DEMONSTRATIO MATHEMATICA
Vol. XXX No 3 1997

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THE GENERALIZED REISSNER-NORDSTRÖM SPACE-TIME

1. Introduction

The following 1-parameter family of Robertson-Walker space-times with metric tensor

$$diag\left(-1,\ \frac{R^2}{1-kr^2},\ R^2r^2,\ R^2r^2sin^2\theta\right), \tag{1}$$

where $R = R(t)$, $k \in \langle -1, 1\rangle$ is a parameter and $(x^1, x^2, x^3, x^4) = (t, r, \theta, \varphi)$, is well-known. Such family of space-times was applied in the general relativity theory for parameter values $k = -1, 0, 1$. Resolving the Einstein equation, Friedman has obtained three models of Universe: the open model for $k = -1$, the flat model for $k = 0$ and the closed one for $k = 1$. The space-times for other values of parameter $k \in \langle -1, 1\rangle$ were not applied in physics.

The Reissner-Nordström space-time (denoted shortly by R-N.) has the following metric tensor [1], [2]:

$$diag(-E,\ E^{-1},\ r^2,\ r^2sin^2\theta), \tag{2}$$

where

$$E = 1 - \frac{r_0}{r} + \frac{Kr_0^2}{r^2}, \tag{3}$$

$K = const$, $r_0 = const \geq 0$, $(x^1, x^2, x^3, x^4) = (t, r, \theta, \varphi)$.

The R-N space-time with metric tensor (2) has the scalar curvature equal to zero. The purpose of the present paper is to generalize the R-N space-time to one with nonvanishing scalar curvature.

2. The family of generalized Reissner-Nordström space-times

We introduce the generalized R-N space-times by means of the following metric tensor

$$(4) \qquad diag(-E^{a+1},\ E^{a-1},\ r^2,\ r^2 sin^2\theta)$$

where a is a parameter, $a \in \langle -1, 1\rangle$, E has the form (3) and K is also a parameter, $K \in (\frac{1}{4}, \frac{9}{32})$. Therefore we obtain the 2-parameter family of space-times with respect to parameters a and K. Then the scalar curvature T takes the form [3]:

$$(5) \qquad T = \frac{ar^2E'' - 2E^a + 2}{r^2E^a} \quad \left(:= \frac{ar^2E'' + 2}{r^2E^a} - \frac{2}{r^2}\right).$$

For $a = 0$ the generalized R-N space-times with metric tensor (4) reduces to the space with metric tensor (2), i.e. to the ordinary R-N space -times. In this case the scalar curvature T of (5) vanishes.

We investigate the properties of the family (4) for $K = \frac{1+\varepsilon}{4}$ (then in (2) $E > 0$ and for $\varepsilon = 0$, $K = \frac{1}{4}$, $E = 0$ at the point $\frac{r_0}{2}$).

3. The special cases of values of the scalar curvature

In the case $a = 1$, the metric tensor (4) takes the form

$$(6) \qquad diag(-E^2, 1,\ r^2,\ r^2 sin^2\theta).$$

The formula (5) for $K = \frac{1+\varepsilon}{4}$ takes the form

$$(7) \qquad T = \frac{4(1+\varepsilon)r_0^2}{r^2[4(r-\frac{r_0}{2})^2 + \varepsilon r_0^2]}$$

and its derivative

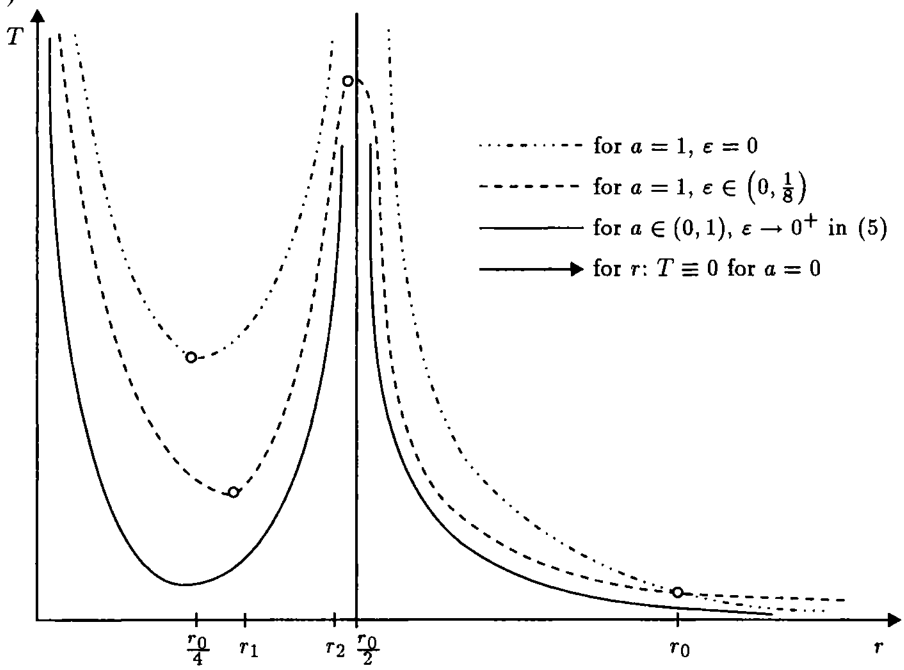
$$(8) \qquad T' = \frac{-8(1+\varepsilon)r_0^2[8r^2 - 6r_0 r + (1+\varepsilon)r_0^2]}{r^3[4(r-\frac{r_0}{2})^2 + \varepsilon r_0^2]^2}.$$

The zeros of the derivative T' are given by

$$r_1 = \frac{3 - \sqrt{1-8\varepsilon}}{8} r_0, \qquad r_2 = \frac{3 + \sqrt{1-8\varepsilon}}{8} r_0.$$

The graphs of the scalar curvature T given by (7) with $\varepsilon \in (0, \frac{1}{8})$ as well as for the other cases are illustrated below:

(9)



In the case $a = -1$ the metric tensor (4) takes the form

$$diag(-1,\ E^{-2}\ r^2,\ r^2 sin^2\theta). \tag{10}$$

Using (5) we find that the scalar curvature is

$$T = \frac{2}{r^2}[E(E + 2E'r) - 1]. \tag{11}$$

In the case $K = \frac{1+\varepsilon}{4}$, $\varepsilon \in (0, \frac{1}{8})$ the scalar curvature (11) has the form

$$T = \frac{-r_0^2}{8r^6}[8r^2(3+\varepsilon) - 16r(1+\varepsilon)r_0 + 3(1+\varepsilon)^2 r_0^2] \tag{12}$$

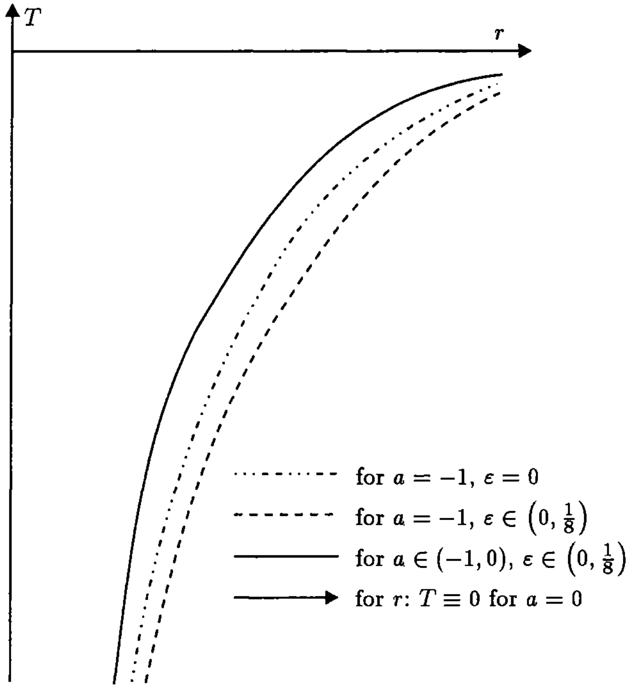
and its derivative T' is equal to

$$T' = \frac{1}{4r^7}[16r^2(3+\varepsilon)r_0^2 - 40r(1+\varepsilon)r_0^3 + 9(1+\varepsilon)^2 r_0^4], \tag{13}$$

where $\Delta = -64(1+\varepsilon)^2(2+9\varepsilon)r_0^6 < 0$ and the function T in (12) has no local extremum. The graphs of the scalar curvature T given by (5) with $a \in (-1, 0)$, $K = \frac{1+\varepsilon}{4}$, $\varepsilon \in (0, \frac{1}{8})$ and curvature (12) for $\varepsilon = 0$ and $\varepsilon \in (0, \frac{1}{8})$

and for $a = 0$ are as follow

(14)



4. Distributions of scalar curvatures in the family of generalized Reissner-Nordström space-times and their semi-groups

Let us consider the formula (5) with $a \in (0,1)$, $K = \frac{1+\varepsilon}{4}$, $\varepsilon \in (0, \frac{1}{8})$ at $r = \frac{r_0}{2}$ and at $r \neq \frac{r_0}{2}$. Then one has

(15) $$\lim_{a \to 0^+} \big(\lim_{\varepsilon \to 0^+,\ r \neq \frac{r_0}{2}} T \big) = 0, \qquad \lim_{a \to 0^+} \big(\lim_{\varepsilon \to 0^+,\ r = \frac{r_0}{2}} T \big) = +\infty.$$

We see that the scalar curvature (5) of the family of generalized R-N space-times gives rise for $\varepsilon \to 0^+$, $a \to 0^+$ to the special Schwarz distribution, namely the Dirac delta at $\frac{r_0}{2}$ [4]

(16) $$\delta(r) = \begin{cases} 0 & \text{for } r \neq \frac{r_0}{2}, \\ +\infty & \text{for } r = \frac{r_0}{2}. \end{cases}$$

For the scalar curvature T of the form (5) with $a \in (-1, 0)$, $K = \frac{1+\varepsilon}{4}$, $r \neq 0$ or $r = 0^+$ one has

(17) $$\lim_{a \to 0^-} \big(\lim_{\varepsilon \to 0^+,\ r \neq 0} T \big) = 0, \qquad \lim_{a \to 0^-} \big(\lim_{\varepsilon \to 0^+,\ r = 0^+} T \big) = -\infty.$$

We see that similarly as in (16)

$$\delta(r) = \begin{cases} 0 & \text{for } r \neq 0, \\ -\infty & \text{for } r = 0^+. \end{cases} \tag{18}$$

Hence the family of scalar curvatures (5) gives rise for $\varepsilon \to 0^+, a \to 0^-$ to the special Schwarz distribution, the Dirac delta at 0^+.

We define the multiplication $*$ in families of all curvatures (5) for $a \in \langle 0, 1 \rangle$ and for $a \in < -1, 1 >$ by the formula

$$T_1 * T_2 = \frac{a_1 a_2 r^2 E'' + 2}{r^2 E^{a_1 a_2}} - \frac{2}{r^2}. \tag{19}$$

For $a \in \langle 0, 1 \rangle$ as well as for $a \in \langle -1, 1 \rangle$ the pair $(T, *)$ is the abelian semi-group. The unity of this semi-group is the curvature T given by (5) for $a = 1$ and the zero is the scalar curvature of the classical R-N space-time, i.e. the curvature T for $a = 0$. The set of curvatures T for $a \in (0, 1)$ forms the ideal of this semi-group which is also an abelian group.

Similarly, for $a \in (-1, 1)$ one obtain the ideal and abelian semi-group.

We hope that the above family of generalized R-N space-times and the scalar curvature (5) with the structure (19) can be applied in cosmology.

References

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Received April 19, 1996.