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DISCRETE HYPERGEOMETRIC FUNCTIONS AND THEIR PROPERTIES

1. Introduction

The study of basic hypergeometric series (also called q-hypergeometric series or q-series) acquired an independent status, when Heine [10] in 1878 converted a simple observation that $\lim_{q\to1}[(1-q^{a})/(1-q)] = a$ into a systematic investigation of the *basic hypergeometric series* defined by

$$(1.1)\qquad {}_2\varphi_1(\alpha,\beta;\gamma;z) = 1 + \frac{(1-q^{\alpha})(1-q^{\beta})}{(1-q)(1-q^{\gamma})}z + \frac{(1-q^{\alpha})(1-q^{\alpha+1})(1-q^{\beta})(1-q^{\beta+1})}{(1-q)(1-q^{2})(1-q^{\beta})(1-q^{\gamma+1})}z^{2} + \dots$$

which is a q-analogue of the classical Gauss hypergeometric series since

$$(1.2)\qquad \lim_{q\to1} {}_2\varphi_1(\alpha,\beta;\gamma;z) = {}_2F_1(\alpha,\beta;\gamma;z).$$

A life long program of developing the theory of basic hypergeometric series in a systematic manner, studying q-differentiation and q-integration and deriving q-analogues of the hypergeometric summation theorems and transformation formulas that were discovered by A. C. Dixon, J. Dougall, L. Saalachütz, F. J. W. Whipple and others was gathered by F. H. Jackson (see for example [11]–[16]).

The q-different functions are essentially those involving arguments of multiplicative rather than additive character. For example

$$(1.3)\qquad \Delta_q f(x) = f(x) - f(qx),\ q \text{ a parameter (or 'base');}$$

is a q-difference operator. A very extensive development of the theory of q-difference equations was carried out from 1909–1950 by F. H. Jackson (see for example [11]–[14], [16]). In 1910 he introduced the concept of q-integration which he defined as the inverse of the q-difference operator

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad |q| \neq 1, \tag{1.4}$$

and denoted by

$$D_q^{-1} f(x) = \frac{1}{1-q} \int f(x) d(q,x). \tag{1.5}$$

It is obvious that, if f is differentiable, then

$$\lim_{q \to 1} D_q f(x) = \frac{df}{dx}. \tag{1.6}$$

Harman [7] in 1978, introduced the concept of q-analyticity of a function by replacing derivatives by q-difference operators $D_{q,x}$ and $D_{q,y}$ defined as

$$D_{q,x}[f(z)] = \frac{f(z) - f(qx,y)}{(1-q)x}, \tag{1.7}$$

$$D_{q,y}[f(z)] = \frac{f(z) - f(x,qy)}{(1-q)iy}, \tag{1.8}$$

where f is a discrete function.

The two operators involve a *basic triad* of points denoted by

$$T(z) = \{(x,y),(qx,y),(x,qy)\}. \tag{1.9}$$

Let D be a discrete domain. Then a discrete function f is said to be *q-analytic* at $z \in D$, if

$$D_{q,x}[f(z)] = D_{q,y}[f(z)]. \tag{1.10}$$

If an addition (1.10) holds for every $z \in D$ such that $T(z) \subseteq D$, then f is said to be *q-analytic* in D. In this case, for simplicity, the common operator D_q is used, where

$$D_q \equiv D_{q,x} \equiv D_{q,y}. \tag{1.11}$$

The function z^n is of basic importance in complex analytic, since its use in infinite series leads to the Weierstrassian concept of an analytic function. Harman [8] defined, for a non-negative integer n, a q-analytic function $z^{(n)}$ to denote the discrete analogue of z^n, if it satisfies the following conditions

$$D_q[z^{(n)}] = \frac{1-q^n}{1-q} z^{(n-1)}, \quad z^{(0)} = 1, \quad 0^{(n)} = 0, \quad n > 0. \tag{1.12}$$

The operator C_y given by

$$C_y \equiv \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D_{q,x}^j, \tag{1.13}$$

when applied to the real function x^n, yields $z^{(n)}$. In fact, Harman [8] defined $z^{(n)}$, with a non-negative integer n, by

$$z^{(n)} \equiv C_y(x^n) = \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j D_{q,x}^j (x^n), \tag{1.14}$$

which, on simplification, yields

$$z^{(n)} = \sum_{j=0}^{\infty} \begin{bmatrix} n \\ j \end{bmatrix}_q x^{n-j} (iy)^j, \tag{1.15}$$

where

$$\begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{[n]!}{[j]![n-j]!},$$

or, alternatively,

$$z^{(n)} = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q x^j (iy)^{n-j}. \tag{1.16}$$

To justify that $z^{(n)}$ is a proper analogue of z^n, Harman [8] proved that $z^{(n)}$ is a q-analytic function and satisfies the three requirements of (1.12).

We shall also use the following notations due to Hahn [3]. Let

$$f(x) = \sum_{r=0}^{\infty} a_r x^r \tag{1.17}$$

be a power series in x. Then

$$f([x-y]) = \sum_{r=0}^{\infty} a_r (x-y)_r, \tag{1.18}$$

$$f\left(\frac{t}{[x-y]}\right) = \sum_{r=0}^{\infty} a_r \frac{t^r}{(x-y)_r}, \tag{1.19}$$

where

$$(x-y)_\alpha = x^\alpha \prod_{n=0}^{\infty} \left\{ \frac{1-(y/x)q^n}{1-(y/x)q^{\alpha+n}} \right\}. \tag{1.20}$$

For various other definitions, notations and results used in this paper one is referred to remarkable books on q-hypergeometric series by Exton [1], Gasper and Rahman [2] and Slater [19].

2. Discrete hypergoemetric functions

Using Harman's discrete analogue $z^{(n)}$ for the classical function z^n, we now introduce a discrete analogue ${}_rM_s[(a_r);(b_s);q,z]$ of the q-hypergeometric function ${}_r\varphi_s^{(q)}[(a_r);(b_s);z]$. It is well known that

$$D_q\{{}_r\varphi_s^{(q)}[(a_r);(b_s);x]\} = \frac{(1-q^{a_1})\dots(1-q^{a_r})}{(1-q)(1-q^{b_1})\dots(1-q^{b_s})}{}_r\varphi_s^{(q)}[1+(a_r);1+(b_s);x],$$

and so it seems reasonable to assume that, for a non-negative integer n, a q-analytic function ${}_rM_s[(a_r);(b_s);q,z]$ will denote the discrete analogue of ${}_r\varphi_s^{(q)}[(a_r);(b_s);z]$, if it satisfies the following conditions

$$(2.1)\quad \begin{cases} \text{(i) } D_q\{{}_rM_s[(a_r);(b_s);q,z]\} \\ \qquad = \dfrac{(1-q^{a_1})\dots(1-q^{a_r})}{(1-q)(1-q^{b_1})\dots(1-q^{b_s})}{}_rM_s[1+(a_r);1+(b_s);q,z], \\ \text{(ii) the first term of the series is 1,} \\ \text{(iii) } {}_rM_s[(a_r);(b_s);q,0]=1. \end{cases}$$

Such a function is obtained by applying the operator C_y defined in (1.13) to the q-hypergeometric function ${}_r\varphi_s[(a_r);(b_s);x]$ with real number x. In fact ${}_rM_s[(a_r);(b_s);q,z]$ is defined by

$$(2.2)\quad {}_rM_s[(a_r);(b_s);q,z] \equiv C_y\{{}_r\varphi_s{}^{(q)}[(a_r);(b_s);x]\} = \sum_{n=0}^{\infty}\frac{(q^{(a_r)})_n z^{(n)}}{(q)_n(q^{(b_s)})_n} = \sum_{n=0}^{\infty}\sum_{n=0}^{\infty}\frac{(q^{(a_r)})_{n+k}x^n(iy)^k}{(q)_n(q)_k(q^{(b_s)})_{n+k}}.$$

The following theorem shows that ${}_rM_s[(a_r);(b_s);q,z]$ satisfies (2.1) and hence can be taken as a discrete analogue of ${}_r\varphi_s^{(q)}[(a_r);(b_s);z]$.

THEOREM 1. *${}_rM_s[(a_r);(b_s);q,z]$ is q-analytic and satisfies the requirements of* (2.1).

Proof. We have

$$\begin{aligned} {}_rM_s[(a_r);(b_s);q,z] &= \sum_{n=0}^{\infty}\frac{(q^{(a_r)})_n z^{(n)}}{(q)_n(q^{(b_s)})_n} \\ &= \sum_{n=0}^{\infty}\frac{(q^{(a_r)})_n}{(q)_n(q^{(b_s)})_n}\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_q x^{n-j}(iy)^j, \end{aligned}$$

and hence

$$D_{q,x}\{{}_rM_s[(a_r);(b_s);q,z]\}$$
$$= \sum_{n=0}^{\infty} \frac{(q^{(a_r)})_n}{(q)_n(q^{(b_s)})_n} \sum_{j=0}^{n-1} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{1-q^{n-j}}{1-q} x^{n-j-1}(iy)^j$$
$$= \frac{1}{1-q} \sum_{n=1}^{\infty} \frac{(q^{(a_r)})_n z^{(n-1)}}{(q)_{n-1}(q^{(b_s)})_n}$$
$$= \frac{(1-q^{a_1})\dots(1-q^{a_r})}{(1-q)(1-q^{b_1})\dots(1-q^{b_s})} {}_rM_s[1+(a_r);1+(b_s);q,z].$$

Similarly,

$$D_{q,y}\{{}_rM_s[(a_r);(b_s);q,z]\}$$
$$= \frac{(1-q^{a_1})\dots(1-q^{a_r})}{(1-q)(1-q^{b_1})\dots(1-q^{b_s})} {}_rM_s[1+(a_r);1+(b_s);q,z].$$

Hence ${}_rM_s[(a_r);(b_s);q,z]$ is q-analytic and satisfies the condition (i) of (2.1). Since $z^{(0)}=1$ and $0^n=0$, $n>0$, by definition, so ${}_rM_s[(a_r);(b_s);q,z]$ satisfies (ii) and (iii) of (2.1). This proves Theorem 1.

It is of interest to note the similarity of ${}_rM_s[(a_r);(b_s);q,z]$ to the function ${}_r\varphi_s{}^{(q)}[(a_r);(b_s);[x+y]]$ defined by Jackson [15] as follows

(2.3) $${}_r\varphi_s{}^{(q)}[(a_r);(b_s);[x+y]]$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(q^{(a_r)})_{m+n} x^m (iy)^n q^{\frac{1}{n}n(n-1)}}{(q)_m(q)_n(q^{(b_s)})_{m+n}}$$
$$= \sum_{N=0}^{\infty} \frac{(q^{(a_r)})_N}{(q)_N(q^{(b_s)})_N}(x+iy)(x+iqy)\dots(x+iq^{N-1}y).$$

The discrete hypergeometric function defined in (2.2) can be written in one of the following forms

(2.4) $${}_rM_s[(a_r);(b_s);q,z] = \sum_{n=0}^{\infty} \frac{(q^{(a_r)})_n(iy)^n}{(q)_n(q^{(b_s)})_n} {}_r\varphi_s{}^{(q)}[(a_r)+n;(b_s)+n;x],$$

(2.5) $${}_rM_s[(a_r);(b_s);q,z] = \sum_{n=0}^{\infty} \frac{(q^{(a_r)})_n x^n}{(q)_n(q^{(b_s)})_n} {}_r\varphi_s{}^{(q)}[(a_r)+n;(b_s)+n;iy].$$

We observe that, from (2.4) and (2.5), a discrete hypergeometric function can be regarded as a generating function for the q-hypergeometric functions of the form

$${}_r\varphi_s{}^{(q)}[(a_r)+n;(b_s)+n;x] \text{ or } {}_r\varphi_s^{(q)}[(a_r)+n;(b_s)+n;iy].$$

We further observe that for $x = 0$, the function ${}_rM_s[(a_r);(b_s);q,z]$ is reduced to ${}_r\varphi_s{}^{(q)}[(a_r);(b_s);iy]$, while for $y = 0$ it becomes ${}_r\varphi_s{}^{(q)}[(a_r);(b_s);x]$.

3. Particular cases

As particular cases of (2.4) and (2.5) we have the following interesting results

(3.1) $${}_0M_0[-;-;q,z] = e_q(x)e_q(iy),$$

(3.2) $${}_2M_1[a,b;c;q,z] = \frac{1}{(1-x)_{a+b-c}}\sum_{n=0}^{\infty}\frac{(q^a)_n(q^b)_n(iy)^n}{(q)_n(q^c)_n(xq^{a+b-c})_n}{}_2\varphi_1\left[\begin{matrix}q^{c-a},q^{c-b};xq^{a+b-c+n}\\ q^{c+n};\end{matrix}\right],$$

(3.3) $${}_0M_1[-;a;q,z] = (q)_{a-1}\left(-\frac{1}{\sqrt{x}}\right)^{a-1}\sum_{n=0}^{\infty}\frac{(y/\sqrt{x})^n}{(q)_n}{}_qj_{a+n-1}(2i\sqrt{x}).$$

Further, summing up the ${}_r\varphi_s$-functions by means of known summation theorems, we have

(3.4) $${}_1M_0[a;-;q,z] = \frac{1}{(1-x)_a}{}_1\varphi_0\left[q^a;-;\frac{iy}{[1-xq^a]}\right],$$

(3.5) $${}_1M_0[a;-;q,z] = \frac{1}{(1-x)_a}{}_1\varphi_1[q^a;xq^a;iy],$$

(3.6) $${}_1M_0[a;-;q,z] = \frac{1}{(1-iy)_a}{}_1\varphi_0\left[q^a;-;\frac{x}{[1-iyq^a]}\right],$$

(3.7) $${}_1M_0[a;-;q,z] = \frac{1}{(1-iy)_a}{}_1\varphi_1[q^a;iyq^a;x],$$

(3.8) $${}_2M_1[a,-n;b;q,(q,y)] = \frac{(q^{b-a})_n q^{an}}{(q^b)_n}{}_2\varphi_1\left[\begin{matrix}q^a,q^{-n};-iyq^{1-b}\\ q^{1+a-b-n};q^{-1}\end{matrix}\right].$$

4. Integral representations

We also note the following simple integral representations

(4.1)
$${}_2M_1[a,b;c;q,z] = \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)}\int_0^1 t^{b-1}(1-qt)_{c-b-1}\,{}_1M_0[a;-;q,zt]d(t;q),$$

provided $Reb > 0$, $|x| < 1$, $|y| < 1$, and

(4.2) $${}_3M_2[a,b,c;d,e;q,z] = \frac{\Gamma_q(d)\Gamma_q(e)}{\Gamma_q(b)\Gamma_q(c)\Gamma_q(d-b)\Gamma_q(e-c)}$$
$$\times\int_0^1\int_0^1 t^{b-1}(1-qt)_{d-b-1}v^{c-1}(1-qv)_{e-c-1}\;{}_1M_0[a;-;ztv]d(t;q)d(v;q),$$

provided $Reb > 0$, $Rec > 0$, $|x| < 1$, $|y| < 1$.

One can similarly write down the integral representation for ${}_rM_s$-function.

5. Contiguous discrete hypergeometric functions

Any two discrete hypergeometric functions ${}_rM_s[(a_r);(b_s);q,z]$ and ${}_rM_s[(a'_r);(b'_s);q,z]$ are said to be contiguous, when all their parameters are equal except one pair and this pair of the parameters differs only by unity.

If we use the notations

$$(\alpha_r,+i)_n = (\alpha_1)_n(\alpha_2)_n\dots(\alpha_{i-1})_n(\alpha_i+1)_n(\alpha_{i+1})_n\dots(\alpha_r)_n$$

and

$$(\alpha_r,-i)_n = (\alpha_1)_n(\alpha_2)_n\dots(\alpha_{i-1})_n(\alpha_i-1)_n(\alpha_{i+1})_n\dots(\alpha_r)_n,$$

for $1 \le i \le r$, and similar notations for (β_s), we have

$$\begin{aligned}(5.1)\quad & {}_rM_s[(\alpha_r,+i);(\beta_s);q,z] \\ &= \frac{1}{1-q^{\alpha_i}}\{{}_rM_s[(\alpha_r);(\beta_s);q,z] - q^{\alpha_i}\,{}_rM_s[(\alpha_r);(\beta_s);q,qz]\},\end{aligned}$$

$$\begin{aligned}(5.2)\quad & {}_rM_s[(\alpha_r,-i);(\beta_s);q,z] \\ &= (1-q^{\alpha_i-1})\sum_{n=0}^{\infty} q^{n(\alpha_i-1)}\,{}_rM_s[(\alpha_r);(\beta_s);q,q^nz],\end{aligned}$$

$$(5.3)\quad {}_rM_s[(\alpha_r);(\beta_s,+j);q,z] = (1-q^{\beta_j})\sum_{n=0}^{\infty} q^{n\beta_j}\,{}_rM_s[(\alpha_r);(\beta_s);q,q^nz],$$

$$\begin{aligned}(5.4)\quad & {}_rM_s[(\alpha_r);(\beta_s,-j);q,z] \\ &= \frac{1}{1-q^{\beta_j-1}}\{{}_rM_s[(\alpha_r);(\beta_s);q,z] - q^{\beta_j-1}\,{}_rM_s[(\alpha_r);(\beta_s);q,qz]\}.\end{aligned}$$

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