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WEYL–VON NEUMANN TYPE THEOREMS WITH THE PERTURBATION VANISHING ON A GIVEN SUBSPACE, I

1. Introduction

The theory of approximation of operators in a Hilbert space by diagonal (and block-diagonal) operators was widely developed. One says that any normal operator F is diagonal if $F = \sum_i \lambda_i \widehat{e}_i, \lambda_i \in C$, for some orthonormal basis (e_i). We denote

$$\widehat{e}\cdot = \langle \cdot, e\rangle e \text{ for } e \in H, \ \|e\| = 1.$$

Many problems in differential equations and dynamically systems described by Hamiltonians are connected with such approximations (see, for example, [7]). Let us mention (as the most interesting results in that direction) the following classical results. The Weyl–von Neumann theorem states that, for any selfadjoint operator A, a perturbation Y for which $A+Y$ is diagonal can be found with the arbitrarily small Hilbert–Schmidt norm $\|Y\|_2$ ([7], [8] compare also [1], [4], [5], [6]). Voiculescu in his theorem states the same for any normal operator A. His proof is more algebraic (and more complicated). Both those results have a number of extensions and applications [9]. Some other aspects of compact perturbations can be found in [2].

In that note we show, by an extension of the method of von Neumann, that his classical result can be improved in the following way. One can require that the perturbation Y of a selfadjoint operator A (with $\|Y\|_2$ arbitrarily small and $A+Y$ diagonal) can additionaly satisfy $YP = 0$ for any given finite-dimensional orthogonal projection O (Theorem 2.3).

We also show that the classical von Neumann theorem gives an analogous approximation for any projection I (not necessarily selfadjoint). Namely, there exists an operator Y such that $\|Y\|_2$ is arbitrarily small and

$$I+Y=\sum_i \langle\cdot,e_i\rangle(e_i+\lambda_i f_i),\lambda_i\in C$$

for some orthogonal system $\{e_i\}\cup\{f_i\}$, i.e., $I+Y$ is a block-diagonal projection, as simple as possible (Theorem 3.1).

It seems interesting to obtain the perturbation Y in the Voiculescu theorem (and in our Theorem 3.1 for projections) with the additional requirement that $YP=0$ for a given finite-dimensional orthogonal projection P. This will be done in the next part of the paper.

2. Extension of the Weyl–von Neumann theorem

Let H be a separable Hilbert space.

2.1. THEOREM (*Weyl–von Neumann [6], [7], [8]*). *If A be a selfadjoint (possibly unbounded) operator on H and if $\varepsilon>0$, then there exists a selfadjoint operator Y with the Hilbert–Schimdt norm $\|Y\|_2<\varepsilon$, such that $A+Y$ is a diagonal (selfadjoint) operator.*

The following lemma is just an extension of the classical technical proposition, crucial in the von Neumann reasoning ([7], X §2, Lemma 2.2).

2.2. LEMMA. *Let A be a selfadjoint operator in H. For any $\mu>0$ and an orthonormal system $f_1,\dots,f_n$, there exist an operator Y and a finite-dimensional orthogonal projection T, such that:*

(i) $\dim Y\le 4\frac{n}{\mu},\|Y\|<\mu$,

(ii) $T^{\perp}f_i=0,i=1,\dots,n$,

(iii) $A+Y$ *is reduced by* TH,

(iv) $Yf_i=0,i=1,\dots,n$.

Proof. Assume that $A=\int_0^1\lambda dE(\lambda)$ is the spectral representation of A. This is a harmless assumption involving no loss of generality. Let m be a positive integer (to be determined later).

Write $g_i=Af_i$, $i=1,\dots,n$. Let E_j be the projection $E([\frac{j-1}{m},\frac{j}{m}])$, $1\le j<m$ and $E_m=E([\frac{m-1}{m},1])$ and let $f_{ij}=E_jf_i$, $g_{ij}=E_jg_i$, $j=1,\dots,m,i=1,\dots,n$. Let $T_j,j=1,\dots,m$, be the orthogonal projection whose range is the span of the vectors $f_{1j},\dots,f_{nj}$, $g_{1j},\dots,g_{nj}$. For any $j=1,\dots,m$ we have $E_jT_j=T_j$, that is,

$$\left\|AT_j-\frac{j}{m}T_j\right\|\le\frac{1}{m}.$$

We shall denote $T=T_1+T_2+\ldots+T_n$. Observe that T is the orthogonal projection in the subspace spanned by f_{ij}, g_{ij}, $i=1,\dots,n$, $j=1,\dots,m$.

Let $-Y = TAT^{\perp} + T^{\perp}AT$. We have

$$\dim Y \leq 2\dim T \leq 4nm. \tag{1}$$

By the definition of Y, the perturbed operator $A + Y$ is reduced by TH.

As $Af_{ij} = g_{ij}$, $j = 1, \ldots, m$, $i = 1, \ldots, n$ we have $Yf_{ij} = 0$, thus $Yf_i = \sum_j Yf_{ij} = 0$.

It remains to prove (i). Observe that $YE_j = (-TAT^{\perp} - T^{\perp}AT)E_j = E_jY$, hence $Y = \sum_j^m E_jY = \sum_j E_jYE_j$ and $\|Y\| = \max_j \|E_jYE\| \leq 2\max_j \|E_j(A-1)E_j\| \leq 2\frac{1}{m}$.

If $m > \frac{2}{\mu}$, then, by (1), we have (i).

2.3. THEOREM. *Let A be a selfadjoint bounded operator in H. For any $\varepsilon > 0$ and a finite-dimensional orthogonal projection P, there exists a selfadjoint operator Y in H with the Hilbert–Schmidt norm $\|Y\|_2 < \varepsilon$, such that $A + Y$ is a diagonal operator (i.e., the operator $A + Y$ has a pure point spectrum), and $YP = 0$.*

Proof. One may assume that $A = \int_0^1 \lambda dE(\lambda)$ is the spectral representation of A. Let P be an orthogonal projection, $\dim P = n$ and $f_1, \ldots, f_n$ an orthonormal system in PH. Suppose that $\varepsilon > 0$.

Step 1. Apply Lemma 2.2 to μ such that $\sqrt{n\mu} < \varepsilon$. The result is the finite-dimensional orthogonal projection T_0 and the operator Y_0, such that $A + Y_0$ is reduced by T_0H,

$$\|Y_0\| < \mu;\ \dim Y_0 \leq 8\frac{n}{\mu},\ T_0^{\perp} f_i = 0, i = 1, \ldots, n;$$
$$\|Y_0\|_2 \leq \|Y_0\|\sqrt{\dim Y_0} < \frac{\varepsilon}{2}.$$

Step 2. Apply the Weyl–von Neumann theorem to the part of the bounded operator $A + Y_0$, acting in the subspace $(1 - T_0)H$. The result is the selfadjoint operator Y_1 in $(1-T_0)H_1$ with the Hilbert–Schmidt norm $\|Y_1\|_2 < \varepsilon/2$ and the diagonal operator $A + Y_0 + Y_1$ in $(1 - P_0)H$. Define $Y = Y_0 + Y_1$ to finish the proof.

2.4. Remark. The perturbation Y in Theorem 2.3 can satisfy $\|Y\|_p < \varepsilon$ for any Schatten norm $\|\cdot\|_p$, $p > 1$, (instead of $\|Y\|_2 < \varepsilon$). As usual $\|Y\|_p$ denotes $(\sum_n |\lambda_n|^p)^{1/p}$ according to the representation of a compact operator $Y\cdot = \sum_n \lambda_n \langle\cdot, e_n\rangle \sigma_n$ for some orthonormal sets $\{e_n\}$ and $\{\sigma_n\}$ in H [11]. For details (and other extensions) see [7, X §2.2]. In consequence, it is obvious that, in our Theorem 2.3, any Schatten norm $\|\cdot\|_p$, $p > 1$ can be used instead of the Hilbert–Schmidt norm $\|\cdot\|_2$.

3. Approximation of a projection

In a Hilbert space H, projections which are not necessarily selfadjoint play an important role. It is enough to remark that an operator A similar to a selfadjoint one ($A = TBT^{-1}$ with $B = B^*$; $T, T^{-1} \in B(H)$ has the form $\int_{-a}^{a} \lambda dE(\lambda)$, where $E(\lambda)$ is a spectral family of projections (i.e. $\|E(\lambda)\|$ are uniformly bounded and $E(\lambda)E(\lambda_1) = E(\lambda_1) = (E(\lambda_1))^2$ for $\lambda_1 \le \lambda$).

By analogy to a selfadjoint diagonal operator $A = \sum_i \lambda_i \widehat{e}_i$, $\lambda_i \in R$, where $\{e_j\}$ is an orthonormal system, we shall consider the projection of the form

$$I(\cdot) = \sum_i \langle \cdot, e_i \rangle (e_i + \lambda_i f_i), \tag{2}$$

where $\{e_i\} \cup \{f_i\}$ is an orthonormal system, to have an equally elementary form.

In this section, as a corollary from the Weyl–von Neumann theorem we obtain

3.1. THEOREM. *Let I be a projection in H. For each $\varepsilon > 0$ there exists an operator Y in H with the Hilbert-Schmidt norm $\|Y\|_2 < \varepsilon$, an orthonormal system $\{e_i\} \cup \{f_i\}$ in H and a sequence of positive numbers $\{a_i\}$, such that*

$$I + Y = \sum_{i \in N} \langle \cdot, e_i \rangle (e_i + a_i f_i).$$

Let us begin with some elementary properties of projections. An arbitrary projection I is uniquely described by two closed subspaces $\widetilde{K}$, $\widetilde{F}$ of the space H which are, respectively, the kernel and the set of fixed points of the operator I:

$$Ix = x \Leftrightarrow x \in \widetilde{F},$$
$$Ix = 0 \Leftrightarrow x \in \widetilde{K},$$

for all $x \in H$. Then $\widetilde{K} + \widetilde{F} = H$, $\widetilde{K} \cap \widetilde{F} = \emptyset$.

Two orthogonal projections P, Q are said to have a generic position if

$$P \wedge Q = P \wedge Q^{\perp} = P^{\perp} \wedge Q = P^{\perp} \wedge Q^{\perp} = 0.$$

3.2. THEOREM (Halmos). ([3, Theorem 2], [10, V, 1]). *If orthogonal projections have a generic position, then $H = L \oplus L$ and, according to this representation, we have*

$$P = \begin{bmatrix} 1_L & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} c^2 & sc \\ sc & s^2 \end{bmatrix}$$

for some operators s, c in L, $0 \le s$, $0 \le c$, $\ker s = \ker c = 0$ and $s^2 + c^2 = 1_L$.

Let $\widetilde{F} = FH$ and $\widetilde{K} = KH$ be, respectively, the set of fixed points and the kernel of some projection I, with some orthogonal projections F and K.

Then

$$F = 1 \oplus 0 \oplus F_3,$$
$$K = 0 \oplus 1 \oplus K_3$$

according to some representation

$$H = H_1 \oplus H_2 \oplus H_3$$

and, moreover, F_3, K_3 have a generic position in H_3. Therefore it is sufficient to deal with an approximation of a projection I for which F, K have a generic position. We shall call such an I generic position.

It is easy to verify that a generic projection can be represented in the form

$$I = \begin{bmatrix} 1_L & -\frac{c}{s} \\ 0 & 0 \end{bmatrix} \tag{3}$$

according to a suitable representation $H = L \oplus L$. Here $\frac{s}{c}$ denotes one selfadjoint unbounded operator in L, $\frac{s}{c} = \int_0^1 \frac{\lambda}{\sqrt{1-\lambda^2}} dE(\lambda)$.

The spectral measure satisfies $E((0,1)) = 1$.

3.3. Proof of Theorem 3.1. Let I be of the form (3). Apply the Weyl–von Neumann theorem to the operator $A = -\frac{c}{s}$ in the space L. The result is Y' in H such that $\|Y'\|_2 < \varepsilon$ and $-\frac{c}{s} + Y' = \sum_i a_i f_i'$.

It is obvious that we may assume a_i to be positive. It is enough to denote

$$Y = \begin{bmatrix} 0 & Y' \\ 0 & 0 \end{bmatrix}.$$

Then

$$I = Y = \begin{bmatrix} 1_L & -\frac{c}{s} + Y' \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1_L & \sum_i a_i f_i' \\ 0 & 0 \end{bmatrix} = \sum_i \langle \cdot, e_i \rangle (e_i + a_i f_i)$$

where $e_i = [\begin{smallmatrix} f_i' \\ 0 \end{smallmatrix}]$, $f_i = [\begin{smallmatrix} 0 \\ f_i' \end{smallmatrix}]$.

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