DEMONSTRATIO MATHEMATICA
Vol. XXX No 3 1997

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ON THE DERIVATIONS IN THE DEFORMATION ALGEBRA OF TWO LINEAR CONNECTIONS

Introduction

In this note we study the derivations in a Weyl algebra and the derivations in deformation algebras associated to the tangent bundle TM and to the tangent bundles of order $r, T_r(M)$, of a differentiable manifold M. Manifolds, mappings, tensor fields and connections we discuss here are always assumed to be C^∞.

Let M be an n-dimensional manifold.We denote by $\Im^r_s(M)$ the module of all tensor fields of type (r,s) in M over $F(M)$-the ring of all real functions on M.

Let ∇ and $\bar{\nabla}$ be two linear connections on M and $A = \bar{\nabla} - \nabla \in \Im^1_2(M)$.

Defining the product of two vector fields X and Y as

$$X \star Y = A(X,Y) \tag{0.1}$$

we can easily verify that the module $\Im^1_0(M)$ becomes an algebra over $F(M)$. This algebra is called the deformation algebra of the pair of connections $(\nabla, \bar{\nabla})$ and is denoted by $U(M,A)$ [7].

DEFINITION. An element $F \in \Im^1_1(M)$ is called derivation in the $U(M,A)$ *algebra* if

$$F(A(X,Y)) = A(F(X),Y) + A(X,F(Y)) \quad \forall X,Y \in \Im^1_0(M). \tag{0.2}$$

1. Derivations in a Weyl algebra

Let g be a Riemannian metric on a differentiable manifold M and let $\hat{g}$ be the conformal structure generated by g, that is

$$\hat{g} = \{e^u g \mid u \in F(M)\}. \tag{1.1}$$

Let W be a Weyl structure on the conformal manifold $(M, \hat{g})$, that is a mapping

$$W : \hat{g} \longrightarrow \Im_1^0(M)$$

which verifies [1]

$$W(e^u g) = W(g) - du, \quad \forall u \in F(M). \tag{1.2}$$

Let us consider the Weyl manifold $(M, \hat{g}, W)$. A linear connection ∇ on M is called compatible with the Weyl structure W if for every $X \in \Im_0^1(M)$ we have [1] : $\nabla_X g + W(g)(X)g = 0$.

It is know there is the only one symmetric linear connection ∇ on M compatible with the Weyl structure W. The connection ∇ is given by the formula [1]

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\ &+ Wg(X)g(Y,Z) + Wg(Y)g(X,Z) - Wg(Z)g(X,Y) \\ &+ g([X,Y],Z) + g([Z,X],Y) - g([Y,Z],X) \end{aligned} \tag{1.3}$$

for every $X, Y, Z \in \Im_0^1(M)$. Let $\dot{\nabla}$ be the Levi-Civita connection associated to g and let ∇ be the symmetric connection of Weyl, given by the formula (1.3). We denote by $A = \nabla - \dot{\nabla}$ the deformation tensor associated to the pair $(\dot{\nabla}, \nabla)$. The algebra $U(M, A)$ is called Weyl algebra [6].

THEOREM 1.1. *Let us consider the Weyl algebra $U(M, A)$, where $n \geq 3$. Let us suppose that M is connected. Thus the following statements are equivalent:*

1) *All the elements of the $F(M)$-module $\Im_1^1(M)$ are derivations in the Weyl algebra $U(M, A)$. (Remark. This holds if and only if $A = 0$.)*

2) *$\dot{\nabla}$ and ∇ determine the same curvature tensor, if the Ricci tensor of the metric g is nondegenerate.*

Proof. The implication 1)$\Longrightarrow$ 2) is obvious.

2) $\Longrightarrow$ 1). Let R, respectively $\dot{R}$, be the curvature tensor of the connection ∇, respectively $\dot{\nabla}$. From the equality $R = \dot{R}$ we have

$$(\dot{\nabla}_X R)(Y, Z, V) = (\dot{\nabla}_X \dot{R})(Y, Z, V), \quad \forall X, Y, Z, V \in \Im_0^1(M). \tag{1.4}$$

Since $A = \nabla - \dot{\nabla}$, using (1.4) it follows

$$\begin{aligned} (\nabla_X R)(Y, Z, V) - A(X, R(Y,Z)V) + R(A(X,Y), Z)V \\ + R(Y, A(X,Z))V + R(Y,Z)A(X,V) = \\ = (\dot{\nabla}_X \dot{R})(Y, Z, V), \quad \forall X, Y, Z, V \in \Im_0^1(M). \end{aligned} \tag{1.5}$$

Using the identities of Bianchi, from (1.5) we obtain

$$A(X, \dot{R}(Y,Z)V) + A(Y, \dot{R}(Z,X)V) + A(Z, \dot{R}(X,Y)V) = \dot{R}(Y,Z)A(X,V) + \dot{R}(Z,X)A(Y,V) + \dot{R}(X,Y)A(Z,V), \quad \forall X,Y,Z,V \in \Im^1_0(M). \tag{1.6}$$

Let A^i_{jk}, respectively $\dot{R}^i_{jkl}$, be the components of A, respectively of R, in a local coordinates system. Then the relation (1.6) becomes:

$$A^r_{is}\dot{R}^s_{hjk} + A^r_{js}\dot{R}^s_{hki} + A^r_{ks}\dot{R}^s_{hij} = A^s_{ih}\dot{R}^r_{sjk} + A^s_{jh}\dot{R}^r_{ski} + A^s_{kh}\dot{R}^r_{sij}. \tag{1.6'}$$

Let g_{ij} be the local components of g. In local coordinates, the relation (1.3) can be written as

$$A^i_{jk} = \delta^i_j\psi_k + \delta^i_k\psi_j - g_{jk}\psi_i \tag{1.7}$$

where $2\psi_k$ are the local coordinates of the 1-form Wg, and $\psi^i = g^{ij}\psi_j$. From (1.7) and (1.6'), we have

$$(g_{jr}\dot{R}^i_{skh} + g_{jk}\dot{R}^i_{shr} + g_{jh}\dot{R}^i_{srk} + \delta^i_r\dot{R}_{sjkh} + \delta^i_k\dot{R}_{sjhr} + \delta^i_h\dot{R}_{sjrk})\psi^s = 0 \tag{1.8}$$

where $\dot{R}_{ijkh} = g_{is}\dot{R}^s_{jkh}$. If we transveit the relation (1.8) with g^{jr}, we obtain

$$(n-3)\dot{R}^i_{skh}\psi^s + (\delta^i_k\dot{R}_{hs} - \delta^i_h\dot{R}_{ks})\psi^s = 0 \tag{1.9}$$

where $\dot{R}_{ij} = \dot{R}^k_{ikj}$ are the components of the Ricci tensor associated to g. Contracting (1.9) with respect to i and k, we obtain

$$(n-2)\dot{R}_{sk}\psi^s = 0. \tag{1.10}$$

Because $n \geq 3$ and $det(\dot{R}_{sk}) \neq 0$, (1.10) results $\psi^s = 0$; and by (1.7) we have $A^i_{jk} = 0$. Therefore $A = 0$, which implies that all the elements of the $F(M)$-module $\Im^1_1(M)$ are derivations in the Weyl algebra $U(M, A)$.

2. Derivations on the tangent bundle

Let M be an n-dimensional differentiable manifold and $T_p(M)$ the tangent space at a point p of M. A system of coordinates (U, x^h) which covers the base space M, induces on the tangent bundle $TM = \bigcup_{p\in M} T_p(M)$ a system of coordinates $(\pi^{-1}(U), x^h, y^h)$, where $\pi : TM \longrightarrow M$ is the natural projection (cf. e.g.[8]).

Let ∇ be a connection on M, with local components Γ^i_{jk}.

A vector field X on M, having the local components X^h, induces on TM three vector fields [8]:

1) the vertical lift $X^v = (0, X^i)$;
2) the complete lift $X^c = (X^i, y^i\frac{\partial X^h}{\partial x^i})$;
3) the horizontal lift $X^H = (X^h, -y^h\Gamma^i_{hj}X^j)$.

The connection ∇ on M induces on TM two connections [8]:

1) the complete lift ∇^c, so that

$$(2.1)\quad \begin{aligned} &\nabla^c_{X^c}Y^c = (\nabla_X Y)^c, \nabla^c_{X^v}Y^v = 0,\\ &\nabla^c_{X^c}Y^v = (\nabla_X Y)^c, \nabla^c_{X^v}Y^c = (\nabla_X Y)^v, (\forall)X, Y \in \Im^1_0(M). \end{aligned}$$

2) the horizontal lift ∇^H, so that

$$(2.2)\quad \begin{aligned} &\nabla^H_{X^H}Y^H = (\nabla_X Y)^H, \nabla^H_{X^H}Y^v = (\nabla_X Y)^v,\\ &\nabla^H_{X^v}Y^H = 0, \nabla^H_{X^v}Y^v = 0, \forall X, Y \in \Im^1_0(M). \end{aligned}$$

A tensor field F of type (1.1) on M induces on TM three fields [8] :

1) the complete lift F^c, so that

$$F^c(X^v) = (F(X))^v, F^c(X^c) = (F(X))^c,$$

2) the vertical lift F^v, so that

$$(2.3)\quad F^v(X^v) = 0, F^v(X^c) = F^v(X^H) = (F(X))^v,$$

3) the horizontal lift F^H, so that

$$F^H(X^v) = (F(X))^v, F^H(X^H) = (F(X))^H (\forall) X \in \Im^1_0(M).$$

A distribution D of dimension r on M, determined by the projection tensor $m \in \Im^1_1(M), m^2 = m$, $m(TM) = D$ and the rank of m is r, induces two distributions in TM [8] :

1) the complete lift D^c, determined by the projection tensor m^c;

2) the horizontal lift D^H, determined by the projection tensor m^H.

Let $\bar{\nabla}$ be a linear connection on M and D a distribution on M, so that $A(X,Y) \in D, (\forall)X, Y \in D$, where $A = \bar{\nabla} - \nabla$. The deformation algebra of the pair of connections $(\bar{\nabla}, \nabla)$ is denoted by $U(D, A)$.

Proposition 2.1. *If F is a derivation in the algebra $U(D, A)$ then:*

1) F^H is a derivation in the algebra $U(D^H, \bar{\nabla}^H - \nabla^H)$;

2) F^v is a derivation in the algebra $U(D^c, \bar{\nabla}^c - \nabla^c)$;

3) F^c is a derivation in the algebra $U(D^c, \bar{\nabla}^c - \nabla^c)$.

Proof. In truth, because $A = \bar{\nabla} - \nabla$, we have the relation

$$(2.4)\quad F(A(X,Y)) = A(F(X), Y) + A(X, F(Y)), \quad (\forall)X, Y \in D$$

or

$$(2.4')\quad F(\bar{\nabla}_X Y) - F(\nabla_X Y) = \bar{\nabla}_{F(X)}Y - \nabla_{F(X)}Y + \bar{\nabla}_X F(Y) - \nabla_X F(Y)$$

for every $X, Y \in D$.

Taking into account that the horizontal lift D^H is generated by X^v and X^H, X being an arbitrary vector field of D, and F^H, $A^H = \bar{\nabla}^H - \nabla^H$ are tensor fields, it is sufficient to verify the relation

$$F^H(A^H(U,V)) = A^H(F^H(U),V) + A^H(U,F^H(V)) \tag{2.5}$$

where (U,V) is equal to (X^H,Y^H) or (X^H,Y^v), (X^v,Y^H), (X^v,Y^v) with $X,Y \in D$. For $(U,V) = (X^H,Y^H)$, the relation (2.5) is equivalent to the relation

$$F^H(\bar{\nabla}^H_{X^H}Y^H) - F^H(\nabla^H_{X^H}Y^H) = \bar{\nabla}^H_{F^H X^H}Y^H - \\ -\nabla^H_{F^H X^H}Y^H + \bar{\nabla}^H_{X^H}F^H Y^H - \nabla^H_{X^H}F^H Y^H$$

which follows from (2.4') and (2.2).

The other cases and the properties 2) and 3) can be demonstrated similarly, taking into account that the complete lift D^c of the distribution D is generated by X^v and X^c, where $X \in D$.

Remark. F^c is in general not a derivation in the algebra $U(D^H, \bar{\nabla}^H - \nabla^H$ when F is a derivation in the algebra $U(D, \bar{\nabla} - \nabla)$ because from the relations [7]

$$F^c(X^H) = (FX)^H + (\nabla_\gamma F)X^H, \quad (\nabla_\gamma F)X^H \in TM, \quad X \in D$$

we obtain that $F^c X^H$ is not a field from D^H. Morever F^v is not a derivation in $U(D^H, \bar{\nabla}^H - \nabla^H)$ in general.

3. Derivations in deformation algebras associated to the tangent bundle of order r

Let M be an n-dimensional differentiable manifold, R the real line, r a fixed integer so that $r \geq 1$. The tangent bundle $T_r(M)$ of order r over M is the set of all r-jets $j^r_p(f)$ determined by a mapping $F : R \longrightarrow M$ so that $F(0) = p$ [2, 8].

For $\lambda = 0, 1, \ldots, r$, the λ-lift of a vector field X on M is a vector field $X^{(\lambda)}$ on $T_r(M)$, and λ-lifts of a tensor field of type (1,1) S in M is a tensor field $S^{(\lambda)}$ in $T_r(M)$, and we have

$$S^{(\lambda)}X^{(\mu)} = (SX)^{(\lambda+\mu-r)}, \quad \forall S \in \Im^1_1(M), \quad X \in \Im^1_0(M) \\ \text{and } \lambda, \mu = 0, 1, \ldots, r. \tag{3.1}$$

An affine connection ∇ in M determines globally an affine connection in $T_r(M)$ which is called the lift of the affine connection ∇ and denoted by ∇^*. We have (see [2, 8])

$$\nabla^*_{X^{(\lambda)}}Y^{(\mu)} = (\nabla_X Y)^{(\lambda+\mu-r)}, \quad \forall X, Y \in \Im^1_0(M) \\ \text{and } \lambda, \mu = 0, 1, \ldots, r. \tag{3.2}$$

PROPOSITION 3.1. *If S is a derivation in the algebra $U(TM, \bar{\nabla} - \nabla)$ then $S^{(\lambda)}$, for $\lambda = 0, 1, \ldots, r$ is a derivation in algebra $U(T_r(M), \bar{\nabla}^* - \nabla^*)$.*

Proof. In truth, for every $X, Y \in \Im^1_0(M)$ the relation

$$S^{(\lambda)}(A^*(X^{(\mu)}, Y^{(\nu)}) = A^*(S^{(\lambda)}X^{(\mu)}, Y^{(\nu)}) + A^*(X^{(\mu)}, S^{(\lambda)}Y^{(\nu)}),$$

where $A^* = \bar{\nabla}^* - \nabla^*$ and $\lambda, \mu, \nu = 0, 1, \ldots, r$, is equivalent to

$$S^{(\lambda)}(\bar{\nabla}^*_{X^{(\mu)}}Y^{(\nu)} - \nabla_{X^{(\mu)}}Y^{(\nu)}) = A^*((SX)^{(\lambda+\mu-r)}, Y^{(\nu)}) + A^*(X^{(\mu)}, (SY)^{(\lambda+\nu-r)})$$

or

$$\begin{aligned} S^{(\lambda)}((\bar{\nabla}_X Y)^{(\mu+\nu-r)} - (\nabla_X Y)^{(\mu+\nu-r)}) &= \bar{\nabla}^*_{(SX)^{(\lambda+\mu-r)}}Y^{(\nu)} - \nabla^*_{(SX)^{(\lambda+\mu-r)}}Y^{(\nu)} \\ &+ \bar{\nabla}^*_{X^{(\mu)}}(SY)^{(\lambda+\nu-r)} - \nabla^*_{X^{(\mu)}}(SY)^{(\lambda+\nu-r)}. \end{aligned}$$

The last relation can be written

$$(S(\bar{\nabla}_X Y) - S(\nabla_X Y))^{(\lambda+\mu+\nu-2r)} = (\bar{\nabla}_{SX}Y - \nabla_{SX}Y + \bar{\nabla}_X SY - \nabla_X(SY)^{(\lambda+\mu+\nu-2r)},$$

which follows from the relation (3.2).

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Received February 23, 1996.