\frac{1}{2} + \frac{p}{\alpha}\right)}{\sqrt{\pi}\Gamma\left(1 + \frac{p}{\alpha}\right)} & \text{for } \quad \alpha \geq p. \end{cases} \tag{7}$$

The result is sharp.

Proof. From the Theorem 2 we get $\beta(p,\alpha) \geq 0$ for $\alpha \in R$. Using condition (6) we obtain

$$\beta(1,\alpha) = 0 \quad \text{for } -2 \leq \alpha \leq 0. \tag{8}$$

By Lemma 3 we have

$$\beta(1,\alpha) = 0 \quad \text{for } 0 < \alpha < 1. \tag{9}$$

and by Lemma 4 we obtain

$$\beta(1,\alpha) = \frac{\Gamma\left(\frac{1}{2} + \frac{1}{\alpha}\right)}{\sqrt{\pi}\Gamma\left(1 + \frac{1}{\alpha}\right)} \quad \text{for } \alpha \geq 1. \tag{10}$$

Now, let $f \in M_p(\alpha)$. From Theorem 1 we have $f = h^p$, where $h \in M\left(\frac{\alpha}{p}\right)$. Thus

$$\frac{zf'(z)}{f(z)} = p\frac{zh'(z)}{h(z)},$$

which yields

$$\beta(p,\alpha) = p\beta\left(1, \frac{\alpha}{p}\right). \tag{11}$$

The conditions (8)-(11) give (7).

The extremal functions are the functions $k(z) = \frac{z^p}{(1+z)^{2p}}$ for $-2p \le \alpha \le 0$ and $k_{p,\alpha}$ defined by (4) for $\alpha > 0$.

Remark. For $\alpha \ge p$ Theorem 3 was proved in a different way in [1] by the author and J. Stankiewicz.

Putting $\alpha = 1$ in Theorem 3 we obtain:

COROLLARY 3. *If* $f \in S_p^c$, $p \le 2$, *then* $f \in S_p^*$.

The result is sharp. It means that the order of starlikeness for the class S_p^c is equal to zero.

The Corollary 3 implies, that Nunokawa's conjecture, which states that the order of starlikeness for the class S_p^c is equal to $\alpha(p) = \frac{\Gamma(p+\frac{1}{2})}{\sqrt{\pi}\Gamma(p)} > 0$, is false. In the proof of this conjecture there is an error. Namely, M. Nunokawa maintains, that the extremal function f_1 in his conjecture satisfies the condition

$$\frac{zf_1'(z)}{f_1(z)} = (p - \alpha(p))\frac{1+z}{1-z} + \alpha(p),$$

where $\alpha(p)$ denotes the order of starlikeness for the class S_p^c. Thus the function f_1 has to be the p-valent Koebe function of order α, $f_1(z) = \frac{z^p}{(1+z)^{2(p-\alpha)}}$, $z \in K$. But this function for $p \ge 2$ isn't an extremal function.

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