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ON EXTENDING AN UNBOUNDED ORTHOGONAL VECTOR MEASURE IN A SEMIFINITE VON NEUMANN ALGEBRA TO A WEIGHT

The aim of this paper is to extend the result of the paper [2] (on the possibility of the extension of a Hilbert-space-valued unbounded orthogonal vector measure on all projections on a Hilbert space to a vector weight) to the case of an arbitrary semifinite von Neumann algebra.

Throughout the paper, let M be a von Neumann algebra which acts on a Hilbert space, endowed with an inner product $< .,. >$. We will denote by X^{pr} and X^+ the sets of all orthoprojections and positive operators in $X(\subset M)$, respectively. We will examine measures on projections with values in a Hilbert space K complemented with an improper element ∞. The following assumptions will be needed in this case:

$$f+\infty \equiv \infty \ (f \in K), \quad \infty + \infty = \infty, \quad \lambda \cdot \infty \equiv \infty (\lambda > 0), \quad 0 \cdot \infty = \theta$$

(here θ denotes the zero vector in K). We first give a definition of a scalar unbounded measure on projections (see Definition 2.2 [6]).

DEFINITION 1. Let M be a von Neumann algebra acting on a Hilbert space H. A mapping $m : M^{pr} \to [0, +\infty]$ is said to be a *semifinite measure* if

(i) there exists a net $(p_\alpha)_{\alpha \in A} \subset M^{pr}$ with $p_\alpha \nearrow 1$ and $m(p_\alpha) < +\infty$ for $\alpha \in A$;

(ii) $m(\sum p_i) = \sum m(p_i)$ for any family (p_i) of mutually orthogonal projections in M^{pr}.

We will deal with unbounded orthogonal vector measures (see Definition 3.3 [6] and [3]):

DEFINITION 2. Let M be a von Neumann algebra acting on a Hilbert space H and K be a Hilbert space complemented with an improper element

∞. A mapping $\mu : M^{pr} \to K \cup \{\infty\}$ is said to be a *semifinite orthogonal vector measure* (*K-valued measure* for short) if

(i) there exists a net $(p_\alpha)_{\alpha\in A} \subset M^{pr}$ with $p_\alpha \nearrow 1$ and $\mu(p_\alpha) \in K$ for $\alpha \in A$;

(ii) $pq = 0,\ \mu(p), \mu(q) \in K \Rightarrow \langle \mu(p), \mu(q)\rangle = 0$;

(iii) $p \doteq \sum p_i$ $(p, p_i \in M^{pr})$ implies

$$\mu(p) = \begin{cases} \sum \mu(p_i) & \text{if the series } \sum \mu(p_i) \text{ converges} \\ & \text{in the norm topology on } K, \\ \infty & \text{otherwise.} \end{cases}$$

(The convergence of an uncountable family of summands means that there exists a limit of the net of finite sums.)

A measure μ is said to be *unbounded* if $\mu(1) = \infty$. Our definition agrees with the one given in [1] whenever $\mu(1) \in K$.

In [1], the problem of the extension of a bounded K-valued measure to an orthogonal vector field was affirmatively solved. The latter was defined as a linear mapping $F : M \to K$ that is continuous (for the ultraweak topology on M and the weak topology on K) and satisfies $\langle F(p), F(q)\rangle = 0$ when $p, q \in M^{pr}$ and $pq = 0$.

THEOREM 1 [1]. *Let M be a von Neumann algebra without type I_2 direct summands and let $\mu : M^{pr} \to K$ be a K-valued measure. Then there is an orthogonal vector field $F : M \to K$ such that $F|M^{pr} = \mu$.*

Let us introduce an unbounded analogue to the orthogonal vector field. However we will impose no continuity requirements.

DEFINITION 3. We call a mapping $F : M^+ \to K \cup \{\infty\}$ *an orthogonal vector weight* if satisfies the following conditions:

(i) $F(x + y) = F(x) + F(y)$, $F(\lambda x) = \lambda F(x) (x, y \in M^+, \lambda \geq 0)$;

(ii) $pq = 0 (p, q \in M^{pr})$, $F(p), F(q) \in K \Rightarrow \langle F(p), F(q)\rangle = 0$.

The following theorem is the main result of the paper:

THEOREM 2. *Let M be a semifinite von Neumann algebra without type I_2 direct summands and let $\mu : M^{pr} \to K \cup \{\infty\}$ be an unbounded K-valued measure. Then there exists an orthogonal vector weight $F : M^+ \to K \cup \{\infty\}$ such that $F|M^{pr} = \mu$.*

Proof. The proof will be devided into 5 steps. We first prove (steps 1–4) the theorem for finite von Neumann algebras.

Step 1. According to Proposition 1 [3] for a given unbounded K-valued measure we can construct an unbounded semifinite scalar measure

$m : M^{pr} \to [0, +\infty]$ putting

$$(1) \qquad m(p) = \begin{cases} \|\mu(p)\|^2 & \text{if } \mu(p) \in K, \\ \infty & \text{otherwise .} \end{cases}$$

The following statement is a sharpened version of Proposition 1 [4].

PROPOSITION 1. *Let M ba a finite von Neumann algebra with no type I_2 direct summands and let $m : M^{pr} \to [0, +\infty]$ be a semifinite measure. Then there exists a normal weight $\omega : M^+ \to [0, +\infty]$ whose restriction to M^{pr} is m.*

Proof. Let τ be a faithful normal semifinite trace on M (it exists because M is finite and hence semifinite). Suppose now that $p_\alpha \nearrow 1$, where $(p_\alpha)_{\alpha \in A} \subset M^{pr}$ is a net satisfying (i) of Definition 1. As $m_\alpha \equiv m_{p_\alpha} | M^{pr}_{p_\alpha}$ is a finite measure on projections in the reduced algebra M_{p_α}, it extends to a normal state defined by a selfadjoint operator $T_\alpha \geq 0$ (the density operator) which acts in the Hilbert space $p_\alpha H$ and is affiliated with M_{p_α}. Moreover, m_α is defined as $m_\alpha(p) = \tau_\alpha(T_\alpha p)$ ($p \in M^{pr}_{p_\alpha}$), where τ_α is the trace τ reduced to M_{p_α}. The family (T_α) is compatible in the sense that $p_\alpha \leq p_\beta$ implies $p_\alpha T_\beta p_\alpha = T_\alpha$. Let us define a linear operator T on H by

$$D(T) = \bigcup_\alpha (D(T_\alpha) \cap p_\alpha H) \text{ and } Tf = T_\alpha f \text{ for } f \in D(T_\alpha) \cap p_\alpha H.$$

Obviously, T is positive and affiliated with M. So, its closure (for which we use the same letter T) is affiliated with M as well. Since M is finite, it follows that T is measurable with respect to M and hence positive and selfadjoint. Define a normal weight ω putting

$$\omega(x) = \tau(Tx) \quad (x \in M^+),$$

where the right side is understood in the known regularized sense (see [5]):

$$\tau(Tx) \equiv \lim_{\varepsilon \to 0+} \tau(T_\varepsilon^{1/2} x T_\varepsilon^{1/2}), \quad T_\varepsilon \equiv T(I + \varepsilon T)^{-1}, \quad \varepsilon > 0.$$

The proof is completed by showing that $\omega | M^{pr} = m$. Since M is finite, it follows that every $p \in M^{pr}$ is representable as an orthogonal sum $p = \sum p_\gamma$, where each projection p_γ is majorized by a suitable p_α, $\alpha = \alpha(\gamma)$. We thus get

$$\omega(p) = \sum \omega(p_\gamma) = \sum \tau(Tp_\gamma) = \sum \tau_{\alpha(\gamma)}(T_{\alpha(\gamma)} p_\gamma) = \sum m(p_\gamma) = m(p)$$

and the proof is complete.

Let $\omega : M^+ \to [0, +\infty]$ be a weight which extends the measure m defined by (1). We consider the following sets:

$$m_\omega \equiv \{x \in M^+ | \omega(x) < +\infty\};$$
$$n_\omega \equiv \{y \in M | y^* y \in m_\omega\} \quad \text{(a left ideal in } M);$$
$$lin_{\mathbb{C}} m_\omega = n_\omega^* n_\omega = (n_\omega^* \cap n_\omega)^2.$$

Moreover, $(n_\omega^* n_\omega)^+$ is a hereditary subcone of M^+. Define a positive bilinear form on n_ω by

$$\langle x, y \rangle_\omega \equiv \omega(y^* x) \quad (x, y \in n_\omega).$$

We will denote by $\|x\|_\omega \equiv \langle x, x \rangle_\omega^{\frac{1}{2}}$ $(x \in n_\omega)$ the corresponding seminorm.

Step 2. We introduce the set

$$\mathfrak{M} \equiv \{p \in M^{pr} | \mu(p) \in K\} = \{p \in M^{pr} | m(p) < +\infty\}.$$

According to J. Hamhalter [1], the measure $\mu_p \equiv \mu | M_p^{pr}$ extends to an orthogonal vector field $F_p : M_p^+ \to K$. Furthemore, standard arguments based on the spectral theorem show that $x \in M_p^+ \cap M_q^+$ $(p, q \in \mathbb{R})$ implies $F_p(x) = F_q(x)$.

Remark 1. A consequence of Proposition 1 is that if $p \in \mathfrak{M}$, then the restriction $\omega_p \equiv \omega | M_p$ is a normal state of M_p.

Let A be the set of indices as in Definition 2. Denote $\mathfrak{M}_A \equiv \{p \in M^{pr} | \exists \alpha \in A \ (p \leq p_\alpha)\}$ and $M_A^+ \equiv \{x \in m_\omega) | \exists \alpha \in A \ (rp(x) \leq p_\alpha)\}$ (a hereditary subcone of M^+). Here $rp(x)$ denotes the range projection of the operator x.

Put $\mu(\mathfrak{M}_A) \equiv \{\mu(p) | p \in \mathfrak{M}_A\}$. We will consider $K_0 \equiv [lin_{\mathbb{R}} \mu(\mathfrak{M}_A)]^-$ as a real Hilbert space with the inner product $\langle \xi, \eta \rangle_0 \equiv Re\langle \xi, \eta \rangle$ $(\xi, \eta \in K_0)$. We extend $\eta | \mathfrak{M}_A$ to an additive and positively homogeneous mapping by

$$\varphi(k) \equiv F_{rp(k)}(k) \quad (k \in M_A^+).$$

It should be noted that $lin_{\mathbb{R}} \{\varphi(k) | k \in M_A^+\}$ is dense in K_0.

We next claim that φ well extends to a real linear mapping (again denoted by φ) from $M_A^{sa} \equiv M_A^+ - M_A^+$ into K_0.

Step 3. Take an arbitrary $y \in n_\omega$, and let Φ_y be the real linear functional on the linear subspace $\varphi(M_A^{sa}) \subset K_0$ (that is dense in K_0) defined by

$$\Phi_y(\varphi(k)) \equiv Re\langle k, y \rangle_\omega \quad (k \in M_A^{sa}).$$

Put $M_0 \equiv \{k \in M_A^{sa} | k = \sum_i \lambda_i p_i (p_i p_j = 0, i \neq j)$ is a finite sum$\}$. Note that $\varphi(M_0)$ is dense in K_0. For $k = \sum_i \lambda_i p_i \in M_0$ we have

$$\begin{aligned}(2) \qquad \|k\|_\omega^2 &= \omega(k^2) = \sum_i \lambda_i^2 m(p_i) = \sum_i \lambda_i^2 \|\mu(p_i)\|^2 \\ &= \sum_i \lambda_i^2 \langle F_{rp(k)}(p_i), F_{rp(k)}(p_i)\rangle \\ &= \langle F_{rp(k)}(k), F_{rp(k)}(k)\rangle = \|\varphi(k)\|^2 \\ &= \|\varphi(k)\|_0^2.\end{aligned}$$

Therefore,

$$|\Phi_y(\varphi(k))| = |Re\langle k, y\rangle_\omega| \leq \|y\|_\omega \|k\|_\omega = \|y\|_\omega \|\varphi(k)\|_0.$$

Hence Φ_y is a continuous linear functional on K_0 and, by the Riesz theorem, there exists $\varphi^\sim(y) \in K_0$ such that

$$\Phi_y(b) = \langle b, \varphi^\sim(y)\rangle_0 \quad (b \in K_0).$$

Finally, define the desired orthogonal vector weight $F : M^+ \to K \cup \{\infty\}$ extending μ by

$$(3) \qquad F(x) = \begin{cases} \varphi^\sim(x) & \text{if } x \in (n_\omega^* n_\omega)^+, \\ \infty & \text{otherwise .}\end{cases}$$

Step 4. We will verify that $F|M_A^+ = \varphi$ and F extends μ. First suppose that $k \in M_A^+$ and $k_\alpha \nearrow k$, where $k_\alpha \in M_0 \cap \{k\}''$ (i.e., $k_\alpha \in M_0$ are operators from the commutative von Neumann algebra generated by k). From (2) and Remark 1, if follows that

$$\begin{aligned}\|k\|_\omega^2 &= \omega(k^2) = \lim_\alpha \omega(k_\alpha^2) = \lim_\alpha \|\varphi(k_\alpha)\|^2 = \lim_\alpha \|F_{rp(k_\alpha)}(k_\alpha)\|^2 \\ &= \lim_\alpha \|F_{rp(k)}(k_\alpha)\|^2 = \|F_{rp(k)}(k)\|^2 = \|\varphi(k)\|^2 = \|\varphi(k)\|_0^2.\end{aligned}$$

This gives that $\|t\|_\omega^2 = \|\varphi(t)\|_0^2$ for all $t \in M_A^{sa}$. We have

$$\begin{aligned}Re\langle t, k\rangle_\omega &= \frac{1}{4} Re \sum_{n=0}^{3} i^n \langle t + i^n k, t + i^n k\rangle_\omega \\ &= \frac{1}{4}\{\langle t + k, t + k\rangle_\omega - \langle t - k, t - k\rangle_\omega\} \\ &= \frac{1}{4}\{\langle \varphi(t + k), \varphi(t + k)\rangle - \langle \varphi(t - k), \varphi(t - k)\rangle\} \\ &= Re\langle \varphi(t), \varphi(k)\rangle = \langle \varphi(t), \varphi(k)\rangle_0 \quad (t \in M_A^{sa}).\end{aligned}$$

The equality

$$\langle \varphi(t), \varphi^\sim(k)\rangle_0 = \Phi_k(\varphi(t)) = Re\langle t, k\rangle_\omega = \langle \varphi(t), \varphi(k)\rangle_0 \quad (t \in M_A^{sa})$$

yields $F(k) = \varphi^\sim(k) = \varphi(k)$.

Finally, let $p \in \mathfrak{M}$ and (p_α) be a net of projections satisfying the condition (i) of Definition 2. Since M is finite it follows that $p \wedge p_\alpha \nearrow p$. Then for every $k \in M_A^{sa}$ it holds

$$|\langle k,p\rangle_\omega - \langle k,p\wedge p_\alpha\rangle_\omega| = |\langle k,p-p\wedge p_\alpha\rangle_\omega|$$
$$= |\omega((p-p\wedge p_\alpha)k)| \le \omega(p-p\wedge p_\alpha)^{\frac{1}{2}}\cdot\omega(k^2)\to 0.$$

Hence

$$\langle\varphi(k),\varphi^\sim(p)\rangle_0 = \Phi_p(\varphi(k)) = Re\langle k,p\rangle_\omega = \lim_\alpha\langle\varphi(k),\varphi(p\wedge p_\alpha)\rangle_0$$
$$= \lim_\alpha\langle\varphi(k),\mu(p\wedge p_\alpha)\rangle_0 = \langle\varphi(k),\mu(p)\rangle_0 \quad (k\in M_A^{sa}).$$

In the last equality, we have used the following property: $p_\alpha \nearrow p$ ($p_\alpha \in M_A^{pr}$) implies $\mu(p_\alpha)\to\mu(p)$, which (as $m|M_p$ extends to a normal state on M_p) is equivalent to (iii) of Definition 2. So, we may conclude that $\varphi^\sim(p)=\mu(p)$.

Step 5. (The end of the proof). Let τ be a faithful normal semifinite trace on M. Put

$$\mathfrak{M}_{\tau,\mu}\equiv\{p\in M^{pr}|p\in\mathfrak{M},\tau(p)<+\infty\}.$$

Let $r_\sigma=\bigvee_{q\in\sigma}q$, where σ is a finite subset of $\mathfrak{M}_{\tau,\mu}$. In this case $r\nearrow I$. The family (M_{r_σ}) is an increasing net of finite reduced von Neumann subalgebras of M. By making use of arguments similar to those in [4], we obtain that $\mu_\sigma\equiv\mu|M_{r_\sigma}^{pr}$ is a semifinite orthogonal measure for every σ. Denote by F_σ the orthogonal vector weight constructed as in Step 3 which extends the measure μ_σ.

Let $\omega: M^+\to[0,+\infty]$ again be a weight extending the measure m defined by (1). Its existence is guaranteed by [4] together with the above-proved Proposition 1. In addition, the weight ω has the following property: if $(x_\alpha)\subset M^+$ is a net increasing to $x\in M^+$ with $\omega(x)<+\infty$, then $\omega(x_\alpha)\to\omega(x)$.

Note (using the notation of Steps 1 and 2) that

$$(n_\omega^*n_\omega)^+\supset\bigcup_\sigma(n_{\omega_\sigma}^*n_{\omega_\sigma})^+,\ \text{where } \omega_\sigma\equiv\omega_{r_\sigma}.$$

Put $\varphi_\sigma^\sim(x)\equiv F_\sigma(x), x\in(n_{\omega_\sigma}^*n_{\omega_\sigma})^+$.

For an arbitrary $y\in n_\omega$ we consider the real linear functional Φ_y defined on $\varphi(\bigcup_\sigma M_{A_\sigma}^{sa})$ by

$$\Phi_y(\varphi(k))\equiv Re\langle k,y\rangle_\omega \quad (k\in M_{A_\sigma}^{sa}).$$

Here, $\varphi(\bigcup_\sigma M_{A_\sigma}^{sa})$ is dense in $K_0\equiv[lin_{\mathbb{R}}\mu(\bigcup_\sigma\mathfrak{M}_{A_\sigma})]^-$. Observe that the sets $\mathfrak{M}_{A_\sigma}, M_{A_\sigma}^{sa}$ and the mapping φ can be constructed in finite von Neumann algebras M_{r_σ} by the method of Step 2. According to arguments similar to Step 3, we conclude that Φ_y is a bounded linear functional on K_0. We define the orthogonal vector weight $F: M^+\to K\cup\{\infty\}$ by (3). We will show that F extends μ.

Let $p \in \mathfrak{M}$ and $p_\gamma \nearrow p$ ($p_\gamma \in \mathfrak{M}_{\tau,\mu}$). Then

$$\begin{aligned}\langle\varphi(k), F(p)\rangle_0 &= \Phi_p(\varphi(k)) = Re\langle k, p\rangle_\omega = \lim_\gamma Re\langle k, p_\gamma\rangle_\omega \\ &= \lim_\gamma\langle\varphi(k), \varphi^\sim_{\sigma\cup\{\gamma\}}(p_\gamma)\rangle_0 = \lim_\gamma \langle\varphi(k), \mu(p_\gamma)\rangle_0 \\ &= \langle\varphi(k), \mu(p)\rangle_0 \quad (k \in M^{sa}_{A_\sigma}).\end{aligned}$$

Hence $F(p) = \mu(p)$, and the proof is complete.

Let us add a result associated with the nature of the continuity of the obtained weight.

THEOREM 3. *The orthogonal vector weight constructed as in Theorem 2 has the following property: if (x_α) is a net increasing to x with $F(x_\alpha), F(x) \in K$, then $F(x_\alpha) \to F(x)$ in the norm of K.*

Proof. Given a net (x_α) put $y_\alpha = x - x_\alpha$. In this case $(y_\alpha) \subset m_\omega$, $y_\alpha \leq x$ and $y_\alpha \searrow 0$. It is sufficient to show that $F(y) \to \theta$. We have (in the notations of Theorem 2) $\|y_\alpha\|^2_\omega = \omega(y^2_\alpha) \leq \|x\|\omega(y_\alpha) \to 0$ and

$$\begin{aligned}\|F(y_\alpha)\| = \|\varphi^\sim(y_\alpha)\| &= \sup_{\|\varphi(k)\|_0=1,\ k\in\bigcup_\sigma \mathcal{M}^{sa}_{A_\sigma}} |\Phi_{y_\alpha}(\varphi(k))| \\ &\leq \sup_{\|k\|_\omega=1, k\in\bigcup_\sigma \mathcal{M}^{sa}_{A_\sigma}} |\langle k, y_\alpha\rangle_\omega| \leq \|y_\alpha\|_\omega.\end{aligned}$$

The theorem follows.

Finally, we observe that the constructed orthogonal weight has the following sharpened property of orthogonality (in comparision with (ii) of Definition 3):

COROLLARY. *The orthogonal vector weight F constructed as in Theorem 2 has the following property: if $rp(x)$ and $rp(y)(x, y \in M^+)$ are orthogonal and $F(x), F(y) \in K$, then $\langle F(x), F(y)\rangle = 0$.*

Proof. According to the spectral theorem we approximate x and y ($x, y \in M^+$) from the below by Riemann's integral sums x_n and y_n, respectively,

$$x_n = \sum_i \lambda_{in} p_{in} \nearrow x \quad \text{and} \quad y_n = \sum_j \mu_{jn} q_{jn} \nearrow y,$$

(here, p_{in} and q_{jn} are spectral projections of x and y, respectively). Since $rp(x)$ and $rp(y)$ are orthogonal, it follows that $p_{in}q_{jn} = 0$ for all i, j, n. Then we have $\langle F(p_{in}), F(q_{jn})\rangle =$ for all i, j, n. From Theorem 3, we obtain $\langle F(x), F(y)\rangle = \lim_n\langle F(x_n), F(y_n)\rangle = 0$.

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