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A NOTE ON TWO WEAK FORMS OF OPEN MAPPINGS AND BAIRE SPACES

1. Introduction

Z. Frolik [2] stated without proof the following Proposition 1.1 in order to use it in the proof of the next Theorem 1.2 given also in [2].

PROPOSITION 1.1. *If f is one-to-one feebly-continuous mapping from a space X onto a space Y, then f is almost-continuous.*

THEOREM 1.2. *Let f be a one-to-one feebly-continuous and feebly-open mapping from a space X onto a space Y. Then X is a Baire space if and only if Y is a Baire space.*

Later, T.Neubrunn [9] gave a counter example which shows that Proposition 1.1 is not true in general and proved that Theorem 1.2 is true, while Proposition 1.1 is false.

In 1968 M.K.Singal and A.R. Singal [7] defined almost-open mappings which contain the class of open mappings and in 1984 D.A.Rose [6] introduced a new class of mappings called weakly-open mappings which contain the class of open mappings.

The purpose of the present note is to give some sufficient conditions which insure that Proposition 1.1 is true (Theorem 2.4) and to point out the supposition that "feebly-open mapping" in the "if" part of Theorem 1.2 can be replaced by "almost-open mapping" (Theorem 3.3) and then to show that the suppositions of Theorem 1.2 is also valid for feebly-continuous and weakly-open mapping in the case when the space X is almost-regular (Corollary 3.4).

The most frequently used notations are following: Let A be a subset of a topological space X. The closure of A in X and interior of A in X will be denoted by $\overline{A}$ and $\text{int}A$, respectively. The complement of A in X is $X - A$. Throughout this paper by X and Y will be always denoted topological spaces on which no separation axioms are assumed unless stated explicitly.

No mapping is assumed to be continuous unless stated.

2. Preliminaries

DEFINITION 2.1. *A* mapping $f : X \longrightarrow Y$ is called almost-open [7] (resp. weakly-open [6]) if the image of every regularly open subset of X is an open subset of Y (resp. for every open set U of X, $f(U) \subset \text{int}(f(\overline{U}))$).

Remark 2.1. Every almost-open mapping is weakly-open (see [6] Theorem 4), but the converse of this statement may not be true in general as is shown by the following example.

EXAMPLE 2.1. Let $X = \{a,b,c,d\}$, $\mathcal{T} = \{X, \emptyset, \{b\}, \{d\}, \{b,d\}, \{b,c,d\}\}$ and $Y = \{1,2,3\}$,$\mathcal{T}' = \{Y, \emptyset, \{1\}, \{1,2\}, \{1,3\}\}$. Let $f : (X, \mathcal{T}) \longrightarrow (Y, \mathcal{T}')$ be given by $f(a) = 1, f(b) = 2, f(c) = f(d) = 3$. Then f is weakly-open, but since $\{b\}$ is regular open in $(X, \mathcal{T})$ and $f(\{b\}) = \{2\}$ is not open in $(Y, \mathcal{T}')$, then f is not almost-open.

DEFINITION 2.2. *A* topological space X is said to be almost-regular [8] if for each $x \in X$ and each regular open set G containing x there exists a regular open set V such that $x \in V \subset \overline{V} \subset G$.

The following result which gives the sufficient condition for a weakly-open mapping to be almost-open, will be used in the sequel.

THEOREM 2.1. *If X is almost-regular space and $f : X \longrightarrow Y$ is weakly-open, then f is almost open.*

Proof. Let G be any regular open subset of X and let y be arbitrary point of $f(G)$. Then there exists a point x of G such that $f(x) = y$. Since X is almost-regular, then there exists a regular open subset U of X containing x such that $x \in U \subset \overline{U} \subset G$. Hence $f(x) \in f(\overline{U}) \subset f(G)$. Since f is weakly-open, then $f(U) \subset int(f(\overline{U})) \subset f(G)$ by Definition 2.1.

Thus $f(G)$ is neigbourhood of $f(x)$. This shows that $f(G)$ is open in Y. Consequently f is almost-open.

DEFINITION 2.3. A mapping $f : X \longrightarrow Y$ is called almost-continuous [2] if, for every open set V of Y, $f^{-1}(V) \neq \emptyset$ implies $f^{-1}(V) \subset \overline{int(f^{-1}(V))}$. A mapping $f : X \longrightarrow Y$ is called feebly-continuous [2] if, for every nonempty open set V in Y, $f^{-1}(V) \neq \emptyset$ implies $int(f^{-1}(V)) \neq \emptyset$. A mapping $f : X \longrightarrow Y$ is called feebly-open [2] if, for every nonempty open set U in X, the set int$f(V)$ is nonempty.

Remark 2.2. Example 2.1 shows that weakly-open mapping is not needed to be feebly open. Indeed, since $\{b\}$ is open in $(X, \mathcal{T})$ and $int(f(\{b\}))$ $= int(\{2\}) = \emptyset$, then f is not feebly-open.

DEFINITION 2.4. A being a subset of X is said to be semi-open [5] if there exists an open subset U of X such that $U \subset A \subset \overline{U}$.

THEOREM 2.2. (cf. [5]). *A subset A in X is semi-open if and only if $A \subset \overline{intA}$.*

DEFINITION 2.5. A mapping $f : X \longrightarrow Y$ is said to be semi-continuous [5] if, for each open subset V in Y, $f^{-1}(V)$ is semi-open in X.

The following characterization will be used in the sequel

THEOREM 2.3. (cf.[4], Theorem 1). *For the mapping $f : X \longrightarrow Y$ the following statements are equivalent*

(i) *f is semi-continuous,*

(ii) *$int(\overline{f^{-1}(B)}) \subset f^{-1}(\overline{B})$ for each subset B of Y,*

(iii) *$f(int\overline{A}) \subset \overline{f(A)}$ for each subset A of X.*

Remark 2.3. (i) It appears that Frolik's almost-continuity in Definition 2.3 is precisely Levin's semi-continuity in Definition 2.5, since $\emptyset \subseteq A$ for any set A.

(ii) Obviously every almost-continuous mapping is feebly-continuous, but the converse of this statement is not necessarily true as shown by the following example due to T.Neubrunn [9].

EXAMPLE 2.2. Let X and Y be the set of real numbers with usual topology. Let the mapping $f : X \longrightarrow Y$ be defined as follows $f(x) = x$, if $x \neq 0$ and $x \neq 1$; $f(0) = 1, f(1) = 0$. Then f is one-to-one feebly-continuous and feebly-open, but it is not almost-continuous. Indeed, since $G = (-\frac{1}{2}, \frac{1}{2})$ is an open subset in Y and $f^{-1}(G) = (-\frac{1}{2}, 0) \cup (0, \frac{1}{2}) \cup \{1\}$, then $\overline{int(f^{-1}(G))} = [-\frac{1}{2}, \frac{1}{2}]$, consequently f is not almost-continuous by Definition 2.3.

However we have the following result which gives the sufficient condition for a feebly-continuous mapping to be almost-continuous in order to use it for our proof in the sequel.

THEOREM 2.4. *Let f be one-to-one mapping from a space X into a space Y. If f is feebly-continuous and weakly-open, then f is almost-continuous.*

Proof. Let B be any open subset of Y. To prove that f is almost-continuous it is enough to show that $f^{-1}(B) \neq \emptyset$ implies $f^{-1}(B) \subset \overline{int(f^{-1}(B))}$ by Definition 2.3. Let x be arbitrary point of $f^{-1}(B)$; then we have $f(x) \in B$. Let U be arbitrary open neighbourhood of x. Since f is weakly-open, $f(x) \in f(U) \subset int(f(\overline{U}))$. Hence the set $W = B \cap int(f(\overline{U}))$ is an open neighbourhood of $f(x)$. Since f is feebly-continuous, $int(f^{-1}(B \cap int(f(\overline{U}))) \neq \emptyset$. This shows that $int(f^{-1}(B \cap f(\overline{U}))) \neq \emptyset$. Moreover, since f is injective, we obtain $int(f^{-1}(B)) \cap \text{int}\overline{U} \neq \emptyset$. From this we have $int(f^{-1}(B)) \cap \overline{U} \neq \emptyset$. This implies that $\text{int}(f^{-1}(B)) \cap U \neq \emptyset$. Hence x belongs to the closure of the set $int(f^{-1}(B))$, that is $x \in \overline{int(f^{-1}(B))}$.

COROLLARY 2.5. *Let f be one-to-one mapping from a space X into a space Y. If f is feebly-continuous and almost-open, then f is almost continuous.*

The proof follows from Remark 2.1 and Theorem 2.4.

Remark 2.4. Almost-open mapping does not need to be feebly open in general, as the following example shows.

EXAMPLE 2.3. Let $X = Y = R$ be the set of real numbers and let $\mathcal{U}$ be the usual topology of the set of real numbers. Let the topology $\mathcal{U}^*$ on X be generated by $\mathcal{U} \cup \{U \cap (R - Q) | U \in \mathcal{U}\}$, where Q is the set of rational numbers and let $\mathcal{T}$ be lower limit topology on Y generated by the right half-open intervals $[a, b)$, $a, b \in R$. Now $(X, \mathcal{U}^*)$ is a Baire space and the identity mapping $i : (X, \mathcal{U}^*) \to (Y, \mathcal{T})$ is almost-open and feebly continuous, but it is not feebly-open. To see that i is not feebly-open, let A be the set of irrational numbers between 0 and 1. Then A is nonempty open subset of X. However interior of $i(A) = A$ (with respect to the lower limit topology on Y) is empty. Consequently i is not feebly-open.

The following definitions, which will be used in the sequel, can be found in [1],[3].

DEFINITION 2.6. Let A be a subset of a topological space X. Then A is nowhere dense in X if $\text{int}(\overline{A}) = \emptyset$. If A is not nowhere dense in X, then it is called somewhere dense in X.

DEFINITION 2.7. A subset A of a space X is of first category (also called meager) in X if it is the union of countable family of nowhere dense subsets of X. A subset A of X is of second category (also called nonmeager) in X if it is not of first category in X.

DEFINITION 2.8. A Baire space is a topological space such that every nonempty open subset is of the second category.

DEFINITION 2.9. A mapping $f : X \longrightarrow Y$ is called δ-open (cf.[3],p.45) if, for every nowhere dense subset N of Y, $f^{-1}(N)$ is a nowhere dense subset of X, or equivalently if, for every somewhere dense subset A of X, $f(A)$ is somewhere dense subset of Y.

THEOREM 2.6. *If f is a $\delta-$open mapping from a space of second category X on a space Y, then Y is a space of second category (cf.[3], Proposition 4.6 p.46).*

3. Some Results

THEOREM 3.1. *If $f : X \longrightarrow Y$ is one-to-one feebly-continuous and almost-open mapping, then f is δ-open.*

Proof. Let A be somewhere dense of X. That is $\text{int}(\overline{A}) \neq \emptyset$. Let us put $G = \text{int}(\overline{A})$. Hence G is regular-open subset of X. Since f is almost-open, $f(G)$ is nonempty open subset in Y, by Definition 2.1. On the other hand, since f is almost-continuous by Corallary 2.5, consequently f is semi-continuous, by the case (i) of Remark 2.3, then $f(G) = f(\text{int}(\overline{A})) \subset \overline{f(A)}$, by the case (iii) of Theorem 2.3. This shows that $int(\overline{f(A)}) \neq \emptyset$. Hence f is δ-open.

COROLLARY 3.2. *If X is almost-regular and $f : X \longrightarrow Y$ is feebly-continuous and weakly-open injection, then f is δ-open.*

The proof follows from Theorems 2.1 and 3.1.

THEOREM 3.3. *Let f be one-to-one feebly-continuous and almost-open mapping from a space X onto a space Y. If X is Baire space, then Y is Baire space.*

Proof. Suppose that Y is not a Baire space. Then there exists a non-empty open first category subset B of X, by Definition 2.8. That is $B = \bigcup_{n=1}^{\infty} B_n$ where each B_n is nowhere dense subset in Y, by Definition 2.7. On the other hand, since f is δ-open, by Theorem 3.1, $f^{-1}(B) = \bigcup_{n=1}^{\infty} f^{-1}(B_n)$ is of the first category subset of X, by Definition 2.9. Moreover, since f is feebly-continuous, $\text{int}(f^{-1}(B)) \neq \emptyset$. Let us put $G = \text{int}(f^{-1}(B))$. Thus G is an open first category subset in X. This contradicts the assumption that X is Baire space.

COROLLARY 3.4. *Let f be one-to-one feebly-continuous and weakly-open mapping from an almost-regular space X onto a space Y. If X is a Baire space, then Y is Baire space.*

The proof follows from Theorems 2.1 and 3.3.

Note that Theorem 3.3 and Corollary 3.4 remain to be true if the statement "if X is a Baire space, then Y is a Baire space" is substituted by "if X is a space of second category, then Y is a space of second category". For this case the proof follows from Theorem 2.6.

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