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THE SEMILINEAR WAVE EQUATION ASSOCIATED WITH A NONLINEAR BOUNDARY

1. Introduction

We consider the following boundary-initial value problem

$$u_{tt} - \Delta u + f(u, u_t) = 0, \quad 0 < x < 1, \quad 0 < t < T, \tag{1.1}$$

$$u_x(0,t) = H(u(0,t)) + g(t), \tag{1.2}$$

$$u(1,t) = 0, \tag{1.3}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \tag{1.4}$$

The boundary condition (1.2) is nonlinear, in general nonhomogeneous, and the term $H(u(0,t))$ is supposed to be of the same sign as $u(0,t)$. The nonlinear term $f(u, u_t)$ is supposed to be Hölder continuous with respect to every variable and non-decreasing with respect to the second variable.

The equation (1.1) has the same form as that from [3], but the smoothness of the nonlinear term $f(u, u_t)$ and that of the initial values $u_0(x), u_1(x)$ are less than in [3]. Then the linearization method used for the problems from [3],[7] cannot be here used. In [1] is given a theorem of existence and uniqueness of a global solution of the problem (1.1)–(1.4) in the case of $H = 0$ and

$$f(u, u_t) = |u_t|^{\alpha-1} u_t, \quad 0 < \alpha < 1. \tag{1.5}$$

Such a problem governs the motion of a linear viscoelastic bar with nonlinear elastic constraints. In [4] we consider the existence, uniqueness and continuous dependence (with respect to the parameter h) of the solution to the problem (1.1)–(1.4) with

$$H(s) = hs, \quad h > 0. \tag{1.6}$$

In this paper, we consider two main parts. The first deals with the global existence and the uniqueness of the solution to the problem (1.1)–(1.4). Sometimes some hypotheses on f are abandoned comparing to [4]. The main tool is the Galerkin method associated with a nonlinear integral equation of Volterra type and the monotone operator generated by the nonlinear term $f(u, u_t)$. In the second part we consider the problem (1.1)–(1.4) with the linear boundary condition

$$u_x(0,t) = hu(0,t) + g(t), \quad h > 0, \tag{1.7}$$

instead of (1.2), afterwards, we study the behavior of the solution to such a problem as h tends to 0_+. In section 4 we present some numerical results.

2. The existence and uniqueness theorem

In this paper, we consider the equation (1.1) as an ordinary differential equation in the Banach space for $u(t)$ which stands for $u(x,t)$ so that we shall write

$$u' = u_t = \frac{\partial u}{\partial t}, u'' = u_{tt} = \frac{\partial^2 u}{\partial t^2}.$$

Put $\Omega = (0,1)$, $Q_T = \Omega \times (0,T), T > 0$, $L^q = L^q(\Omega)$, $H^1 = H^1(\Omega)$, where H^1 is the usual Sobolev space on Ω; denote by $\langle .,. \rangle$ the scalar product in L^2 or dual product of a continuous linear functional with an element of a corresponding function space, by $\|.\|$ the norm in L^2, by $\|.\|_X$ the norm in Banach space X and by X' dual of X. Denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$, the space of measurable real functions $f : (0,T) \to X$ such that

$$\|f\|_{L^p(0,T;X)} = \left(\int_0^T \|f(t)\|_X^p \, dt \right)^{1/p} < \infty \quad \text{for} \quad 1 \leq p < \infty$$

or

$$\|f\|_{L^\infty(0,T;X)} = \operatorname{ess}\sup_{0<t<T} \|f(t)\|_X \quad \text{for} \quad p = \infty.$$

Let

$$V = \{v \in H^1 / v(1) = 0\}, \tag{2.1}$$

$$a(u,v) = \left\langle \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x} \right\rangle = \int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} dx. \tag{2.2}$$

The following lemma is easy to prove.

LEMMA 2.1. *$\|v\|_{C^0(\bar{\Omega})} \leq \|v\|_V$ for all $v \in V$.*

We admit the following hypotheses:

(A) $u_0 \in H^1$, $u_1 \in L^2$,

(G) $g \in H^1(0,T)$ for all $T > 0$ and $g(0)$ exists,

(H) $H \in C^1(\mathbb{R})$, $H(0) = 0$, $sH(s) > 0$ for all $s \neq 0$,

$f : \mathbb{R}^2 \to \mathbb{R}$ satisfies the condition

(F_1) $f(0,0)=0, (f(u,v)-f(u,\tilde{v}))(v-\tilde{v}) \geq 0$ for all $u, v, \tilde{v} \in \mathbb{R}$,

and there exist constants $\alpha, \beta \in (0,1]$ and two continuous functions $B_1, B_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

(F_2) B_1 is non-decreasing function,

(F_3) $B_2(|v|) \in L^2(Q_T)$, for all $v \in L^2(Q_T)$ and $T > 0$,

(F_4) $|f(u,v) - f(u,\tilde{v}| \leq B_1(|u|)|v-\tilde{v}|^\alpha$ for all $u, v, \tilde{v} \in \mathbb{R}$,

(F_5) $|f(u,v) - f(\tilde{u},v)| \leq B_2(|v|)|u-\tilde{u}|^\beta$ for all $u, \tilde{u}, v \in \mathbb{R}$.

Then, we have the following theorem.

THEOREM 2.2. *Suppose that* (A), (G), (H), (F) *hold. Then, for* $T > 0$ *the boundary initial value problem* (1.1)–(1.4) *has at least a weak solution* u *on* $(0,T)$ *such that*

$$u \in L^\infty(0,T;V), \tag{2.3}$$

$$u_t \in L^\infty(0,T;L^2) \quad \textit{and} \quad u_t(0,t) \in L^2(0,T). \tag{2.4}$$

Furthermore, if $\beta = 1$ *in* (F_5) *and the function* H *satisfies, in addition,*

(H_1) $H \in C^2(\mathbb{R}), H'(s) > -1$ *for all* $s \in \mathbb{R}$,

then the problem (1.1)–(1.4) *has the only solution satisfying* (2.3), (2.4).

Remark. This result is stronger than that in [4]. Indeed,the following assumptions – made in [4] for the problem (1.1)–(1.4) with (1.6) – are not needed here:

$$\begin{cases} (i) & B_1(|u|) \in L^{2/(1-\alpha)}(Q_T) \text{ for all } u \in L^\infty(0,T;V) \text{ and } T > 0, \\ (ii) & B_2 \text{ is a non-decreasing function.} \end{cases} \tag{2.5}$$

Proof of Theorem 2.2.. The first step (**the Galerkin approximation**). Consider a special base in V

$$w_j(x) = \sqrt{\frac{2}{1+\lambda_j^2}} \cos(\lambda_j x), \quad \lambda_j = (2j-1)\frac{\pi}{2}, \quad j \in \mathbb{N},$$

constructed by the eigenfunctions of the Laplacian operator $\Delta = \dfrac{\partial^2}{\partial x^2}$.

Considering (1.1) as an ordinary differential equation with unknown function $u(t)$, put

$$u_n(t) = \sum_{j=1}^n c_{nj}(t) w_j, \tag{2.6}$$

where c_{nj} satisfy the following system of nonlinear differential equation

$$\begin{aligned} &\langle u_n''(t), w_j\rangle + a(u_n(t), w_j) + (H(u_n(0,t)) + g(t))w_j(0) \\ &\quad + \langle f(u_n(t), u_n'(t)), w_j\rangle = 0, \qquad 1 \leq j \leq n, \end{aligned} \tag{2.7}$$

with conditions

$$(2.8)\qquad \begin{cases} u_n(0) = u_{0n} \\ u_n'(0) = u_{1n}, \end{cases}$$

where

$$(2.9)\qquad \begin{cases} u_{0n} = \sum_{j=1}^{n} \alpha_{nj} w_j \longrightarrow u_0 & \text{in } H^1 \text{ strongly,} \\ u_{1n} = \sum_{j=1}^{n} \beta_{nj} w_j \longrightarrow u_1 & \text{in } L^2 \text{ strongly.} \end{cases}$$

For fixed $T > 0$, by the hypotheses of the theorem, the problem (2.7), (2.8) has the solution $u_n(t)$ on an interval $[0, T_n]$. Owing to the estimations which will follow, we can take $T_n = T$ for every n.

The second step (**the estimations a priori**). Multiplying the *j-th* equation in (2.7) by $c_{nj}'(t)$, summing with respect to j, then integrating with respect to time variable from 0 to t, by (G), (F_1), we have

$$(2.10)\quad S_n(t) \leq S_n(0) + 2g(0)u_{0n}(0) - 2g(t)u_n(0,t) + 2\int_0^t g'(\tau)u_n(0,\tau)d\tau - 2\int_0^t < f(u_n(\tau),0), u_n'(\tau) > d\tau,$$

where

$$(2.11)\qquad S_n(t) = \|u_n'(t)\|^2 + \|u_n(t)\|_V^2 + 2\widehat{H}(u_n(0,t))$$

with

$$(2.12)\qquad \widehat{H}(\eta) = \int_0^\eta H(s)ds.$$

Remark that the function (2.12) has the property

$$(2.13)\qquad 0 < \widehat{H}(\eta) \leq |\eta| \max_{|s| \neq |\eta|} |H(s)| \quad \text{for all} \quad \eta \neq 0.$$

Then, using (2.9), (2.11), (2.13) and Lemma 2.1, we have

$$(2.14)\qquad S_n(0) + 2|g(0)u_{0n}| \leq \tfrac{1}{2}C_1 \quad \text{for all} \quad n,$$

where C_1 is a constant depending on u_0, u_1 and H. We still use Lemma 2.1, then from (2.10), (2.14) and (F_5) we have

$$(2.15)\qquad S_n(t) \leq \tilde{g}(t) + \int_0^t \widetilde{K}(S_n(\tau))d\tau,$$

where

$$(2.16)\qquad \tilde{g}(t) = C_1 + 4g^2(t) + \int_0^t g'^2(\tau)d\tau,$$

$$(2.17)\qquad \widetilde{K}(s) = s + 4B_2(0)s^{(1+\beta)/2}.$$

Because $H^1(0,T) \hookrightarrow C^0([0,T])$, the function $\tilde{g}(t)$ is bounded almost everywhere on $[0,T]$ by a constant M_T depending on T. Then, from (2.15) we have

$$S_n(t) \leq M_T + \int_0^t \widetilde{K}(S_n(\tau))d\tau, \quad 0 \leq t < T_n = T. \tag{2.18}$$

The function $\widetilde{K}(s)$ is continuous and non-decreasing for $s \geq 0$, hence

$$S_n(t) \leq S(t), \quad \forall t \in [0,T], \quad \forall T > 0, \tag{2.19}$$

where $S(t)$ is the maximal solution of the Volterra integral equation

$$S(t) = M_T + \int_0^t \widetilde{K}(S(\tau))d\tau \tag{2.20}$$

with the kernel $\widetilde{K}$ non-decreasing on the interval $[0,T]$ (see [6]).

Now we need an estimation of the term $\int_0^t |u_n'(0,\tau)|^2 d\tau$. Put

$$K_n(t) = \sum_{j=1}^{n} \frac{\sin(\lambda_j t)}{\lambda_j}, \tag{2.21}$$

$$\begin{aligned} \gamma_n(t) = {} & \sum_{j=1}^{n} w_j(0)\left[\alpha_{nj}\cos(\lambda_j t) + \beta_{nj}\frac{\sin(\lambda_j t)}{\lambda_j}\right] \\ & - \sqrt{2}\sum_{j=1}^{n}\int_0^t \frac{\sin[\lambda_j(t-\tau)]}{\lambda_j}\left\langle f(u_n(\tau), u_n'(\tau)), \frac{w_j}{\|w_j\|}\right\rangle d\tau, \end{aligned} \tag{2.22}$$

$$\delta_n(t) = 2\int_0^t K_n(t-\tau)H(u_n(0,\tau))d\tau. \tag{2.23}$$

Then $u_n(0,t)$ can be rewritten as

$$u_n(0,t) = \gamma_n(t) - \delta_n(t) - 2\int_0^t K_n(t-\tau)g(\tau)d\tau. \tag{2.24}$$

To prove Theorem 2.2 we need the following lemmas.

LEMMA 2.3 (see [1]). *There exist a constant $C_2 > 0$ and such a function $D(t)$ continuous, positive and independent of n that*

$$\int_0^t |\gamma_n'(\tau)|^2 d\tau \leq C_2 + D(t)\int_0^t \|f(u_n(\tau), u_n'(\tau))\|^2 d\tau, \tag{2.25}$$

$$\forall t \in [0,T], \forall T > 0.$$

LEMMA 2.4 (see [4], Lemma 2). *There exists a constant $M_T^{(1)} > 0$ depending on T such that*

$$\int_0^t \left| \int_0^s K_n'(s-\tau)g(\tau)d\tau \right|^2 ds \leq M_T^{(1)}, \ \forall t \in [0,T], \quad \forall T > 0. \tag{2.26}$$

LEMMA 2.5. *There exist two positive constants $M_T^{(2)}$, $M_T^{(3)}$ such that*

$$\int_0^t |\delta_n'(\tau)|^2 d\tau \leq M_T^{(2)} + M_T^{(3)} \int_0^t ds \int_0^s |u_n'(0,\tau)|^2 d\tau, \quad \forall t \in [0,T], \ \forall T > 0. \tag{2.27}$$

Proof of Lemma 2.5. . From (2.23) we have

$$\delta_n'(t) = 2K_n(t)H(u_{0n}(0)) + 2\int_0^t K_n(t-\tau)H'(u_n(0,\tau))u_n'(0,\tau)d\tau. \tag{2.28}$$

Since $u_{0n} \to u_0$ strongly in H^1, we have

$$|u_{0n}(0)| \leq \|u_{0n}\|_{C^0(\bar\Omega)} \leq C_3, \quad \forall n. \tag{2.29}$$

Consequently

$$|H(u_{0n}(0))|^2 \leq \max_{|s|\leq C_3} H^2(s) = C_4, \quad \forall n. \tag{2.30}$$

Besides, from (2.19)) we have the inequalities

$$|u_n(0,\tau)| \leq \|u_n(\tau)\|_V \leq \sqrt{\max_{0\leq t\leq T} S(t)} = M_T^{(4)}, \quad \forall n, \tag{2.31}$$

$$|H'(u_{0n}(0,\tau))|^2 \leq \max_{|s|\leq M_T^{(4)}} |H'(s)|^2 = M_T^{(5)}, \quad \forall n, \tag{2.32}$$

where $M_T^{(4)}$, $M_T^{(5)}$ are constants depending on T. From (2.28), (2.30), (2.32) we obtain after some rearrangements

$$\int_0^t |\delta_n'(s)|^2 ds \leq 8C_4 \int_0^T K_n^2(s)ds + 8M_T^{(5)} \int_0^T K_n^2(\tau)d\tau \int_0^t ds \int_0^s |u_n'(0,\tau)|^2 d\tau, \tag{2.33}$$

and (2.27) follows since $K_n \to K$ in $L^2(0,T)$ strongly for every $T > 0$.

LEMMA 2.6. *There exists $M_T > 0$ such that*

$$\int_0^T |u_n'(0,\tau)|^2 d\tau \leq M_T, \quad \forall T > 0. \tag{2.34}$$

Proof of Lemma 2.6. From (2.24) we have

$$\int_0^t |u_n'(0,s)|^2 ds \le 3\int_0^t |\gamma_n'(s)|^2 ds + 3\int_0^t |\delta_n'(s)|^2 ds + 12\int_0^t ds \left| \int_0^s K_n'(s-\tau) g(\tau) d\tau \right|^2. \tag{2.35}$$

Besides, from the hypotheses $(F_2)-(F_5)$ and (2.19) we have

$$\|f(u_n(t), u_n'(t))\|^2 \le 2B_1^2(\sqrt{S(t)})S^\alpha(t) + 2B_2^2(0)S^\beta(t). \tag{2.36}$$

At last from Lemmas 2.3–2.5 and (2.35), (2.36) we obtain the inequality

$$\int_0^t |u_n'(0,s)|^2 ds \le M_T^{(6)} + 3M_T^{(3)} \int_0^t ds \int_0^s |u_n'(0,\tau)|^2 d\tau \tag{2.37}$$

which implies (2.34), by Gronwall's lemma.

The third step (**passing to limit**). Owing to (2.19), (2.36) and (2.34), $\{u_n\}$ has a subsequence still denoted $\{u_n\}$ such that

$$u_n \to u \quad \text{in} \quad L^\infty(0,T;V) \quad \text{weak}^*, \tag{2.38}$$

$$u_n' \to u' \quad \text{in} \quad L^\infty(0,T;L^2) \quad \text{weak}^*, \tag{2.39}$$

$$u_n(0,t) \to u(0,t) \quad \text{in} \quad L^\infty(0,T) \quad \text{weak}^*, \tag{2.40}$$

$$u_n'(0,t) \to u'(0,t) \quad \text{in} \quad L^2(0,T) \quad \text{weak}, \tag{2.41}$$

$$f(u_n, u_n') \to X \quad \text{in} \quad L^\infty(0,T;L^2) \quad \text{weak}^*. \tag{2.42}$$

Owing to (2.19) and (2.34), and besides, by (2.38) and (2.39), we can extract from $\{u_n\}$ a subsequence still denoted $\{u_n\}$ such that (see [5])

$$u_n(0,t) \to u(0,t) \quad \text{uniformly in} \quad C^\circ([0,T]), \tag{2.43}$$

$$u_n \to u \quad \text{strongly in} \quad L^2(Q_T). \tag{2.44}$$

Since H is continuous, from (2.43) we have

$$H(u_n(0,t)) \to H(u(0,t)) \quad \text{uniformly in} \quad C^\circ([0,T]). \tag{2.45}$$

Owing to (2.38), (2.39), (2.42), (2.45) and passing to limit in (2.7) , we have $u(t)$ satisfying the equation

$$\frac{d}{dt}\langle u'(t), v\rangle = a(u(t), v) + (H(u(0,t)) + g(t))v(0) + \langle X(t), v\rangle = 0, \quad \forall v \in V. \tag{2.46}$$

We can prove in a similar manner as in [4] that

$$u(0) = u_0, \quad u'(0) = u_1. \tag{2.47}$$

To prove the existence of solution u, we have to show that $X = f(u, u')$. Then we need the following lemma (see [1]) .

LEMMA 2.7. *Suppose that u is a solution of the problem*

$$u_{tt} - \Delta u + X = 0, \tag{2.48}$$

$$u_x(0,t) = H(u(0,t)) + g(t), \quad u(1,t) = 0, \tag{2.49}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \tag{2.50}$$

with

$$u \in L^\infty(0,T;V) \quad and \quad u_t \in L^\infty(0,T;L^2). \tag{2.51}$$

Then we have

$$\frac{1}{2}\|u'(t)\| + \frac{1}{2}\|u(t)\|_V^2 + \int_0^t [H(u(0,\tau)) + g(\tau)]\, u_t(0,\tau)d\tau + \int_0^t \langle X(\tau), u'(\tau)\rangle d\tau \geq \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|u_0\|_V^2 \quad \text{for a.e. } t \in (0,T). \tag{2.52}$$

Remark. If $u_0 = u_1 = 0$, then the equality occurs in (2.52). Now, from (2.7), (2.8) we have

$$\int_0^t \langle f(u_n(\tau), u_n'(\tau)), u_n'(\tau)\rangle d\tau = \frac{1}{2}\|u_{1n}\|^2 + \frac{1}{2}\|u_{0n}\|_V^2 - \frac{1}{2}\|u_n'(t)\|^2 - \frac{1}{2}\|u_n(t)\|_V^2 - \int_0^t [H(u_n(0,\tau)) + g(\tau)]\, u_n'(0,\tau)d\tau. \tag{2.53}$$

Passing to limit and then using (2.52), we get

$$\lim_{n\to\infty} \sup \int_0^t \langle f(u_n(\tau), u_n'(\tau)), u_n'(\tau)\rangle d\tau \leq \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\|u_0\|_V^2 - \frac{1}{2}\|u'(t)\|^2 - \frac{1}{2}\|u(t)\|_V^2 - \int_0^t [H(u(0,\tau)) + g(\tau)]\, u_t(0,\tau)d\tau \leq \int_0^t \langle X(\tau), u'(\tau)\rangle d\tau \quad a.e. \quad t \in (0,T). \tag{2.54}$$

By using the same arguments as in [4], we can show that $X = f(u, u')$ a.e. in Q_T. The existence of the solution to the problem (1.1)–(1.4) is proved.

The fourth step (**the uniqueness**). Having now $\beta = 1$ in (F_5) and H satisfying (H_1), suppose that u, v are two solutions of the boundary-initial value problem (1.1)–(1.4) satisfying $u, v \in L^\infty(0,T;V)$, $u_t, v_t \in L^\infty(0,T;L^2)$, $u_t(0,t), v_t(0,t) \in L^2(0,T)$. Then $w = u - v$ satisfies the problem

$$(2.55)\qquad \begin{cases} w_{tt} - \Delta w + X_1 = 0, \\ w_x(0,t) = H_1(t), \quad w(1,t) = 0, \\ w(x,0) = w_t(x,0) = 0, \\ w \in L^\infty(0,T;V),\ w_t \in L^\infty(0,T;L^2),\ w_t(0,t) \in L^2(0,T) \end{cases}$$

with

$$(2.56)\qquad \begin{cases} X_1 = f(u,u_t) - f(v,v_t), \\ H_1 = H(u(0,t)) - H(v(0,t)). \end{cases}$$

Applying Lemma 2.7 (see Remark), we have

$$(2.57)\qquad \frac{1}{2}\|w'(t)\|^2 + \frac{1}{2}\|w(t)\|_V^2 + \int_0^t H_1(\tau)w'(0,\tau)d\tau + \int_0^t \langle X_1(\tau), w'(\tau)\rangle d\tau = 0 \quad \text{for a.e. } t \in (0,T).$$

Put

$$(2.58)\qquad \sigma(t) = \|w'(t)\|^2 + \|w(t)\|_V^2.$$

Since the function f is monotonically increasing with respect to the second variable, we obtain

$$(2.59)\qquad -\int_0^t \langle X_1(\tau), w'(\tau)\rangle d\tau \le -\int_0^t \langle f(u(\tau),v'(\tau)) - f(v(\tau),v'(\tau)), w'(\tau)\rangle d\tau$$
$$\le -\int_0^t \|f(u(\tau),v'(\tau)) - f(v(\tau),v'(\tau))\|.\|w'(\tau)\|d\tau.$$

Using the hypothesis (F_5) with $\beta = 1$, we have

$$(2.60)\qquad \|f(u,v') - f(v,v')\| \le \|B_2(|v'|)\|.\|w\|_V.$$

Consequently, from (2.57)–(2.60) we get

$$(2.61)\qquad \sigma(t) + 2\int_0^t H_1(\tau)w'(0,\tau)d\tau \le \int_0^t \|B_2(|v'(\tau)|)\|\sigma(\tau)d\tau.$$

Put

$$(2.62)\qquad \begin{cases} M = \max\{\|u\|_{L^\infty(0,T;V)},\ \|v\|_{L^\infty(0,T;V)}\}, \\ m_1 = \min\limits_{|s|\le M} H'(s), \\ m_2 = \max\limits_{|s|\le M} |H''(s)|. \end{cases}$$

From the hypothesis (H_1) we have

(2.63) $$m_1 > -1.$$

From (2.56) we obtain

(2.64)
$$\begin{aligned}
&2\int_0^t H_1(\tau)w'(0,\tau)d\tau \\
&= 2\int_0^t \left[\int_0^1 \frac{d}{d\theta}H(v(0,\tau)+\theta w(0,\tau))d\theta\right] w_t(0,\tau)d\tau \\
&= w^2(0,\tau)\int_0^1 H'(v(0,t)+\theta w(0,t))d\theta \\
&\quad - \int_0^t w^2(0,\tau)d\tau \int_0^1 H''(v(0,\tau)+\theta w(0,\tau))(v_t(0,\tau)+\theta w_t(0,\tau))d\theta \\
&\geq m_1 w^2(0,t) - m_2\int_0^t w^2(0,\tau)(|u_t(0,\tau)|+|v_t(0,\tau)|)d\tau,
\end{aligned}$$

by using integration by parts and (2.62). From (2.61), (2.64) we have

(2.65)
$$\begin{aligned}
\sigma(t) + m_1 w^2(0,t) &\leq m_2\int_0^t w^2(0,\tau)(|u_t(0,\tau)|+|v_t(0,\tau)|)d\tau \\
&\quad + \int_0^t \|B_2(|v'(\tau)|)\|\sigma(\tau)d\tau.
\end{aligned}$$

We remark that, by (2.58),

(2.66) $$w^2(0,t) \leq \|w(t)\|_V^2 \leq \sigma(t).$$

Consequently

(2.67) $$(1+m_1)w^2(0,t) \leq \sigma(t) + m_1 w^2(0,t).$$

Multiplying two members of (2.67) by a number $k > 0$ and adding to (2.65), we have

(2.68)
$$\begin{aligned}
&\sigma(t) + [m_1 + k(1+m_1)]w^2(0,t) \\
&\leq (1+k)m_2\int_0^t w^2(0,\tau)(|u_t(0,\tau)|+|v_t(0,\tau)|)d\tau \\
&\quad + (1+k)\int_0^t \|B_2(|v'(\tau)|)\|\sigma(\tau)d\tau.
\end{aligned}$$

Choose

$$k > \max\left\{0, \frac{1-m_1}{1+m_1}\right\} \tag{2.69}$$

and denote

$$q(t) = (1+k)\left[m_2(|u_t(0,t)| + |v_t(0,t)|) + \|B_2(|v_t(t)|)\|\right]. \tag{2.70}$$

Then from (2.68)–(2.70) we have

$$\sigma(t) + w^2(0,t) \leq \int_0^t q(\tau)[\sigma(\tau) + w^2(0,\tau)]d\tau. \tag{2.71}$$

This implies $\sigma(t) + w^2(0,t) \equiv 0$ and $w \equiv 0$ by the definition (2.58) of $\sigma(t)$. This ends the proof of Theorem 2.2.

A special case of H gives us the following result which is stronger than that in [4] about the global existence, the hypotheses on the function f being less.

THEOREM 2.8. *Suppose that* (A), (G), (F), *(1.6) hold. Then for every* $T > 0$ *the problem* (1.1)–(1.4) *has at least a weak solution* u *on* $(0,T)$ *satisfying* (2.3), (2.4). *Furthermore, if* $\beta = 1$, *then this solution is unique.*

The following theorem is a generalization of the result in [1].

THEOREM 2.9. *Suppose that* (A), (G), (H), (1.5) *hold. Then for every* $T > 0$ *the problem* (1.1)–(1.4) *has at least a weak solution* u *on* $(0,T)$ *satisfying* (2.3), (2.4). *Furthermore, if* H *satisfies* (H_1) , *then this solution is unique.*

3. The continuous dependence of solution

In this section we study the problem (1.1)–(1.4) with linear boundary condition

$$u_x(0,t) = hu(0,t) + g(t), \quad h > 0, \tag{3.1}$$

(replacing (1.3)), and we also study the behavior of solution as $h \to 0_+$. Suppose that, in addition, $\beta = 1$ and (1.6) holds. Such a problem, by Theorem 2.8, has the only solution

$$u(x,t) = u_h(x,t), \quad h > 0. \tag{3.2}$$

This result generalizes those of [1], [4]. At first, for every $T > 0$ we put

$$W(Q_T) = \left\{v \in L^\infty(0,T;V)/v_t \in L^\infty(0,T;L^2)\right\}. \tag{3.3}$$

$W(Q_T)$ is Banach space with respect to norm (see [5])

$$\|v\|_{W(Q_T)} = \left(\|v\|^2_{L^\infty(0,T;V)} + \|v_t\|^2_{L^\infty(0,T;L^2)}\right)^{1/2}. \tag{3.4}$$

Further, we want to prove that the family of solutions $\{u_h\}_{h>0}$ converges strongly in $W(Q_T)$ to a function $\tilde{u}$ which is the only solution of the problem (1.1)–(1.4) with $H \equiv 0$. We admit following hypothesis on the function B_2 occuring in (F)

(F_6) $B_2 : L^2 \to L^2$ maps bounded sets into bounded sets.

THEOREM 3.1. *Suppose that* $(A), (G), (H_2), F, (F_6)$ *hold and that* $\beta = 1$. *Then*

(i) *for every* $T > 0$ *the problem* (1.1)–(1.4) *with* $H \equiv 0$ *has the only solution* $\tilde{u}(x,t)$ *satisfying*

$$\tilde{u} \in L^\infty(0,T;V), \quad \tilde{u}_t \in L^\infty(0,T;L^2), \quad \tilde{u}_t(0,t) \in L^2(0,T), \tag{3.5}$$

(ii) *the solution* $u_h(x,t)$ *converges strongly in* $W(Q_T)$ *to* $\tilde{u}(x,t)$ *as* $h \to 0_+$. *Furthermore, we have an estimation*

$$\|u_h - \tilde{u}\|_{W(Q_T)} \le C_T h \quad \text{for } h > 0 \quad \text{small.} \tag{3.6}$$

Proof. (i) Case $H \equiv 0$. We prove in the same manner as in the case of Theorem 2.2 that the following estimations a priori hold

$$\|u_n'(t)\|^2 + \|u_n(t)\|_V^2 \le M_T, \quad \forall t \in [0,T], \quad \forall T > 0, \tag{3.7}$$

$$\int_0^T |u_n'(0,t)|^2 dt \le M_T, \quad \forall t \in [0,T], \quad \forall T > 0, \tag{3.8}$$

and that the limit $\tilde{u}$ of the sequence $\{u_h\}$ defined by (2.6), (2.7), (2.8) satisfies (3.5) and the equation (1.1) with the boundary conditions

$$\tilde{u}_x(0,t) = g(t), \quad \tilde{u}(1,t) = 0 \tag{3.9}$$

and the initial conditions

$$\tilde{u}(x,0) = u_0(x), \quad \tilde{u}_t(x,0) = u_1(x). \tag{3.10}$$

Besides, this solution $\tilde{u}$ is unique.

(ii) Consider $h_0 > 0$ fixed, and two parameters $h, h' \in (0, h_0)$. Let u_h and $u_{h'}$ be solutions of the boundary-initial value problem (1.1)–(1.4) corresponding to the parameters h and h', respectively. Then $Z = u_h - u_{h'}$ satisfies

$$\begin{cases} Z_{tt} - \Delta Z + \tilde{X}_1 = 0, \\ Z_x(0,t) = hZ(0,t) + \tilde{h} u_{h'}(0,t), \\ Z(1,t) = 0, \\ Z(x,0) = Z_t(x,0) = 0, \\ Z \in L^\infty(0,T;V), Z_t \in L^\infty(0,T;L^2), Z_t(0,t) \in L^2(0,T), \end{cases} \tag{3.11}$$

where

$$\tilde{X}_1 = f(u_h, u_h') - f(u_{h'}, u_{h'}'), \quad \tilde{h} = h - h'. \tag{3.12}$$

Proving in the same manner as in the case of Theorem 2.2 with $H(s) = hs$, $0 < h < h_0$, we have the following result:

$$(3.13)\qquad \begin{cases} \text{the sequences } \{u_h\}, \{u'_h\} \text{ and } \{u'_h(0,t\} \text{ are bounded in} \\ L^\infty(0,T;V), L^\infty(0,T;L^2) \text{ and } L^2(0,T), \text{ respectively,} \\ \text{by a constant independent of } h \text{ (depending on } T, h_0). \end{cases}$$

Put

$$(3.14)\qquad \sigma_1(t) = \|Z'(t)\|^2 + \|Z(t)\|_V^2.$$

Using again Lemma 2.7, we have

$$(3.15)\quad \sigma_1(t) + hZ^2(0,t) \leq 2|\tilde{h}|\widetilde{M}_T\|Z\|_{L^\infty(0,T;V)} + \int_0^t \|B_2(|u'_{h'}(\tau)|)\|\sigma_1(\tau)d\tau,$$

where $\widetilde{M}_T$ is a constant independent of h, h' and satisfying

$$(3.16)\qquad \|u_{h'}\|_{L^\infty(0,T;V)} + \int_0^T |u'_{h'}(0,\tau)|d\tau \leq \widetilde{M}_T.$$

Using the hypothesis (F_6) and (3.13), we have

$$(3.17)\qquad \|B_2(|u'_{h'}(\tau)|)\| \leq C_T^{(1)},$$

where $C_T^{(1)}$ is a constant depending only on T. Afterwards, owing to (3.15)–(3.17) and the Gronwall's lemma, we obtain

$$(3.18)\qquad \sigma_1(t) \leq 2|\tilde{h}|\widetilde{M}_T\|Z\|_{L^\infty(0,T;V)}exp(TC_T^{(1)}), \quad \forall t \in [0,T].$$

This implies

$$(3.19)\qquad \|u_h - u_{h'}\|_{W(Q_T)} \leq C_T|h - h'|, \quad \forall h, h' \in (0, h_0).$$

Suppose that $\{h_n\}$ is a sequence of real numbers such that $h_n > 0$ and $h_n \to 0_+$, as $n \to \infty$. From (3.19) we deduce that $\{u_{h_n}\}$ is a Cauchy sequence in $W(Q_T)$. Consequently there exists $u^* \in W(Q_T)$ such that

$$(3.20)\qquad u_{h_n} \to u^* \text{ strongly in } W(Q_T) \text{ as } n \to \infty.$$

By passing to limit similarly as in the proof of Theorem 2.2, we conclude that u^* satisfies the equation

$$(3.21)\quad \frac{d}{dt}\langle u_t^*, v\rangle + a(u^*(t), v) + g(t)v(0) + \langle f(u^*, u_t^*), v\rangle = 0,$$
$$\forall v \in V \quad \text{and for a.e. } t \in (0,T),$$

and the initial conditions

$$(3.22)\qquad u^*(0) = u_0, \quad u_t^*(0) = u_1.$$

From the uniqueness of solution we have

$$u^* = \tilde{u}. \tag{3.23}$$

Consequenlty (3.20) is correct for every sequence of positive numbers $\{h_n\}$ such that $h_n \to 0_+$ which implies that

$$u_h \to \tilde{u} \text{ strongly in } W(Q_T), \text{as } h \to 0_+. \tag{3.24}$$

Furthermore, if in (3.19) we suppose that $h' \to 0_+$, we have (3.6), and the theorem is proved completely.

4. Numerical results

Consider the problem

$$u_{tt} - \Delta u + f(u, u_t) = F(x,t), \quad 0 < x < 1, \tag{4.1}$$

$$u_x(0,t) = \tfrac{1}{4}u(0,t) + g(t), u(1,t) = 0, \tag{4.2}$$

$$u(x,0) = \cos \tfrac{\pi}{2}x, u_t(x,0) = -\cos \tfrac{\pi}{2}x, \tag{4.3}$$

where $g(t) = -\frac{1}{4}e^{-t}(9t^2+1), f(u,u_t) = |u_t|^{1/2} sgn(u_t)$.

To solve numerically the problem (4.1)–(4.3) we consider the system of non-linear differential equations

$$\begin{cases} \dfrac{du_k}{dt} = v_k, \\ \dfrac{dv_k}{dt} = \dfrac{u_{k+1} + u_{k-1} - 2u_k}{h^2} - f(u_k, v_k) + F(x_k, t), \end{cases} \tag{4.4}$$

with initial conditions

$$u_k(0) = \cos \tfrac{\pi}{2}kh, v_k(0) = -\cos \tfrac{\pi}{2}kh, \quad k = 0, 1, \ldots, N-1, \tag{4.5}$$

where $u_k(t) = u(x_k,t)$, $v_k(t) = \frac{du}{dt}(x_k,t)$, $x_k = kh$, $h = 1/N$ are unknown.

To solve (4.4) at the time $t = t_0 + \Delta t$ we use the following linear recursive scheme generated by the non-linear term

$$\begin{cases} \dfrac{du_k^{(n)}}{dt} = v_k^{(n)}, \\ \dfrac{dv_k^{(n)}}{dt} = \dfrac{u_{k+1}^{(n)} + u_{k-1}^{(n)} - 2u_k^{(n)}}{h^2} - f(u_k^{(n-1)}, v_k^{(n-1)}) + F(x_k, t), \\ u_k^{(n)}(t_0) = \alpha_{k,t_0}, \quad v_k^{(n)}(t_0) = \beta_{k,t_0}, \quad k = 0, 1, \ldots, N-1. \end{cases} \tag{4.6}$$

The linear system (4.6) is solved by searching its eigenvalues and eigenfunctions. We study the problem (4.1)–(4.3) in the two following cases with respect to the function $F(x,t)$:

(i) $F(x,t) = e^{-t}\{\cos \frac{\pi}{2}x + (x-1)^2(t^2 - 4t + 2) + \frac{\pi^2}{4}\cos \frac{\pi}{2}x - 2t^2\} + |u_t|^{1/2}\text{sgn} u_t$, with $u_t = e^{-t}[-\cos \frac{\pi}{2}x - t^2(x-1)^2 + 2t(x-1)^2]$. For such a

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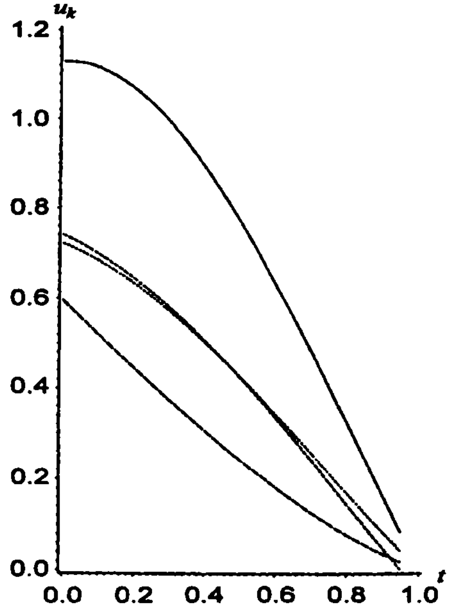

(1) continue line: appr. sol., $T = 0,05$; (2) middle dashes: appr. sol., $T = 0,5$;
(3) short dashes: exact sol., $T = 0,5$; (4) long dashes: appr. sol., $T = 1$.

Fig. 1

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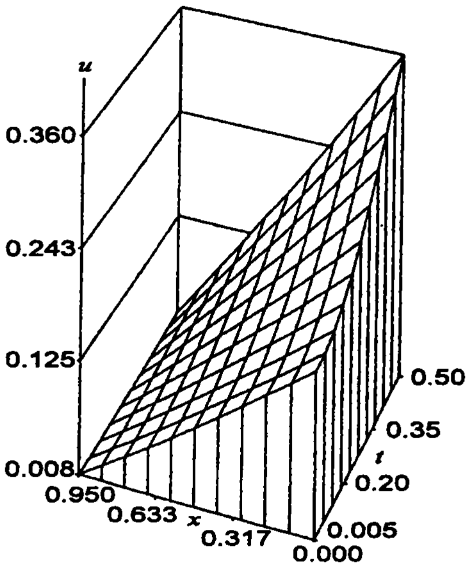

Fig. 2

function $F(x,t)$ the exact solution of the problem (4.1)–(4.3) is $u(x,t) = e^{-t}[cos\frac{\pi}{2}x + t^2(x-1)^2]$. With a step $h = 1/40$ we obtain the curves in Fig. 1 for the approximated solution $u_k(t)$ and the exact solution for $k = 0, 1, \ldots, 39$ and for the times $T = 1/20, 1/2, 1$. The corresponding error is equal to $\frac{10^{-3}}{2}$.

(ii) $F(x,t) = 0$. Always with a step $h = 1/40$ on the interval $[0,1]$ and for $T \in [0; 0,7]$ we have drawn the corresponding surface solution: $(x,t) \to u(x,t)$ in Fig. 2.

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