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THE WEIERSTRASS-TYPE CONDITION FOR A MINIMUM IN A BOLZA OPTIMAL CONTROL PROBLEM

Introduction

The main aim of this paper is to extend the field theory from the classical calculus of variations in order to obtain the Weierstrass-type conditions for a minimum in a Bolza optimal control problem in the case when a functional whose minimum we want to find and all functions occuring in the constraints do not depend explicitly on the parameter t. The analogous problem was considered by V. Veličenko in some of his paper, but very strong assumptions about the smoothness of functions and the assumption that the set U of values of controls is open are too strong for optimal control problems. This paper is a genaralization of chapter II in part II of Young's book [14] and it deals with a new field theory - theory of concourse of flights. This theory is more useful in optimization problems because it is free from a great number of assumptions about smoothness, one-to-one covering and openess of a set of trajectories which satisfy the maximum principle. The Hilbert integral is defined here but in a new, more general sense, and it is shown that the Hilbert integral does not depend on the path of integration when there exists a cocnourse of flights. On this basis, a sufficient condition of Weierstrass-type for a minimum is formulated and conections of the Hilbert integral with the value function, Hamilton-Jacobi equation and the K-function is shown. Moreover, the practice construction of the optimal feedback control is given.

1. Preliminary notes and assumptions

Let L be the family of Lebesgue measurable subsets of the interval $[0,1]$ and B^r — the family of Borel subsets of the space $\mathbb{R}^r$. Denote by $L \times B^r$ the σ-algebra of subsets of $[0,1] \times \mathbb{R}^r$ which is generated by the cartesian product of subsets of the families L and B^r.

Let $f : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ and $f^0 : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$ be functions of variables

(x, u) which are Lipschitz with respect to x when u is fixed, i.e.

$$\|f(x_1, u) - f(x_2, u)\| \le k_1\|x_1 - x_2\| \,,$$
$$\|f^0(x_1, u) - f^0(x_2, u)\| \le k_2\|x_1 - x_2\|$$

for some constants k_1 and k_2 with u fixed, and Borel measurable with respect to u when x is fixed. Let $l : \mathbb{R}^n \to \mathbb{R}$ be a function which is lower semicontinuous. Denote by t_0 any point of the interval $[0, 1]$.

DEFINITION 1.1. A Lebesgue measurable function $u : [t_0, 1] \to U(t)$, where $U(t)$ is any subset of $\mathbb{R}^r$ for each fixed $t \in [t_0, 1]$, will be called an *admissible control* or a *control.*

We assume that the graph of U, i.e. $\{(t, u) \in [t_0, 1] \times \mathbb{R}^r : u \in U(t)\}$, is $L \times B^r$-measurable.

DEFINITION 1.2. The *admissible trajectory corresponding to an admissible control* $u : [t_0, 1] \to U(t)$ is an absolutely continuous function $x : [t_0, 1] \to \mathbb{R}^n$ such that $f^0(x(t), u(t))$ is a summable function, the value $l(x(1))$ is finite and

$$\dot{x}(t) = f(x(t), u(t)) \text{ for } t \in [t_0, 1] \text{ a.e.} \tag{1.1}$$

A pair $(x(t), u(t))$, $t \in [t_0, 1]$, is called an *admissible pair* when $u(t)$ is an admissible control and $x(t)$ — the admissible trajectory corresponding to u.

We shall consider the problem of finding an admissible pair $(x(t), u(t))$, $t \in [0, 1]$, such that the functional

$$I(x, u) = l(x(1)) + \int_0^1 f^0(x(t), u(t))\, dt \tag{1.2}$$

attains its minimum along all admissible pairs for which $x(0) = x_0$ where x_0 is any fixed point in $\mathbb{R}^n$.

DEFINITION 1.3. An admissible pair $(x^*(t), u^*(t))$, $t \in [t_0, 1]$, will be said to satisfy the *maximum principle* if there exists an absolutely continuous conjugate function $y : [t_0, 1] \to \mathbb{R}^n$ such that

$$-\dot{y}(t) \in y(t)\partial_x f(x^*(t), u^*(t)) - \partial_x f^0(x^*(t), u^*(t)) \text{ for } t \in [t_0, 1] \text{ a.e.,} \tag{1.3}$$

$$\begin{aligned} H(x^*(t), y(t), u^*(t)) &= \sup\{H(x^*(t), y(t), u) : u \in U(t)\} \\ &= 0 \qquad \text{for } t \in [t_0, 1] \text{ a.e.,} \end{aligned} \tag{1.4}$$

$$-y(1) \in \partial l(x(1)), \tag{1.5}$$

where $\partial_x f, \partial_x f^0$ are generalized jacobians of the functions $x \to f(x, u(t))$ and $x \to f^0(x, u(t))$, respectively (which exist on the assumption that f, f^0 are Lipschitz functions), $\partial l(x)$ is the generalized gradient of $l(x)$, and $H(x(t), y(t), u(t)) = y(t)f(x(t), u(t)) - f^0(x(t), u(t))$.

From (1.4) we have

$$y(t)f(x^*(t), u^*(t)) - f^0(x^*(t), u^*(t)) = 0 \text{ for } t \in [t_0, 1] \text{ a.e.} \tag{1.6}$$

DEFINITION 1.4. A pair

$$(x(t), u(t)) \text{ for } t \in [t_0, 1]$$

which satisfies the maximum principle will be called a *line of flight*, and a triple

$$(x(t), y(t), u(t)) \text{ for } t \in [t_0, 1]$$

such that the pair $(x(t), u(t))$ is a line of flight and $y(t)$ the conjugate function corresponding to it — a *canonical line of flight.*

An arc of line of flight (canonical line of flight) is any open arc of line of flight (canonical line of flight) i.e. functions $x(t)$, $u(t)$ (resp. $x(t)$, $y(t)$, $u(t)$) defined on an open interval $(t^-, t^+) \subset [t_0, 1]$.

Denote by D a set which is covered by trajectories of lines of flight. For any point $x_1 \in D$, we define a function

$$J(x_1) = l(x(1)) + \int_{t'}^{1} f^0(x(t), u(t))\, dt \tag{1.7}$$

along a line of flight $(x(t), u(t))$, $t \in [t', 1]$, such that $x(t') = x_1$, $t' \in [0, 1]$. Through the point x_1 there can pass more than one trajectory of a line of flight, so from among all lines of flight $(x(t), u(t))$, $t \in [t', 1]$, $x(t') = x_1$, $t' \in (0, 1.]$, we shall consider only those for which functions (1.7) attain the same value. This condition is called synchronization (see [14], p. 266).

2. Spray of flights

Let G be any open subset of $\mathbb{R}^m$. We shall consider two functions $t^-(\sigma)$ and $t^+(\sigma)$ defined on G with values in $[0, 1]$, such that $t^+(\sigma)$ is a C^1-function and, for all $\sigma \in G$, we have $t^-(\sigma) < t^+(\sigma)$.

Denote by S^-, S, S^+ the sets of pairs (t, σ) where $\sigma \in G$ and t satisfies the conditions

$$0 \le t^-(\sigma) = t, \quad t^-(\sigma) < t < t^+(\sigma), \quad t = t^+(\sigma) \le 1,$$

respectively. The notation $|S|$ will be used for the union of the sets S^-, S, S^+.

Denote by S^{*-}, S^*, S^{*+} the sets of triples (t,σ,β) where (t,σ) satisfies the conditions from the definitions of S^-, S, S^+, respectively, and $(\sigma,\beta) \in \tilde{G}$ where G is a standard projection of $\tilde{G}$ in the following sense: for any point $(\sigma^0,\beta^0) \in \tilde{G}$ and any sufficiently small curve $\gamma \subset G$ issuing from σ^0, there exists a continuous function $\beta(\sigma)$ defined on γ such that $\beta(\sigma^0) = \beta^0$ and the point $(\sigma,\beta(\sigma))$ for $\sigma \in \gamma$ lies in $\tilde{G}$ (see [14], p. 266). The notation $|S^*|$ will be used for the union of the sets S^{*-}, S^*, S^{*+}.

Let Z denote a family of arcs of lines of flight defined by the functions

$$x(t,\sigma),\ u(t,\sigma),\ (t,\sigma) \in S. \tag{2.1}$$

The parameter σ is constant along any fixed arc of line of flight defined on the interval $t^-(\sigma) < t < t^+(\sigma)$.

Let Z^* denote a family of canonical line of flight defined by the functions

$$x(t,\sigma),\ y(t,\sigma,\beta),\ u(t,\sigma),\ (t,\sigma,\beta) \in S^*, \tag{2.2}$$

such that $(x(t,\sigma),u(t,\sigma)) \in Z$ and $y(t,\sigma,\beta)$ is the conjugate function corresponding to it. The parameter β appears here because the conjugate function defined by (1.3) with condition (1.5) is not unique.

Let us assume that functions (2.1) and (2.2) can be extended to the sets $|S|$ and $|S^*|$, respectively. This means defining them for $t = t^-(\sigma)$ and $t = t^+(\sigma)$, $\sigma \in G$.

Denote by E^-, E, E^+, $|E|$ the sets of values of the functions $x = x(t,\sigma)$ with (t,σ) from S^-, S, S^+, $|S|$, respectively, and by E^{*-}, E^*, E^{*+}, $|E^*|$ the sets of values of those triples $(x(t,\sigma),y(t,\sigma,\beta),u(t,\sigma))$ for which (t,σ,β) belongs to S^{*-}, S^*, S^{*+}, $|S^*|$, respectively.

For $(t,\sigma) \in |S|$, let us put

$$\begin{aligned}\tilde{f}^0(t,\sigma) &= f^0(x(t,\sigma),u(t,\sigma)),\\ \tilde{f}(t,\sigma) &= f(x(t,\sigma),u(t,\sigma)),\\ \tilde{J}^+(\sigma) &= J(x(t^+(\sigma),\sigma)).\end{aligned}$$

Assume that the following hypotheses are satisfied:

H1. The functions $\tilde{f}^0(t,\sigma)$, $\tilde{f}(t,\sigma)$ are continuous in $|S|$ and there exist continuous derivatives $\tilde{f}^0{}_\sigma(t,\sigma)$, $\tilde{f}_\sigma(t,\sigma)$ in $|S|$ and derivatives $\frac{\partial}{\partial\sigma} f^0(x,u(t,\sigma))$, $\frac{\partial}{\partial\sigma} f(x,u(t,\sigma))$ for each $x = x(t,\sigma)$, which satisfy the conditions

$$\begin{aligned}\tilde{f}^0{}_\sigma(t,\sigma) &= \frac{\partial}{\partial\sigma} f^0(x,u(t,\sigma)) + f^0{}_x(x,u(t,\sigma))x_\sigma(t,\sigma),\\ \tilde{f}_\sigma(t,\sigma) &= \frac{\partial}{\partial\sigma} f(x,u(t,\sigma)) + f_x(x,u(t,\sigma))x_\sigma(t,\sigma),\end{aligned}$$

and the functions $x \to f^0(x, u(t,\sigma))$, $x \to f(x, u(t,\sigma))$ are strictly differentiable in $x = x(t,\sigma)$ for $(t,\sigma) \in S$.

H2. The function $y(t,\sigma,\beta)$ is continuous in $|S^*|$.

H3. The function $x(t,\sigma)$ is a C^1-function in $|S|$ and $u(t,\sigma)$ is a Borel function in $|S|$.

H4. The mappings $S^- \to E^-$, $S \to E$ defined by $(t,\sigma) \to x(t,\sigma)$ are descriptive, i.e. the following condition is satisfied for any point $(t^0,\sigma^0) \in S^-$ ($(t^0,\sigma^0) \in S$): for any rectifiable curve $C \subset E^-$ (resp. $C \subset E$) issuing from $x(t^0,\sigma^0)$, there exists a rectifiable curve $\Gamma \subset S^-$ ($\Gamma \subset S$) issuing from (t^0,σ^0) such that any sufficiently small arc of the curve C issuing from $x(t^0,\sigma^0)$ is the image of a sufficiently small arc of the curve Γ issuing from (t^0,σ^0) under the mapping $(t,\sigma) \to x(t,\sigma)$ (see [14], p. 266).

DEFINITION 2.1. Let us assume that the conditions from the definitions of the functions $t^-(\sigma)$ and $t^+(\sigma)$ and the sets G, $\tilde{G}$ as well as hypotheses H1 — H4 are satisfied. Then the family Z will be called a *spray of flights from E^- to E^+* and the family Z^* — a *canonical spray of flights from E^- to E^+*.

3. The Hilbert integral

Denote by D, as in section 1, a set covered by trajectories of lines of flight.

DEFINITION 3.1. For any subset $A \subset D$, the set $\tilde{A} \subset \mathbb{R}^{2n}$ of points (x,y) will be called the *canonical set corresponding to A* if any point (x,y) lies on a canonical line of flight and $x \in A$.

For $x \in D$, denote by $Y(x)$ the set of those values of the conjugate function y for which (x,y) is a point of the canonical set $\tilde{D}$ corresponding to D. In this way, $Y(x)$ can be a multifunction. Denote by $y(x)$ a single-valued function defined in D which has its values in $Y(x)$ for all x. For any fixed spray of flights Z, we denote by $Y_Z(x)$, for $x \in |E|$, the set of values of the function $y(t,\sigma,\beta)$, $(t,\sigma,\beta) \in |S^*|$, such that the pair $(x(t,\sigma), y(t,\sigma,\beta))$ lies in the canonical spray of flights Z^*, and $x = x(t,\sigma)$. Denote by $y_Z(x)$, $x \in |E|$, a single-valued function which has its values in $Y_Z(x)$ for all x. The single-valued functions $y(x)$ and $y_Z(x)$, defined as above, will be called *selections*.

DEFINITION 3.2. Any rectifiable curve $C \subset D$ will be called a *bounded curve* if the function $J(x)$ defined in (1.7) is bounded along C.

For any bounded rectifiable curve $C \subset D$ with the description $x = v(s)$, $0 \leq s \leq b$, where s is the arc length parameter, we define a curvilinear integral

$$\int_C y(x)\,dx = \int_0^b y(v(s))\frac{dv}{ds}\,ds \tag{3.1}$$

for a selection $y(x)$ for which $y(v(s))\frac{dv}{ds}$ is a measurable function of the arc length s along C. Integral (3.1) will be called the *Hilbert integral.*

Our main aim is to show the circumstances when integral (3.1) does not depend on the choice of the bounded rectifiable curve C lying in D with fixed endpoints x_1, x_2 in D or on the choice of the selection $y(x)$ with values in $Y(x)$. Note that if the expression $y(v(s))\frac{dv}{ds}$ takes the same value for all selections $y(x)$ with values in $Y(x)$, $x \in D$, on a fixed bounded rectifiable curve C, then Hilbert integral (3.1) does not depend on the choice of selection $y(x)$ with values in $Y(x)$.

DEFINITION 3.3. a) For all $x \in D$, a direction Θ such that, for all selections $y(x)$ with values in $Y(x)$, a projection $y\Theta$ onto this direction is this same will be called a *direction of univalence* (see [14], p. 270).

b) We term *curve of univalence* a rectifiable curve $C \subset D$ such that, for almost all points of C, the direction of the tangent to C is a direction of univalence.

c) We shall call $A \subset D$ a *set of univalence* if all bounded rectifiable curves $C \subset A$ are curves of univalence.

It follows from the introduced definition that, for any bounded rectifiable curve C lying in a set of univalence A, Hilbert integral (3.1) does not depend on the choice of the selection $y(x)$ with values in $Y(x)$, $x \in A$. This integral can be expressed as

$$\int_C y(x)\,dx = \int_0^b y(v(s))\Theta(s)\,ds$$

where $v(s)$ is the arc length parametrization of C and $\Theta = \Theta(s) = \frac{dv}{ds}$ is a direction of univalence almost everywhere on C.

DEFINITION 3.4. A set $A \subset D$ will be called an *exact set* if it is a set of univalence and, for any bounded rectifiable curve $C \subset A$ with endpoints x_1, x_2, we have

$$\int_C y(x)\,dx = J(x_1) - J(x_2)$$

for all selections $y(x)$ with values in $Y(x)$.

Of course, the notations described above can also be carried over to the spray of flights Z.

DEFINITION 3.5. a) For all $x \in |E|$, a direction Θ such that, for all selections $y_Z(x)$ with values in $Y_Z(x)$, a projection $y_Z\Theta$ onto this direction is this same will be called a *direction of relative univalence.*

b) We term *curve of relative univalence* a rectifiable curve $C \subset |E|$ such that, for almost all points of C, the direction of the tangent to C is a direction of relative univalence.

c) We shall call $A \subset |E|$ a *set of relative univalence* if all bounded rectifiable curves $C \subset A$ are curves of relative univalence.

d) A set $A \subset |E|$ will be called a *relative exact set* if it is a set of relative univalence and, moreover, for any bounded rectifiable curve $C \subset A$,

$$\int_C y_Z(x)\,dx = J(x_1) - J(x_2) \tag{3.2}$$

for all selections $y_Z(x)$ with values in $Y_Z(x)$, $x \in |E|$, where $x_1 \in |E|$, $x_2 \in |E|$ are endpoints of the curve C.

4. Auxiliary lemmas

Let us suppose that there exists a spray of flights Z from E^- to E^+ such that E^+ is a relative exact set.

In our next cosiderations we shall take only those curves C which are the images of curves Γ in the (t,σ)-space under the mapping $(t,\sigma) \to x(t,\sigma)$ and those selections $y(x)$ which have the form $y(t,\sigma,\beta)$ on those curves, where $\beta = \beta(\sigma)$ is a continuous function chosen according to the definition of the standard projection.

LEMMA 4.1. *If E^+ is a relative exact set, then there exists a derivative $\tilde{J}^+_\sigma(\sigma)$ in G and*

$$\tilde{J}^+_\sigma(\sigma) = -(\tilde{f}^0(t^+(\sigma),\sigma)t^+_\sigma(\sigma) + y(t^+(\sigma),\sigma,\beta)x_\sigma(t^+(\sigma),\sigma)) \quad \text{for } (\sigma,\beta) \in \tilde{G}. \tag{4.1}$$

Proof. Denote by (t^0,σ^0,β^0) any point of S^{*+} and by Γ any sufficiently small rectifiable curve in S^+ with the description $t = t^+(\sigma_k)$ where σ_k varies from σ^0 to σ^1 along the k-coordinate of σ. Denote by C the image under the mapping $(t,\sigma) \to x(t,\sigma)$ of the curve Γ in E^+, with endpoints $x(t^0,\sigma^0)$ and $x(t^1,\sigma^1)$ where $t^1 = t^+(\sigma^1)$. As E^+ is a relative exact set, by (1.6), for a continuous function $\beta(\sigma)$ chosen according to the definition of the standard projection and for a selection $y(x)$ which has the form $y(t,\sigma,\beta)$ on Γ, we have

$$(4.2)\qquad \tilde J^+(\sigma^0)-\tilde J^+(\sigma^1)=J(x(t^0,\sigma^0))-J(x(t^1,\sigma^1))=\int_C y(x)\,dx$$
$$=\int_\Gamma y(t,\sigma,\beta(\sigma))x_t(t,\sigma)dt+y(t,\sigma,\beta(\sigma))x_\sigma(t,\sigma)\,d\sigma$$
$$=\int_\Gamma \tilde f^0(t,\sigma)dt+y(t,\sigma,\beta(\sigma))x_\sigma(t,\sigma)\,d\sigma$$
$$=\int_\Gamma(\tilde f^0(t,\sigma){t^+}_\sigma(\sigma)+y(t,\sigma,\beta(\sigma))x_\sigma(t,\sigma))\,d\sigma.$$

The function $\tilde f^0(t,\sigma)t^+_\sigma(\sigma)+y(t,\sigma,\beta(\sigma))x_\sigma(t,\sigma)$ is a continuous function on Γ, so there exists a limit

$$\lim_{\sigma^1\to\sigma^0}\frac{\int_\Gamma(\tilde f^0(t,\sigma){t^+}_\sigma(\sigma)+y(t,\sigma,\beta(\sigma))x_\sigma(t,\sigma))\,d\sigma)}{(\sigma^0-\sigma^1)},$$

and hence there exists a limit

$$\lim_{\sigma^1\to\sigma^0}\frac{\tilde J^+(\sigma^0)-\tilde J^+(\sigma^1)}{\sigma^0-\sigma^1}.$$

Thus

$$-{\tilde J^+}_{\sigma_k}(\sigma)=\tilde f^0(t,\sigma){t^+}_{\sigma_k}(\sigma)+y(t,\sigma,\beta(\sigma))x_{\sigma_k}(t,\sigma)$$

at (t^0,σ^0,β^0). Since the point (t^0,σ^0,β^0) and the k-coordinate of σ are any point of S^{*+} and any coordinate of σ, and $t=t^+(\sigma)$ in S^{*+}, therefore (4.1) holds. ■

LEMMA 4.2. *Let Γ be any rectifiable curve in $|S|$ with endpoints (t^0,σ^0), (t^1,σ^1). Then*

$$(4.3)\qquad \int_\Gamma \tilde f^0(t,\sigma)dt-\Big(\int_t^1 \tilde f^0_\sigma(\tau,\sigma)\,d\tau\Big)\,d\sigma=J(x(t^0,\sigma^0))-J(x(t^1,\sigma^1))$$

where

$$(4.4)\qquad \int_t^1 \tilde f^0_\sigma(\tau,\sigma)\,d\tau=\int_t^{t^+(\sigma)}\tilde f^0_\sigma(\tau,\sigma)\,d\tau+\tilde f^0(t^+(\sigma),\sigma){t^+}_\sigma(\sigma)+\tilde J^+_\sigma(\sigma).$$

Proof. Consider the function

$$(4.5)\qquad R(t,\sigma)=\int_{t^0}^1 \tilde f^0(\tau,\sigma^0)\,d\tau-\int_t^1 \tilde f^0(\tau,\sigma)\,d\tau$$

defined in $|S|$, where

$$(4.6) \qquad \int_t^1 \tilde{f}^0(\tau,\sigma)\,d\tau = \int_t^{t^+(\sigma)} \tilde{f}^0(\tau,\sigma)\,d\tau + \tilde{J}^+(\sigma),$$

so

$$\int_t^1 \tilde{f}^0(\tau,\sigma)\,d\tau = J(x(t,\sigma)).$$

In virtue of H1 and Lemma 4.1, the function $R(t,\sigma)$ has the continuous derivatives with respect to t and σ in $|S|$. Moreover,

$$(4.7) \qquad \tilde{f}^0(t,\sigma)dt - \left(\int_t^1 \tilde{f}^0{}_\sigma(\tau,\sigma)\,d\tau\right)d\sigma$$

is an exact derivative of $R(t,\sigma)$ with respect to the variables (t,σ) in $|S|$. Hence

$$\int_\Gamma \tilde{f}^0(t,\sigma)dt - \left(\int_t^1 \tilde{f}^0{}_\sigma(\tau,\sigma)\,d\tau\right)d\sigma$$
$$= R(t^1,\sigma^1) - R(t^0,\sigma^0) = J(x(t^0,\sigma^0)) - J(x(t^1,\sigma^1)). \blacksquare$$

Note that, under the above assumptions and notations, Hilbert integral (3.1) takes the form

$$\begin{aligned}\int_C y(x)\,dx &= \int_\Gamma y(t,\sigma,\beta)x_t(t,\sigma)dt + y(t,\sigma,\beta)x_\sigma(t,\sigma)\,d\sigma\\ &= \int_\Gamma \tilde{f}^0(t,\sigma)dt + y(t,\sigma,\beta)x_\sigma(t,\sigma)\,d\sigma\end{aligned}$$
$$= \int_\Gamma \tilde{f}^0(t,\sigma)dt - \left(\int_t^1 \tilde{f}^0{}_\sigma(\tau,\sigma)\,d\tau\right)d\sigma + \int_\Gamma (y(t,\sigma,\beta)x_\sigma(t,\sigma) + \int_t^1 \tilde{f}^0{}_\sigma(\tau,\sigma)\,d\tau)\,d\sigma.$$

In consequence, by (4.3), the relative exactness condition (3.2) reduces to the vanishing of the expression

$$\int_\Gamma (y(t,\sigma,\beta)x_\sigma(t,\sigma) + \int_t^1 \tilde{f}^0{}_\sigma(\tau,\sigma)\,d\tau)\,d\sigma$$

or, equivalently, to the vanishing of

$$y(t,\sigma,\beta)x_\sigma(t,\sigma) + \int_t^1 \tilde{f}^0{}_\sigma(\tau,\sigma)\,d\tau.$$

LEMMA 4.3. *For all* $(t,\sigma,\beta) \in S^{*+}$, *the expression*

$$y(t,\sigma,\beta)x_\sigma(t,\sigma) + \int_t^1 \tilde{f}^0{}_\sigma(\tau,\sigma)\,d\tau$$

vanishes identically.

Proof. According to (4.1), we have

$$\tilde{J}^+_\sigma(\sigma) = -(\tilde{f}^0(t,\sigma)t^+{}_\sigma(\sigma) + y(t,\sigma,\beta)x_\sigma(t,\sigma))$$

and from (4.4), for $t = t^+(\sigma)$,

$$\tilde{J}^+_\sigma(\sigma) = \int_t^1 \tilde{f}^0{}_\sigma(\tau,\sigma)\,d\tau - \tilde{f}^0(t^+(\sigma),\sigma)t^+{}_\sigma(\sigma).$$

Hence, for $t = t^+(\sigma)$, we have

$$\int_t^1 \tilde{f}^0{}_\sigma(\tau,\sigma)\,d\tau - \tilde{f}^0(t,\sigma)t^+{}_\sigma(\sigma) = -\tilde{f}^0(t,\sigma)t^+{}_\sigma(\sigma) - y(t,\sigma,\beta)x_\sigma(t,\sigma),$$

$$y(t,\sigma,\beta)x_\sigma(t,\sigma) + \int_t^1 \tilde{f}^0{}_\sigma(\tau,\sigma)\,d\tau = 0$$

for $(t,\sigma,\beta) \in S^{*+}$. ∎

LEMMA 4.4. *Let C be a rectifiable curve lying, together with its terminal points, in E^- or E. Then C is a bounded curve. Moreover, there exists a bounded, Borel measurable selection $y_Z(x)$ along C.*

Proof. By H4 for each point of C, there exists a neighbourhood on C such that it is the image of some curve Γ, lying entirely in S^- or S, under the mapping $(t,\sigma) \to x(t,\sigma)$. On any such curve the value of the function

$$J(x(t,\sigma)) = \int_t^{t^+(\sigma)} \tilde{f}^0(\tau,\sigma)\,d\tau + \tilde{J}^+(\sigma)$$

is bounded. Indeed, $\tilde{J}^+(\sigma)$ is a continuous function of a variable σ, and $\int_t^{t^+(\sigma)} \tilde{f}^0(\tau,\sigma)\,d\tau$ by H1, is a continuous function of variables (t,σ); so, on the curve Γ which is a compact set it is bounded. It then follows from Borel covering theorem that $J(x(t,\sigma))$ is bounded on C.

In proving the second assertion, without loss of generality we may assume the curve C lying in E^- or E so small that it is the image of a curve Γ lying in S^- or S, respectively, under the mapping $(t,\sigma) \to x(t,\sigma)$.

Denote by $F(x)$ a multifunction defined on C with values on Γ in the following way

$$F(x) = \{(t,\sigma) \in \Gamma : x = x(t,\sigma)\}.$$

For any fixed $x \in C$, the set $F(x)$ of values of this multifunction is closed in $\mathbb{R}^{1+m}$ as the preimage of the one-element set $\{x\}$ under the continuous mapping. Denote by B_C a family of Borel subsets of $\mathbb{R}^n$ which are entirely contained in C. This family B_C is the σ-algebra of Borel sets on C (see [7], p. 77). For any compact subset $A \subset \Gamma$, the set $F^{-1}(A)$ is compact. Indeed, we have

$$\begin{aligned} F^{-1}(A) &= \{x \in C : F(x) \cap A \neq \emptyset\} \\ &= \{x \in C : \{(t,\sigma) \in \Gamma : x = x(t,\sigma)\} \cap A \neq \emptyset\} \\ &= \{x \in C : x = x(t,\sigma) \text{ and } (t,\sigma) \in A\}, \end{aligned}$$

so $F^{-1}(A)$ is the image of the compact set A under the continuous mapping. In this way, for for any compact set $A \subset \Gamma$, we have $F^{-1}(A) \in B_C$. From proposition 1A from [9], p. 160 we have that F is a Borel measurable mapping. By proposition 1B from [9], p. 161 (see also [2], pp. 64, 74), the condition of the measurability of F is equivalent to the fact that there exists a countable (or finite) family $(t_n(x), \sigma_n(x))$, $n \in T$, of Borel measurable selections such that $(t_n(x), \sigma_n(x)) : C \to \Gamma$ and $F(x) = \overline{\{(t_n(x), \sigma_n(x)),\ n \in T\}}$ for all $x \in C$. As the function $y(t,\sigma,\beta)$, where β is a continuous function of variable σ chosen according to the definition of the standard projection, is continuous, thus the composition $y(t_n(x), \sigma_n(x), \beta(\sigma_n(x))) = y_Z(x)$ is a Borel measurable function. By the above, $y_Z(x)$ is bounded. ∎

LEMMA 4.5. *On each arc of the canonical spray of flights Z^*, the expression*

$$y(t,\sigma,\beta)x_\sigma(t,\sigma) + \int_t^1 \tilde{f}^0{}_\sigma(\tau,\sigma)\,d\tau$$

takes a constant value (for fixed (σ,β) and for all $t \in [t^-(\sigma), t^+(\sigma)]$).

Proof. Let $(\hat{t},\hat{\sigma},\hat{\beta})$ be any point of S^* and $\hat{x}(t)$, $\hat{y}(t)$, $\hat{u}(t)$ the corresponding values of the functions $x(t,\hat{\sigma})$, $y(t,\hat{\sigma},\hat{\beta})$, $u(t,\hat{\sigma})$ for $t \in [\hat{t}, t^+(\hat{\sigma}))$. Let σ_i denote any coordinate of the vector $\sigma \in G$. By integrating (1.1) with respect to t in an interval $[\hat{t}, t)$, $t \in [\hat{t}, t^+(\hat{\sigma}))$, then differentiating in σ_i and again differentiating in t, we have

$$x_{\sigma_i}(t,\sigma) - x_{\sigma_i}(\hat{t},\sigma) = \int_{\hat{t}}^t \frac{\partial}{\partial\sigma_i}\tilde{f}(\tau,\sigma)\,d\tau,$$

and

$$\frac{\partial}{\partial t} x_{\sigma_i}(t,\sigma) = \frac{\partial}{\partial \sigma_i} \tilde{f}(t,\sigma) \tag{4.8}$$

at the point $(t,\hat{\sigma})$, $t \in [\hat{t}, t^+(\hat{\sigma}))$.

Multiplying (1.3) by x_{σ_i}, we have at $(t,\hat{\sigma})$, $t \in [\hat{t}, t^+(\hat{\sigma}))$,

$$x_{\sigma_i}(t,\sigma)\frac{\partial}{\partial t}\hat{y}(t) = f^0{}_x(\hat{x}(t),\hat{u}(t))x_{\sigma_i}(t,\sigma) - \hat{y}(t)f_x(\hat{x}(t),\hat{u}(t))x_{\sigma_i}(t,\sigma). \tag{4.9}$$

Multiplying (4.8) by $\hat{y}(t)$ and adding the result to both sides of (4.9), we obtain at this point

$$\begin{aligned} &\hat{y}(t)\frac{\partial}{\partial t}x_{\sigma_i}(t,\sigma) + \frac{\partial}{\partial t}\hat{y}(t)x_{\sigma_i}(t,\sigma) \\ &\quad = f^0{}_x(\hat{x}(t),\hat{u}(t))x_{\sigma_i}(t,\sigma) + \hat{y}(t)\frac{\partial}{\partial \sigma_i}\tilde{f}(t,\sigma) - \hat{y}(t)f_x(\hat{x}(t),\hat{u}(t))x_{\sigma_i}(t,\sigma) \end{aligned}$$

and, by H1,

$$\begin{aligned} &\frac{\partial}{\partial t}(\hat{y}(t)x_{\sigma_i}(t,\sigma)) \\ &\quad = \frac{\partial}{\partial \sigma_i}\tilde{f}^0(t,\sigma) - \frac{\partial}{\partial \sigma_i}f^0(\hat{x}(t),u(t,\sigma)) + \hat{y}(t)\frac{\partial}{\partial \sigma_i}f(\hat{x}(t),u(t,\sigma)) \end{aligned}$$

so

$$\begin{aligned} &\frac{\partial}{\partial t}(\hat{y}(t)\dot{x}_{\sigma_i}(t,\sigma)) - \frac{\partial}{\partial \sigma_i}\tilde{f}^0(t,\sigma) \\ &\quad = \hat{y}(t)\frac{\partial}{\partial \sigma_i}f(\hat{x}(t),u(t,\sigma)) - \frac{\partial}{\partial \sigma_i}f^0(\hat{x}(t),u(t,\sigma)) \end{aligned} \tag{4.10}$$

at $(t,\hat{\sigma})$ for almost all $t \in [\hat{t}, t^+(\hat{\sigma}))$. The triple $(\hat{x}(t),\hat{y}(t),\hat{u}(t))$ satisfies the maximum principle for almost all $t \in [\hat{t}, t^+(\hat{\sigma}))$, so it satisfies condition (1.4). The supremum on the right-hand side of (1.4) is attained for those u for which $u = u(t,\sigma)$, $\sigma \in G$, when $t \in [\hat{t}, t^+(\hat{\sigma}))$ is fixed, thus the necessary condition for the extremum is satisfied:

$$\hat{y}(t)\frac{\partial}{\partial \sigma_i}f(\hat{x}(t),u(t,\sigma)) - \frac{\partial}{\partial \sigma_i}f^0(\hat{x}(t),u(t,\sigma)) = 0$$

at $(t,\hat{\sigma})$ for almost all $t \in [\hat{t}, t^+(\hat{\sigma}))$. From (4.10) we obtain

$$\frac{\partial}{\partial t}(\hat{y}(t)x_{\sigma_i}(t,\sigma)) - \frac{\partial}{\partial \sigma_i}\tilde{f}^0(t,\sigma) = 0 \tag{4.11}$$

at $(t,\hat{\sigma})$ for almost all $t \in [\hat{t}, t^+(\hat{\sigma}))$. Integrating equality (4.10) in the interval

$[\hat{t}, t^+(\hat{\sigma}))$ and taking account of (4.11), we get

$$\hat{y}(\hat{t})x_{\sigma_i}(\hat{t},\sigma) + \int_{\hat{t}}^{t^+(\hat{\sigma})} \tilde{f}^0{}_{\sigma_i}(t,\sigma)\,dt = \hat{y}(t^+(\hat{\sigma}))x_{\sigma_i}(t^+(\hat{\sigma}),\sigma)$$

at $(t,\hat{\sigma})$, $t \in [\hat{t}, t^+(\hat{\sigma}))$, and, according to (4.4),

$$(4.12)\qquad \hat{y}(\hat{t})x_{\sigma_i}(\hat{t},\sigma) + \int_{\hat{t}}^{1} \tilde{f}^0{}_{\sigma_i}(t,\sigma)\,dt$$
$$= y(t^+(\hat{\sigma}),\hat{\sigma})x_{\sigma_i}(t^+(\hat{\sigma}),\hat{\sigma}) + \tilde{f}^0(t^+(\hat{\sigma}),\hat{\sigma})t^+{}_\sigma(\hat{\sigma}) + \tilde{J}^+_\sigma(\hat{\sigma})$$

where the left-hand side is calculated at the point $(\hat{t},\hat{\sigma})$. The right-hand side of (4.12) does not depend on $\hat{t}$ and depends only on $\hat{\sigma}$ and $\hat{\beta}$. Hence the value of the expression

$$y(\hat{t},\hat{\sigma},\hat{\beta})x_{\sigma_i}(\hat{t},\hat{\sigma}) + \int_{\hat{t}}^{1} \tilde{f}^0{}_{\sigma_i}(t,\sigma)\,dt$$

does not depend on the choice of the point $\hat{t}$. ■

LEMMA 4.6. *If the identity*

$$y(t,\sigma,\beta)x_\sigma(t,\sigma) + \int_t^1 \tilde{f}^0{}_\sigma(\tau,\sigma)\,d\tau \equiv 0$$

holds in S^{*-} (or S^*), then E^- (resp. E) is a relative exact set.

Proof. On account of the similarity in proving both assertions of the lemma, we shall limit ourselves to the first, i.e. we shall show that, under the above assumption, the set E^- is relative exact.

Let C denote a sufficiently small bounded rectifiable curve contained in E^-, with the parametric description $x = v(s)$, $0 \le s \le b$, where s is the arc length parameter. Denote by $\Theta(s)$ the direction of the tangent to the curve C defined for almost all s. Let $s_0 \in [0,b)$ be any point such that the function $\Theta(s)$ is approximately continuous at it i.e. it is a point such that, for each $\epsilon > 0$, there exists a closed set B of values of s such that, for any sufficiently small interval $W = \{s : 0 \le s \le \delta\}$, we have

(i) $|\Theta(s) - \Theta(s_0)| < \epsilon$ for $s \in B \cap W$,

(ii) $meas\,(W - B) < \epsilon\, meas\, W$.

Denote $\hat{x} = v(s_0)$, $\hat{\Theta} = \Theta(s_0)$, $\hat{y}$ — any element of $Y_Z(\hat{x})$, $(\hat{t},\hat{\sigma},\hat{\beta})$ — a point of S^{*-} such that $x(\hat{t},\hat{\sigma}) = \hat{x}$, $y(\hat{t},\hat{\sigma},\hat{\beta}) = \hat{y}$.

Denote by Γ a rectifiable curve in S^- such that small arcs of the curve C, issuing from $\hat{x}$, are, in accordance with H4, the images under the mapping $(t,\sigma) \to x(t,\sigma)$ of small arcs γ of the curve Γ, issuing from the point $(\hat{t},\hat{\sigma})$. Let us find the parametric description of the curve $\Gamma : t = \tilde{t}(\lambda)$, $\sigma = \sigma(\lambda)$, $0 \le \lambda \le p$, so that the point $(\hat{t},\hat{\sigma})$ of Γ should correspond to the value $\lambda_0 \in [0,p)$, where λ is the arc length parameter. Let $s(\lambda)$ denote a continuous increasing function on $[\lambda_0, p]$, such that $s(\lambda_0) = s_0$, which characterizes the arc length along C, i.e. it satisfies

$$v(s(\lambda)) = x(\tilde{t}(\lambda), \tilde{\sigma}(\lambda)) \text{ for } \lambda \in [\lambda_0, p]. \tag{4.13}$$

Let Δs, ΔJ denote the corresponding difference in s and in $J(x)$ at the ends of a sufficiently small curve C issuing from $\hat{x}$, and let $A = \{\lambda : s(\lambda) \in B\}$.

Now, we must only show that

(a) the expression $\frac{\Delta J}{\Delta s}$ is bounded,

(b) for any sufficiently small arc of the curve C, issuing from the point $\hat{x}$, we have $\lim_{\Delta s \to 0} \frac{\Delta J}{\Delta s} = -\hat{y}\hat{\Theta}$.

Actually, condition (b) gives us that $\hat{\Theta}$ is a direction of relative univalence at the point $\hat{x}$. Moreover, for any selection $y_Z(x)$ with its values in $Y_Z(x)$, $x \in |E|$, the equality

$$\frac{dJ(v(s))}{ds} = -y_Z(v(s))\frac{dv}{ds} \tag{4.14}$$

holds almost everywhere along C as the point $\hat{x}$ is any point of C such that the function $\Theta(s)$ is approximately continuous, so it is almost any point of curve C. Besides, by (a), after integrating (4.14) in s we get condition (3.2) from the definition of relative exactness. Let γ be a sufficiently small arc of Γ issuing from the point $(\hat{x},\hat{\sigma})$, described in the interval $P = [\lambda_0, \lambda_1]$, $P \subset [\lambda_0, p]$. Let $\Delta\tilde{J}$ denote the value of integral (4.3) along γ. In virtue of the assumption that $yx_\sigma + \int_t^1 \tilde{f}^0{}_\sigma\, d\tau \equiv 0$ in S^{*-}, for β chosen according to the definition of the standard projection and such that $\beta(\hat{\sigma}) = \hat{\beta}$, we have

$$\begin{aligned}\Delta\tilde{J}(x) &= \int_\gamma \tilde{f}^0(t,\sigma)dt - \left(\int_t^1 \tilde{f}^0{}_\sigma(\tau,\sigma)\, d\tau\right) d\sigma \\ &= \int_P \tilde{f}^0(\tilde{t}(\lambda),\tilde{\sigma}(\lambda))d\tilde{t}(\lambda) + y(\tilde{t}(\lambda),\tilde{\sigma}(\lambda),\beta(\tilde{\sigma}(\lambda)))x_\sigma(\tilde{t}(\lambda),\tilde{\sigma}(\lambda))\, d\tilde{\sigma}(\lambda).\end{aligned}$$

From (1.6) and (1.1) we obtain

$$\begin{aligned}&\tilde{f}^0(\tilde{t}(\lambda),\tilde{\sigma}(\lambda)) \\ &= y(\tilde{t}(\lambda),\tilde{\sigma}(\lambda),\beta(\tilde{\sigma}(\lambda)))f(\tilde{t}(\lambda),\tilde{\sigma}(\lambda)) = y(\tilde{t}(\lambda),\tilde{\sigma}(\lambda),\beta(\tilde{\sigma}(\lambda)))x_t(\tilde{t}(\lambda),\tilde{\sigma}(\lambda)),\end{aligned}$$

so

$$\Delta\tilde{J}(x) = \int_P y(\tilde{t}(\lambda),\tilde{\sigma}(\lambda),\beta(\tilde{\sigma}(\lambda)))x_t(\tilde{t}(\lambda),\tilde{\sigma}(\lambda))d\tilde{t}(\lambda) + y(\tilde{t}(\lambda),\tilde{\sigma}(\lambda),\beta(\tilde{\sigma}(\lambda)))x_\sigma(\tilde{t}(\lambda),\tilde{\sigma}(\lambda))\,d\tilde{\sigma}(\lambda).$$

Now, taking (4.13) into account, we get

$$\Delta\tilde{J}(x) = \int_P y(\tilde{t}(\lambda),\tilde{\sigma}(\lambda),\beta(\tilde{\sigma}(\lambda)))\Theta(s(\lambda))\,ds(\lambda). \tag{4.15}$$

The last integral in (4.15) is the Riemann-Stieltjes integral (the function $s(\lambda)$ is increasing and uniformly continuous). The function $y(t,\sigma,\beta)$ is bounded as a continuous function on the closed interval P, thus there exists a constant $M > 0$ such that $\|y(\tilde{t}(\lambda),\tilde{\sigma}(\lambda),\beta(\tilde{\sigma}(\lambda)))\| \le M$ for $\lambda \in [\lambda_0,\lambda_1]$. Moreover, $\|\Theta(s(\lambda))\| = \|\frac{dv(s(\lambda))}{ds(\lambda)}\| = \|\frac{dv(s)}{ds}\| = 1$ (see Tonelli th., [7], p. 180). Hence and from (4.15) we have $\|\frac{\Delta\tilde{J}(x)}{\Delta s}\| \le \frac{M}{\Delta s}\Delta s$ and, since $-\Delta\tilde{J}(x) = \Delta J(x)$, we obtain $\|\frac{\Delta J(x)}{\Delta s}\| \le M$, so the expression $\frac{\Delta J}{\Delta s}$ is bounded. Moreover

$$\begin{aligned} &\Delta\tilde{J}/\Delta s - \hat{y}\hat{\Theta} \\ &= \Delta s^{-1} \int_P (y(\tilde{t}(\lambda),\tilde{\sigma}(\lambda),\beta(\tilde{\sigma}(\lambda)))\Theta(s(\lambda)) - \hat{y}\hat{\Theta})\,ds(\lambda) \\ &= \Delta s^{-1} \int_{P\cap A} (y(\tilde{t}(\lambda),\tilde{\sigma}(\lambda),\beta(\tilde{\sigma}(\lambda)))\Theta(s(\lambda)) - \hat{y}\hat{\Theta})\,ds(\lambda) \\ &\quad + \Delta s^{-1} \int_{P-A} (y(\tilde{t}(\lambda),\tilde{\sigma}(\lambda),\beta(\tilde{\sigma}(\lambda)))\Theta(s(\lambda)) - \hat{y}\hat{\Theta})\,ds(\lambda). \end{aligned} \tag{4.16}$$

For a sufficiently small P, the function $y\Theta - \hat{y}\hat{\Theta}$ is bounded on P, thus it is bounded on $P - A$ and this set has $s(\lambda)$-measure less than ϵ by (ii). The set $P \cap A$ has $s(\lambda)$-measure at most Δs and, by (i) and the continuity of y, we have

$$\|y\Theta - \hat{y}\hat{\Theta}\| \le \|y\|\|\Theta - \hat{\Theta}\| + \|\hat{\Theta}\|\|y - \hat{y}\|,$$

so the value of this expression is at most a fixed multiple of ϵ. The above implies that the last two terms in (4.16) cannot exceed certain fixed multiples of an arbitrarily small positive ϵ. In consequence, condition (b) is true. ∎

LEMMA 4.7. *Let Z^* be the canonical spray of flights corresponding to a spray of flights Z. Then E^- and E are relative exact sets.*

Proof. From Lemma 4.5 we have that the expression $yx_\sigma + \int_t^1 \tilde{f}^0{}_\sigma\,d\tau$ takes a constant value along each arc of the canonical spray Z^* and from Lemma 4.3 we get that in S^{*+} this expression vanishes identically. The continuity of the function $yx_\sigma + \int_t^1 \tilde{f}^0{}_\sigma \, d\tau$ in $|S^*|$ implies that this expression vanishes identically in S^{*-} and S^*. From Lemma (4.6) we obtain that E^- and E are relative exact sets. ■

5. Chain of flights

Up till now, we have considered the fixed spray of flights Z defined in section 2. Of course, the family of lines of flight may consist of a greater number of sprays of flights satisfying conditions H1 — H4, with trajectories contained in D.

DEFINITION 5.1. A finite or countable sequence of sprays of flights

$$Z_1, Z_2, \ldots, Z_N, \ldots$$

with trajectories contained in D will be called a *chain of flights*, and the corresponding sequence of canonical sprays of flights

$$Z_1^*, Z_2^*, \ldots, Z_N^*, \ldots$$

will be called the *canonical chain of flights* if, for $i = 1, 2, \ldots, N-1, \ldots$, the set E_i^{*-} corresponding to the canonical spray Z_i^* contains the set E_{i+1}^{*+} corresponding to Z_{i+1}^*.

It is required here that not only the arcs of lines of flight between individual sprays should fit together but also the arcs of canonical lines of flight should have this property.

DEFINITION 5.2. The sets E_i^- and E_i corresponding to the spray of flight Z_i will be called the *constituent sets of the chain.*

If the set E_i^- or E_i of the spray Z_i is a relative exact set, we shall call it a *relative exact constituent set* for the given chain.

If the set S_1^{*+} corresponding to E_1^* has the form $S_1^{*+} = \{(t, \sigma, \beta) : t = 1, \ (\sigma, \beta) \in \tilde{G}_1\}$ where $\tilde{G}_1$ is such that its standard projection is G_1 corresponding to spray Z_1, then the chain will be called a *distinguishable chain.*

Note that if, in any fixed spray of flights Z_i, the set E_i^+ is relative exact, then, according to Lemma 4.7, the sets E_i and E_i^- are relative exact. Then the set E_{i+1}^+ corresponding to the spray Z_{i+1} is also relative exact as a subset of E_i^-. It is easy to show by induction that if the set E_1^+ of any fixed chain of flights is relative exact, then all constituent sets of this chain are relative exact.

DEFINITION 5.3. A chain whose all constituent sets are relative exact will be called a *relative exact chain.*

Assume that, apart from hypotheses H1 — H4, the following condition is satisfied:

H5. The function $l^+(\sigma) = l(x(1,\sigma))$ has a continuous derivative for $\sigma \in G_1$.

LEMMA 5.1. *Any distinguishable chain is relative exact.*

Proof. At first, we shall show that, in any distinguishable chain, the expression

$$y(t,\sigma,\beta)x_\sigma(t,\sigma) + \int_t^1 \tilde{f}^0{}_\sigma(\tau,\sigma)\,d\tau$$

is equal to 0 at any point $(t,\sigma,\beta) \in S_1^{*+}$, so it is equal to 0 at any point $(1,\sigma,\beta)$ where $(\sigma,\beta) \in \tilde{G}_1$. Indeed, we have

$$\begin{aligned}
&y(1,\sigma,\beta)x_\sigma(1,\sigma) + \int_t^1 \tilde{f}^0{}_\sigma(\tau,\sigma)\,d\tau \\
&= y(1,\sigma,\beta)x_\sigma(1,\sigma) + \int_{t^+(\sigma)}^{t^+(\sigma)} \tilde{f}^0{}_\sigma(\tau,\sigma)\,d\tau + \tilde{f}^0(t^+(\sigma),\sigma)t^+{}_\sigma(\sigma) + \tilde{J}^+_\sigma(\sigma) \\
&= y(1,\sigma,\beta)x_\sigma(1,\sigma) + l^+{}_\sigma(\sigma).
\end{aligned}$$

From (1.5) it follows (see [4], p. 61) that

$$(-y(1,\sigma,\beta),-1)(x_\sigma(1,\sigma),l^+{}_\sigma(\sigma)) = 0,$$

thus

$$y(1,\sigma,\beta)x_\sigma(1,\sigma) + \int_t^1 \tilde{f}^0{}_\sigma(\tau,\sigma)\,d\tau \equiv 0$$

in S_1^{*+}. From the proof of Lemma 4.7 and from Lemma 4.6 we get that the sets E_1^- and E_1 are relative exact. Taking account of the considerations preceding Definition 5.3, we have that E_1^+ is relative exact. ∎

6. Concourse of flights

Denote by K the family of all bounded rectifiable curves lying in D, and by D_n, $n = 1,2,\ldots$, a finite or countable system of disjoint subsets of D whose union is D.

DEFINITION 6.1. A curve $C \in K$ will be called a *fragment* if its interior lies entirely in some D_n.

The class of such fragments will be denoted by K_0.

DEFINITION 6.2. a) Let the final point of C_1 be the initial point of C_2. We term *fusion of* C_1, C_2 a curve C made up of two adjacent arcs consisting of C_1 and C_2 in that order.

b) Let C_2 be a closed curve intersecting C_1. We term *embellishment of C_1 by C_2* a curve C that describes first an arc of C_1, up to an intersection, then C_2 and then the remaining arc of C_1.

c) We term C_1 the result of *trimming* or *cutting* C_2 from C if C is expressible as the embellishment of C_1 by C_2 or as the fusion of C_1 and C_2, respectively.

We shall assume that K and K_0 are classes such that if any curve belongs to one of them, then all arcs of this curve and all inverse arcs belong to this class. Moreover, we shall assume that the operations of embellishment and trimming can be carried out countable often and the operations of fusion only finitely often under the restriction that from elements of K we shall again obtain elements of K.

Denote by K_1 a class of curves which are obtained from the elements of the class K_0 after finite operations of fusion and countable of embellishment. Denote by K_2 a class of such curves which are obtained by at most countable operations of trimming.

In problem (1.1)–(1.2) we want to find a minimum of the functional $I(x, u)$ in the entire set D. So far, we have had information only about this functional in sets D_n whose union is equal to D. This means that we have information about $I(x, u)$ in some subclass of curves from K_0 when we are interested in this functional in the class K.

The method described in our paper can be applied only when $K_2 = K$.

DEFINITION 6.3. If $K = K_2$, then the class K_0 will be called a *repairable class of fragments* and the decomposition of the set D into disjoint subsets D_n — a *repairable decomposition.* Then the set D will be termed the *unimpaired union of the sets D_n*.

DEFINITION 6.4. We shall term *concourse of flights* a finite or countable system of chains of flights such that D is the unimpaired union of the constituent sets of these chains and $\tilde{D}$ — the unimpaired union of their canonical constituent sets, where by canonical constituent set we understand a set from the canonical chain corresponding to a constituent set from a chain of flights.

7. The sufficient condition for a minimum

Let C be any bounded rectifiable curve in D with the description $x = v(s)$, $0 \leq s \leq b$, where s is the arc length parameter. We suppose the following additional hypothesis:

H6. For any bounded rectifiable curve C in D, there exists a selection $y(x)$ such that the value of the expression $y(v(s))\frac{dv}{ds}$ along C is equal at most to an integrable function $K(s)$ of the arc length s of C.

This assumption, used in the proof of Theorem 7.1, ensures us that the Hilbert integral exists along any bounded rectifiable curve C in D.

THEOREM 7.1 (main theorem). *Let us assume that there exists a concourse of flights. Then the set D is exact.*

The proof of this theorem is analogous to that of theorem (29.1) in [14], p. 280, when we set $\Sigma = Z$, $T(x) = J(x)$.

COROLLARY. *Assume that there exists a concourse of flights. Let C be any arc of an admissible trajectory in D, issuing from $x_1 = x(t_1)$, $t_1 \in [0,1]$, and ending at $x_2 = x(t_2)$, $t_2 \in [0,1]$, $t_1 \le t_2$. Let $y(x)$ be any selection in D. Then*

$$\int_C y(x)\,dx = J(x(t_1)) - J(x(t_2)).$$

Proof. According to Theorem 7.1, we have to show that any admissible trajectory C is bounded. Any admissible trajectory C is the union of a finite number of fragments from K_0. By Lemma 4.4, along any such fragment the function $J(x)$ is bounded, so C is a bounded curve. ∎

The following theorem gives us a sufficient condition for a minimum in our problem.

THEOREM 7.2. *Suppose that there exists a concourse of flights. Let $(x^*(t), u^*(t))$ be a line of flight defined on $[0,1]$ which is a member of this concourse of flights, and $x(0) = x_0$. Then $(x^*(t), u^*(t))$ is the pair which realizes the minimum of the functional $I(x,u)$ relative to all admissible pairs $(x(t), u(t))$ defined on $[0,1]$, such that $x(0) = x_0$ and whose trajectories $x(t)$ are contained in D.*

Proof. Let $(x(t), u(t))$ be any admissible pair defined on $[0,1]$ such that $x(0) = x_0$ and $x(t)$ lies in D for $t \in [0,1]$. We have

$$\begin{aligned} I(x^*,u^*) - I(x,u) &\\ &= l(x^*(1)) + \int_0^1 f^0(x^*(t),u^*(t))\,dt - l(x(1)) - \int_0^1 f^0(x(t),u(t))\,dt \\ &= J(x_0) - l(x(1)) - \int_0^1 f^0(x(t),u(t))\,dt. \end{aligned}$$

From the above corollary we get

$$J(x_0) = \int_C y(x)\,dx + J(x(1)) = \int_C y(x)\,dx + l(x(1))$$

where C is an arc of any admissible trajectory, with endpoints x_0, $x(1)$. In any constituent set there exists a Borel measurable and bounded selection $y_Z(x)$ (by Lemma 4.4), so, along the trajectory $x(t)$, there exists a measurable and bounded selection $y(x)$ (since $x(t)$ is a finite union of arcs which are contained in some constituent sets). The composition $y(x(t))$ is a measurable and bounded function and, according to (1.1), we get

$$J(x_0) = \int_0^1 y(x(t))f(x(t),u(t))\,dt + l(x(1)).$$

Hence

$$I(x^*,u^*) - I(x,u) = \int_0^1 (y(x(t))f(x(t),u(t)) - f^0(x(t),u(t)))\,dt$$

and, by (1.4)

$$I(x^*,u^*) - I(x,u) \le 0. \ \blacksquare$$

8. Conclusions

1. For any fixed subset $T \subset D$, the set $\bar{T} \subset \mathbb{R}^{n+r}$ of points (x,u) will be called a set generated by a line of flight if each point (x,u) of this subset is contained in some line of flight and x lies in T. For $x \in D$, let us denote by $U(x)$ the set of those values of $u(t)$ for which (x,u) is a point of the set $\bar{D}$ generated by D, and by $u(x)$ — a selection defined in D such that its values are contained in $U(x)$ for all x. Note that if there exists a concourse of flights, then the function $u(x)$ described above is an optimal feedback control.

2. In the set D, the value function is defined as

$$S(\bar{x}) = \int_C y(x)\,dx + l(x(1))$$

where C is any bounded rectifiable curve in D with endpoints $\bar{x}$, $x(1)$ and $y(x)$ is a selection in D. In this way we obtain an effective expression for the value function $S(x)$. Up till now, only the conditions for the existence of this function were known (comp. [13]).

3. Let us suppose that D has an non empty interior, and that the value function $S(x)$ is Fréchet differentiable at the point $x \in \operatorname{int} D$. Then $dS/dx = y(x)$. Indeed, from (4.14) we have

$$\frac{dS(v(s))}{dv}\frac{dv}{ds} = y(v(s))\frac{dv}{ds},$$

so

$$\left(\frac{dS(v(s))}{dv} - y(v(s))\right)\frac{dv}{ds} = 0.$$

Therefore

$$dS(x)dx - y(x) = 0.$$

In this case, there exists only one selection $y(x)$ defined in D. Moreover, we have

$$\int_t^1 f^0(x(t), u(x(t)))\, dt = \int_t^1 y(x(t)) f(x(t), u(x(t)))\, dt$$

where $u(x)$ is the optimal feedback control from conclusion 1. From these considerations we obtain the Hamilton-Jacobi equation

$$H(x, y(x), u(x)) = 0.$$

4. Note that the Hilbert integral $\int_C y(x)\, dx$ where C is any bounded rectifiable curve in D and $y(x)$ — a selection in D, satisfies all conditions from the definition of the K-function and, in this way, we have explicit form of the K-function in a Bolza problem.

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