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Roger Yue Chi Ming

A NOTE ON YJ-INJECTIVITY

In [11], p-injective modules are introduced to study von Neumann regular rings and associated rings. In [13], this concept is weakened to YJ-injectivity. Recall that

(1) A left A-module M is p-injective if, for any principal left ideal P of A, every left A-homomorphism of P into M extends to A;

(2) ${}_AM$ is YJ-injective if, for any $0 \neq a \in A$, there exist a positive integer n (depending on a) such that $a^n \neq 0$ and every left A-homomorphism of Aa^n into M extends to A.

A is called left YJ-injective if ${}_AA$ is YJ-injective. P-injectivity and YJ-injectivity are similarly defined on the right side. We may note the following results on YJ-injectivity [I3] : (a) A is left YJ-injective iff for any $0 \neq a \in A$, there exist a positive integer n such that a^nA is a non-zero right annihilator; (b) If A is left YJ-injective, the Jacobson radical coincides with the left singular ideal of A; (c) A is quasi-Frobeniusean iff A is right Artinian, left and right YJ-injective.

Throughout, A denotes an associative ring with identity and A-modules are unital. J, Z will stand respectively for the Jacobson radical and the left singular ideal of A. A left (right) ideal of A is called reduced if it contains no non-zero nilpotent element.

An ideal of A will always mean a two-sided ideal. Following E. H. FELLER, A is called left duo if every left ideal of A is an ideal. A is called left quasi-duo if every maximal left ideal of A is an ideal (S. H. BROWN).

We write «A is MELT» if any maximal essential left ideal of A (if it exists) is an ideal of A. MELT rings generalize effectively left quasi-duo rings and semi-simple Artinian rings.

In this note, we consider various rings whose simple one-sided modules are YJ-injective. This is motivated by a well-known theorem of I. Kaplansky: A commutative ring A is von Neumann regular iff every simple A module is injective.

Injectivity and projectivity, together with various generalizations (including p-injectivity) are considered in [9] . The bibliography of [9] shows that these concepts have drawn the attention of numerous authors (cf. also [2], [3], [6], [8], [10], [15]).

For completeness, we recall the following definitions :

(1) A is von Neumann regular if, for every $a \in A$, there exist $b \in A$ such that $a = aba$.

(2) A left A-module M is flat if, given any monomorphism $N \to Q$ of right A-modules N, Q, the induced homomorphism $N \otimes M \to Q \otimes M$ is also monomorphism.

(3) A is fully left idempotent (resp. fully idempotent) if every left ideal (resp. every ideal) of A is idempotent.

(4) A is semi-prime if A contains no non-zero nilpotent ideal.

(5) Given a left A-module M, a left submodule N, N is called on essential submodule of M (or M is an essential extension of N) if, for any non-zero left submodule Q of M, $Q \cap N \neq 0$; N is called a complement left submodule of M if N has no proper essential extension in M.

If I is a left ideal of A, I is called a complement left ideal if I is a complement left submodule of A.

(6) ${}_AQ$ is injective if, given any left A-monomorphism $g : M \to N$, any left A-homomorphism $f : M \to Q$, there exist a left A-homomorphism $h : N \to Q$ such that $hg = f$;

(7) A is a $P.\ I.$ ring if A satisfies a polynomial identity with coefficients in the centroid and at least one coefficient is invertible.

(8) A is a left MI-ring if A contains an injective maximal left ideal.

(9) A is quasi-Frobeniusean if A is left or right Artinian and left or right self-injective.

(10) A is pseudo-Frobeniusean if ${}_AA$ is an injective cogenerator in the category of left A-modules.

It is well-known that A is von Neumann regular iff every left (right) A-module is flat (iff every left (right) A-module is p-injective). The concepts of complement submodules, essential extensions, quasi-Frobeniusean and pseudo-Frobeniusean rings have been extensively studied for many years (cf. for exemple [5] and [9]). Also note that a well-knoun result of E. P. Armendariz – J. W. Fisher asserts that a P.I.-ring is von Neumann regular iff it is fully idempotent.

LEMMA 1. *Let A be a reduced ring, M a maximal left ideal of A which is an ideal of A. If ${}_AA/M$ is YJ-injective, then A/M_A is flat.*

Proof. Let $0 \neq m \in M$. If $Am = Mm$, then $m = dm$ for some $d \in M$. Suppose now that $Am \neq Mm$. The map $a + M \to am + Mm$ of A/M into

Am/Mm implies that ${}_AA/M \approx {}_AAm/Mm$. Since ${}_AA/M$ is YJ-injective, then so is ${}_AAm/Mm$. For any $0 \neq a \in A$, there exist a positive integer n such that $a^n \neq 0$ and any left A-homomorphism of Aa^n into Am/Mm extends to A. Since $m \in A$, there exist a positive integer t such that $m^t \neq 0$ and any left A-homomorphism of Am^t into Am/Mm extends to A. If $Am^t = Mm^t$, then $m^t = um^t$, $u \in M$, which implies that $1-u \in l(m^t) = l(m)$ (because A is reduced). Then $m = um$ which contradicts $Am \neq Mm$. This proves that $Am^t \neq Mm^t$ and if $h : Am/Mm \to Am^t/Mm^t$, is the map $am + Mm \to am^t + Mm^t$ $(a \in A)$, then h is an isomorphism.

Consequently, given the projection $g : Am^t \to Am^t/Mm^t$, since $h^{-1}g : Am^t \to Am/Mm$ is a left A-homomorphism, there exist $d \in A$ such that $h^{-1}g(m^t) = m^t dm + Mm$. But $h^{-1}g(m^t) = h^{-1}(m^t + Mm^t) = m + Mm$ which yields $m \in Mm$. In that case, $Am = Mm$ which contradicts our hypothesis. Thus $s \in Ms$ for every $s \in M$ which proves that A/M_A is flat by [7, Theorem 3.57].

THEOREM 2. *The following conditions are equivalent:*

(1) *A is strongly regular;*

(2) *A is a reduced MELT ring whose simple left modules are YJ-injective;*

(3) *A is a reduced MELT ring whose simple right modules are YJ-injective or flat;*

(4) *A is a left quasi-duo ring whose simple left modules are YJ-injective;*

(5) *A is a left quasi-duo ring such that $l(b) \subseteq r(b)$ for every nilpotent element b of A and every simple right A-module is either YJ-injective or flat.*

Proof. It is easily seen that (1) implies (2) and (3).

Assume (2). Let M be a maximal left ideal of A. If ${}_AA = {}_AM \oplus {}_AU$, then M is generated by a central idempotent (because A is reduced). If ${}_AM$ is essential in ${}_AA$, then M is an ideal of A by hypothesis. Therefore A is left quasi-duo and (2) implies (4).

Similarly, (3) implies (5).

Assume (4). Suppose there exist $0 \neq b \in A$ such that $b^2 = 0$. If $Ab + l(b) \neq A$, let M be a maximal left ideal containing $Ab + l(b)$. Since ${}_AA/M$ is YJ-injective and $b^2 = 0$, if $f : Ab \to A/M$ is the left A-homomorphism $ab \to a + M$, there exist $y \in A$ such that $f(b) = by + M$. Now $l + M = by + M$ implies that $1 \in M$ (because $by \in M$). This contradicts $M \neq A$ and proves that $Ab + l(b) = A$. If $1 = ab + d$, $a \in A$, $d \in l(b)$, then $b = ab^2 + db = 0$ which contradicts our first hypothesis. We have proved that A must be reduced.

For any maximal left ideal N of A, since ${}_AA/N$ is YJ-injective, by Lemma 1, A/N_A is flat, whence ${}_AA/N$ is injective [12, Lemma 1]. A is therefore a left quasi-duo left V-ring which is strongly regular by [1, Theorem 3.1]. Therefore (4) implies (1).

Assume (5). Again suppose there exist $0 \neq b \in A$ such that $b^2 = 0$. Then $l(b) \subseteq r(b) \neq A$ and if M is a maximal right ideal containing $r(b)$, A/M_A is either YJ-injective or flat. If A/M_A is YJ-injective, define a right A-homomorphism $f : bA \to A/M$ by $f(ba) = a + M$ for all $a \in A$. Then there exist $c \in A$ such that $f(b) = cb + M$. Since $1 + M = cb + M$, $1 \in M$ (because $cb \in l(b) \subseteq r(b) \subseteq M$) which contradicts $M \neq A$. If A/M_A is flat, then $b \in M$ implies that $b = db$ for some $d \in M$ [7, Theorem 3.57]. Now $1 - d \in l(b) \subseteq r(b) \subseteq M$ implies that $1 \in M$, again a contradiction! This proves that A is reduced. Now suppose there exist $u \in A$ such that $Au + l(u) \neq A$. Let R be a maximal left ideal containing $Au + l(u)$. Then R is a maximal right ideal and if A/R_A is YJ-injective, there exist a positive integer n such that $u^n \neq 0$ and if $g : u^n A \to A/R$ is the map $u^n a \to a + R$ for all $a \in A$, then g is a well-defined right A-homomorphism (because A is reduced). Now $g(u^n) = cu^n + R$ for some $c \in A$ which implies that $1 - cu^n \in R$, whence $1 \in R$, contradicting $R \neq A$. If A/R_A is flat, then $u = vu$ for some $v \in R$ [7, Theorem 3.57]. Therefore $1 - v \in l(u) \subseteq R$, which yields $1 \in R$, again a contradiction! This proves that $A = Aa + l(a)$ for all $a \in A$ and hence $a = za^2$ for every $a \in A$. Thus (5) implies (1).

Note that in condition (3) of Theorem 2, neither «reduced» nor «MELT» can be dropped.

In 1974, we proved that if every simple left A-module is p-injective, then A is fully left idempotent (cf. for example, [9, P. 340]).

QUESTION 1: Is A fully left idempotent if every simple left A-module is YJ-injective?

QUESTION 2: Are left quasi-duo rings whose simple right modules are YJ-injective strongly regular?

Rings whose simple right modules are either injective or projective need not be semi-prime (cf. for example [15]).

PROPOSITION 3. *Let A be a ring whose simple right modules are either YJ-injective or projective. If $AbA + (l(AbA) \cap r(AbA))$ is a complement left ideal for every $b \in A$, then A is biregular.*

Proof. Let $d \in A$ such that $(AdA)^2 = 0$. Then $l(AdA)$ is an essential right ideal of A. By hypothesis, $I = l(AdA) \cap r(AdA)$ is a complement left ideal of A. If $I \neq l(AdA)$, there exist a non-zero left subideal K of $l(AdA)$ such that $K \cap I = 0$. Now $AdAK \subseteq K \cap AdA \subseteq K \cap I = 0$ implies that $K \subseteq r(AdA)$, whence $K \subseteq I$. Therefore $K = K \cap I = 0$. This contradiction proves that $I = l(AdA)$, whence $l(AdA) \subseteq r(AdA)$. Therefore $r(AdA)$ is an essential right ideal of A which implies that $AdA \subseteq Y$, where Y is the right singular ideal of A.

Since every simple right A-module is either YJ-injective or projective, then $Y \cap J = 0$ by [14, Proposition 8]. Since $(AdA)^2 = 0$, $AdA \subseteq J$ which yields $AdA \subseteq J \cap Y = 0$. This proves that A is semi-prime. Now for every $b \in A$, $l(AbA) = r(AbA)$, $AbA \cap r(AbA) = 0$ and by hypothesis, $C = AbA \oplus r(AbA)$ is a complement left ideal of A. If $C \cap U = 0$ for some left ideal U of A, $AbAU \subseteq U \cap AbA \subseteq U \cap C = 0$ which implies that $U \subseteq r(AbA)$, whence $U \subseteq C$. Therefore $U = U \cap C = 0$ which proves that ${}_AC$ is essential in ${}_AA$. Since C is a complement left ideal, $C = A$ which implies that AbA is generated by an idempotent which is central (because A is semi-prime). This proves that A is biregular.

Call A a left MI-ring if A contains an injective maximal left ideal. Left MI-rings (even hereditary, Artinian) need not be left self-injective (cf. [15]).

Self-injective rings play an important role in ring theory. Well known examples are quasi-Frobeniusean rings, pseudo-Frobeniusean rings and the maximal quotient rings of non-singular rings.

PROPOSITION 4. *Let A be a left YJ-injective left MI-ring. Then A is left self-injective.*

Proof. $A = M \oplus U$, where M is an injective maximal left ideal, $M = Ae$, $e = e^2 \in A$, $U = Au$, $u = 1 - e$. First suppose that M is not an ideal of A. Then $MU \neq 0$ and if $w \in U$ such that $Mw \neq 0$, we have $Mw = U$. Let $g : M \to U$ be the map $m \to mw$ for all $m \in M$. Then $M/\ker g \approx U$ implies that $M = \ker g \oplus V$ (because ${}_AU$ is projective), where ${}_AU \approx {}_AV$ is injective. Therefore $A = M \oplus U$ is left self-injective. Now suppose that M is an ideal of A. $R = uA = r(M)$ and we claim that R is a minimal right ideal of A. Otherwise, there exist $0 \neq t \in R$ such that $tA \neq R$. By [13, Lemma 3], there exist a positive integer m such that t^mA is a non-zero right annihilator. Then $M = l(t^mA)$ implies that $R = r(M) = r(l(t^mA)) = t^mA$ (because t^mA is a right annihilator). This yields $tA = R$, a contradiction! This proves that R is a minimal right ideal of A and therefore $M_A \cap R_A = 0$. Thus $A_A = M_A \oplus R_A$ which implies that A/M_A is projective. By [12, Lemma 1], ${}_AA/M$ is injective which implies that ${}_AU$ is injective. Therefore $A = M \oplus U$ is again left self-injective.

COROLLARY 5. *If A is a left YJ-injective left MI-ring satisfying the maximum condition on left annihilators, then A is quasi-Frobeniusean.* (Apply [3, Theorem 24.20].)

Semi-prime left p-injective PI-rings need not be regular (cf. [2, Example 4.8]).

QUESTION 3: Is A left self-injective if A is a PI-ring which is left MI? (A is left self-injective regular if A is also semi-prime.)

QUESTION 4: Is A left self-injective regular if A is a left MI-ring whose simple left modules are YJ-injective? (The answer is «yes» if «YJ-injective» is replaced by «p-injective».) We add a remark.

Remark. A is quasi-Frobeniusean iff A is right p-injective right Noetherian left YJ-injective.

Note that if A is von Neumann regular, then every left(right) A module is YJ-injective. But we are unable to say whether the converse is true.

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UNIVERSITE PARIS VII – DENIS DIDEROT
UFR DE MATHEMATIQUES UMR 9994 CNRS
2, Place Jussieu,
75251 PARIS CEDEX 05, FRANCE

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