DEMONSTRATIO MATHEMATICA
Vol. XXX No 3 1997

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NOTE ON GENERALIZATION OF GODUNOVA–LEVIN–OPIAL INEQUALITY

1. Introduction

In 1967 E. K. Godunova and V. I. Levin [3] proved the following theorem:

THEOREM A. *Let u be real-valued absolutely continuous function defined on $[a,b]$ with $u(a)=0$. Let F be real-valued convex, increasing function on $[0,\infty)$ with $F(0)=0$. Then, the following integral inequality holds*

$$\int_a^b F'(|u(t)|)|u'(t)|dt \le F\left(\int_a^b |u'(t)|dt\right).$$

This theorem is a generalization of the well known Opial inequality [5]. The multidimensional generalization is given in 1993 by J. Pečarić [10]:

THEOREM B. *Let u_i, $i=1,\dots,n$ be real-valued absolutely continuous functions defined on $[a,b]$ with $u_i(a)=0$, $i=1,\dots,n$. Let F be nondecreasing function on $[0,\infty)^n$ with $F(0,\dots,0)=0$ such that all its first partial derivatives F_i', $i=1,\dots,n$ are nondecreasing functions. Then, the following integral inequality holds*

$$\int_a^b \left(\sum_{i=1}^n F_i'(|u_1(t)|,\dots,|u_n(t)|)|u_i'(t)|dt\right) \le F\left(\int_a^b |u_1'(t)|dt,\dots,\int_a^b |u_n'(t)|dt\right).$$

In [10] J.Pečarić also proved the following generalization of Theorem B :

THEOREM C. *Let F, F_i', u_i, $i=1,\dots,n$ be defined as in Theorem B. Let p_i, $i=1,..,n$ be real-valued positive functions defined on $[a,b]$ and $\int_a^b p_i(t)dt=1$, $i=1,\dots,n$. Let h_i, $i=1,\dots,n$ be real-valued positive convex and increasing functions on $(0,\infty)$. Then, the following integral inequality* *holds*

$$\int_a^b \Big(\sum_{i=1}^n F_i'\Big(|u_1(t)|,\dots,|u_n(t)|\Big)|u_i'(t)|\Big)dt$$

$$\le F\Big(h_1^{-1}\Big(\int_a^b p_1(t)h_1(\frac{|u_1'(t)|}{p_1(t)})dt\Big),\dots,h_n^{-1}\Big(\int_a^b p_n(t)h_n(\frac{|u_n'(t)|}{p_n(t)})dt\Big)\Big).$$

Some special cases of this theorem are proved in [7] and [8] by B. G. Pachpatte. For example, if we take $F(x_1,\dots,x_n) = \prod_{i=1}^n f_i(x_i)$ and $h_i = h$, for each $i = 1,\dots,n$, we obtain the most general of mentioned results in [7] and [8].

An interesting generalization of Theorem A is given in 1972 by G. I. Rozanova [11]:

THEOREM D. *Let u, F be defined as in Theorem A. Let φ be real-valued convex increasing function on $(0,\infty)$. Let $r(x) \ge 0$, $r'(x) > 0$, $r(a) = 0$. Then, the following integral inequality holds*

$$\int_a^b F'\Big(r(t)\varphi(\frac{|u(t)|}{r(t)})\Big)r'(t)\varphi(\frac{|u'(t)|}{r'(t)})dt \le F\Big(\int_a^b r'(t)\varphi(\frac{|u'(t)|}{r'(t)})dt\Big).$$

The aim of present note is to generalize above mentioned inequalities.

2. The main results

THEOREM 1. *Let u_i, $i = 1,\dots,n$ be real-valued absolutely continuous functions defined on $[a,b]$ with $u_i(a) = 0$, $i = 1,\dots,n$. Let φ_i, $i = 1,\dots,n$ be real-valued positive, convex and increasing functions on $(0,\infty)$. Let $r_i(x) \ge 0$, $r_i'(x) > 0$, $r_i(a) = 0$, $i = 1,\dots,n$. Let F be real-valued nonnegative and continuous function on $[0,\infty)^n$ with $F(0,\dots,0) = 0$ such that all its first partial derivatives F_i', $i = 1,\dots,n$ are nonnegative and nondecreasing on $[0,\infty)^n$. Then, the following integral inequality holds*

$$\int_a^b \Big(\sum_{i=1}^n F_i'\Big(r_1(t)\varphi_1(\frac{|u_1(t)|}{r_1(t)}),\dots,r_n(t)\varphi_n(\frac{|u_n(t)|}{r_n(t)})\Big)r_i'(t)\varphi_i(\frac{|u_i'(t)|}{r_i'(t)})\Big)dt \tag{1}$$

$$\le F\Big(\int_a^b r_1'(t)\varphi_1(\frac{|u_1'(t)|}{r_1'(t)})dt,\dots,\int_a^b r_n'(t)\varphi_n(\frac{|u_n'(t)|}{r_n'(t)})dt\Big).$$

Proof. From the assumptions of Theorem 1, we have

$$u_i(t) = \int_a^t u_i'(s)ds, \quad t \in [a,b], \quad i = 1,\dots,n. \tag{2}$$

and

$$(3)\qquad r_i(t) = \int_a^t r_i'(s)ds, \quad t \in [a,b], \quad i = 1,\ldots,n.$$

From the assumptions of Theorem 1 and from (2) we have

$$(4)\qquad |u_i(t)| \le \int_a^t |u_i'(s)|ds, \quad t \in [a,b], \quad i = 1,\ldots,n.$$

Since φ_i, $i = 1,\ldots,n$ are increasing on $(0,\infty)$, from (3) and (4), we have

$$(5)\qquad \varphi_i\left(\frac{|u_i(t)|}{r_i(t)}\right) \le \varphi_i\left(\frac{\int_a^t \frac{r_i'(s)|u_i'(s)|}{r_i'(s)}ds}{\int_a^t r_i'(s)ds}\right), \quad i = 1,\ldots,n.$$

Using Jensen's inequality [2, p. 133] we obtain from (5) that

$$(6)\qquad \varphi_i\left(\frac{|u_i(t)|}{r_i(t)}\right) \le \frac{1}{r_i(t)}\int_a^t r_i'(s)\varphi_i\left(\frac{|u_i'(s)|}{r_i'(s)}\right)ds, \quad i = 1,\ldots,n.$$

Since F_i', $i = 1,\ldots,n$ are nondecreasing, from (6), for each $t \in [a,b]$, we have

$$(7)\qquad \int_a^b \sum_{i=1}^n F_i'\left(r_1(t)\varphi_1(\frac{|u_1(t)|}{r_1(t)}),\ldots,r_n(t)\varphi_n(\frac{|u_n(t)|}{r_n(t)})\right) r_i'(t)\varphi_i(\frac{|u_i'(t)|}{r_i'(t)})dt$$
$$\le \int_a^b \sum_{i=1}^n \left(F_i'\left(\int_a^t r_1'(s)\varphi_1(\frac{|u_1'(s)|}{r_1'(s)})ds,\ldots\right.\right.$$
$$\left.\left.\ldots,\int_a^t r_n'(s)\varphi_n(\frac{|u_n'(s)|}{r_n'(s)})ds\right) r_i'(t)\varphi_i(\frac{|u_i'(t)|}{r_i'(t)})\right)dt.$$

The right-hand side of the inequality (7) is equal to

$$\int_a^b \frac{d}{dt}F\left(\int_a^t r_1'(s)\varphi_1(\frac{|u_1'(s)|}{r_1'(s)})ds,\ldots,\int_a^t r_n'(s)\varphi_n(\frac{|u_n'(s)|}{r_n'(s)})ds\right)dt$$
$$= F\left(\int_a^b r_1'(t)\varphi_1(\frac{|u_1'(t)|}{r_1'(t)})dt,\ldots,\int_a^b r_n'(t)\varphi_n(\frac{|u_n'(t)|}{r_n'(t)})dt\right).$$

which gives us required inequality (1).

More general result is established in the following theorem:

THEOREM 2. *Let F, F_i', φ_i, u_i, r_i, $i = 1,\ldots,n$ be defined as in Theorem 1. Let p_i, $i = 1,..,n$ be real-valued positive functions defined on $[a,b]$ and $\int_a^b p_i(t)dt = 1$, $i = 1,\ldots,n$. Let h_i, $i = 1,\ldots,n$ be real-valued positive,* *convex and increasing functions on* $(0,\infty)$. *Then, the following integral inequality holds*

$$(8)\quad \int_a^b \Big(\sum_{i=1}^n F_i'\Big(r_1(t)\varphi_1(\frac{|u_1(t)|}{r_1(t)}),\dots,r_n(t)\varphi_n(\frac{|u_n(t)|}{r_n(t)})\Big)r_i'(t)\varphi_i(\frac{|u_i'(t)|}{r_i'(t)})\Big)dt$$
$$\le F\Bigg(h_1^{-1}\Bigg(\int_a^b p_1(t)h_1\Bigg(\frac{r_1'(t)\varphi_1(\frac{|u_1'(t)|}{r_1'(t)})}{p_1(t)}\Bigg)dt\Bigg),\dots$$
$$\dots,h_n^{-1}\Bigg(\int_a^b p_n(t)h_n\Bigg(\frac{r_n'(t)\varphi_n(\frac{|u_n'(t)|}{r_n'(t)})}{p_n(t)}\Bigg)dt\Bigg)\Bigg).$$

Proof. In order to prove Theorem 2, we first observe that having $\int_a^b p_i(t)dt = 1$, $i=1,\dots,n$, we can write

$$(9)\quad \int_a^b r_i'(t)\varphi_i\Big(\frac{|u_i'(t)|}{r_i'(t)}\Big)dt = \frac{\int_a^b \frac{p_i(t)r_i'(t)\varphi_i\left(\frac{|u_i'(t)|}{r_i'(t)}\right)}{p_i(t)}dt}{\int_a^b p_i(t)dt},\quad i=1,\dots,n.$$

Since h_i, $i=1,\dots,n$ are convex, from (9) and using Jensen's inequality for convex functions [2, p. 133], we obtain

$$h_i\Bigg(\int_a^b r_i'(t)\varphi_i\Big(\frac{|u_i'(t)|}{r_i'(t)}\Big)dt\Bigg) \le \int_a^b p_i(t)h_i\Bigg(\frac{r_i'(t)\varphi_i(\frac{|u_i'(t)|}{r_i'(t)})}{p_i(t)}\Bigg)dt,\quad i=1,\dots,n.$$

which implies

$$(10)\quad \int_a^b r_i'(t)\varphi_i\Big(\frac{|u_i'(t)|}{r_i'(t)}\Big)dt \le h_i^{-1}\Bigg(\int_a^b p_i(t)h_i\Bigg(\frac{r_i'(t)\varphi_i(\frac{|u_i'(t)|}{r_i'(t)})}{p_i(t)}\Bigg)dt\Bigg),\quad i=1,\dots,n.$$

Now using Theorem 1, the inequality (10) and the fact that F is nondecreasing, by assumption, we obtain the required inequality (8).

3. Examples

Now it is interesting how these two theorems include the theorems mentioned in our introduction.

For $n=1$, Theorem 1 gives Theorem D.

If we put $\varphi_i(t)=t$ for each $i=1,\dots,n$ in Theorem 1, we will obtain Theorem B.

If we put $\varphi_i(t)=t$ for each $i=1,\dots,n$ in Theorem 2, we will obtain Theorem C.

For $n=2$ we can obtain results of B. G. Pachpatte [9] by putting $F(x_1,x_2)=f(x_1)\cdot g(x_2)$.

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Received January 3, 1996.