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SOME ESTIMATIONS ON ENERGY IN THERMOELASTICITY OF MICROPOLAR BODIES

1. Introduction

Our purpose is to study the asymptotic partition of total energy for the solutions of the mixed initial boundary value problem within the context of the thermoelasticity of micropolar bodies. The asymptotic equipartition property is a familiar notion in the theory of differential equations. In short, this means that the kinetic and the potential energy of a classical solution with finite energy become asymptotically equal in mean as time tends to infinity. We find such a property in various papers for physical systems governed by nondissipative hyperbolic partial differential equations or systems of such equations. Our purpose is to study the asymptotic partition of total energy for the solutions of the mixed initial boundary value problem within the context of the linear thermoelasticity of micropolar bodies.

The system of equations governing this problem consists of hyperbolic equations with dissipation and, therefore, does not belong to one of the categories considered previously in literature on subject. By using the dissipative mechanism of the system, we can prove that the equipartition occurs between the mean kinetic and strain energies. Instead of abstracted version of this question, we prefer to emphasize the technique itself on the thermoelasticity of micropolar bodies.

The plane of the paper is following one. We first write down the mixed initial boundary value problem within context of thermoelasticity of micropolar bodies. Then we establish some Lagrange type identities and also we introduce the Cesaro means of various parts of the total energy associated to the solutions. Finally, we establish, basing on previous estimations, the relations that describe the asymptotic behaviour of the mean energies.

It should be noted that there are many papers which employ the various refinements of the Lagrange identity as Levine (1977), Rionero and Chirita

(1987), Marin (1994), Gurtin (1993). We find also many papers that use the Cesaro means, as Day (1980), Levine (1977) for instance.

2. Basic equations

Let B be an open set domain of three-dimensional Euclidean space occupied by the reference configuration of a homogeneous micropolar body. We assume that B is regular and finite region with boundary ∂B and we denote the closure of B by $\bar{B}$. We use a fixed system of rectangular Cartesian axes and adopt Cartesian tensor notation. Points in B are denoted by x_j and $t \in [0, \infty)$ is temporal variable. Also, the spatial argument and the time argument of a function will be omitted when there is no likehood of confusion. In the absence of the body force, body couple force and heat supply fields, the field for linear thermoelasticity of micropolar bodies are, (see, [7])

$$t_{ij,j} = \varrho \ddot{u}_i, \quad m_{ij,j} + \varepsilon_{ijk} t_{jk} = I_{ij} \ddot{\varphi}_i, \tag{1}$$

$$-q_{i,i} = \varrho \theta_0 \dot{\eta}, \quad (x,t) \in B \times [0, \infty). \tag{2}$$

The relations (1) are the motion equations and (2) is the energy equation. In (1), (2) we use the following notations: u_i- components of displacement, φ_i-components of microrotatia, t_{ij}-components of stress, m_{ij}-components of couple stress, q_i-components of the heat conduction vector, η-the specific entropy, ϱ-the constant reference density, θ_0-the constant reference temperature, I_{ij}-components of inertia and ε_{ijk}-the alternating symbol.

A superposed dot denotes the differentiation with respect to time t, and a subscript preceded by a comma denotes the diferentiation with respect to the corresponding spatial variable. When the reference solid has a centre of symmetry at each point, but is otherwise non-isotropic, then the constitutive equations are

$$\begin{aligned} t_{ij} &= A_{ijmn}\varepsilon_{mn} + B_{ijmn}\gamma_{mn} + D_{ij}(\theta + \alpha\dot{\theta}), \\ m_{ij} &= B_{mnij}\varepsilon_{mn} + C_{ijmn}\gamma_{mn} + E_{ij}(\theta + \alpha\dot{\theta}), \\ q_i &= -\theta_0 K_{ij}\theta_{,j}, \\ \varrho\eta &= a + d\theta + h\dot{\theta} - D_{ij}\varepsilon_{ij} - E_{ij}\gamma_{ij}, (x,t) \in B \times [0, \infty). \end{aligned} \tag{3}$$

In the above equations we use the following geometrical equations

$$\varepsilon_{ij} = u_{j,i} + \varepsilon_{jik}\varphi_k, \quad \gamma_{ij} = \varphi_{j,i}. \tag{4}$$

The tensor coefficients in (3) are constants subject to the symmetry conditions

$$A_{ijmn} = A_{mnij}, \quad C_{ijmn} = C_{mnij}, K_{ij} = K_{ji}. \tag{5}$$

The density ϱ, the coefficients of inertia I_{ij} and temperature θ_0 are given constants which satisfy the conditions

$$\varrho > 0, \quad \theta_0 > 0, \quad I_{ij} > 0. \tag{6}$$

From the entropy production inequality we obtain the following conditions

$$d\alpha - h \geq 0, \quad K_{ij}\xi_i\xi_j \geq 0, \qquad \forall \xi_i. \tag{7}$$

In concordance with the conditions (7) we assume that $A_{ijmn}, C_{ijmn}, K_{ij}$ are positive definite tensors, i.e.

$$\begin{aligned} &A_{ijmn}\xi_{ij}\xi_{mn} \geq k_0\xi_{ik}\xi_{ik}, && k_0 > 0, && \forall \xi_{ij} = \xi_{ji}, \\ &C_{ijmn}\xi_{ij}\xi_{mn} \geq k_1\xi_{ik}\xi_{ik}, && k_1 > 0, && \forall \xi_{ij} = \xi_{ji}, \\ &K_{ij}\xi_i\xi_j \geq k_2\xi_i\xi_i, && k_2 > 0, && \forall \xi_i. \end{aligned} \tag{8}$$

Moreover, according to (Green, Lindsay, 1972), we can assume that

$$\alpha > 0, \quad h > 0, \quad d\alpha - h > 0. \tag{9}$$

Now, we admit the following precsribed boudary conditions

$$\begin{aligned} u_i = 0 \text{ on } \partial B_1 \times [0,\infty), \quad & t_{ij}n_j = 0 \text{ on } \partial B_1^c \times [0,\infty), \\ \varphi_i = 0 \text{ on } \partial B_2 \times [0,\infty), \quad & m_{ij}n_j = 0 \text{ on } \partial B_2^c \times [0,\infty), \\ \theta = 0 \text{ on } \partial B_3 \times [0,\infty), \quad & q_i n_i = 0 \text{ on } \partial B_3^c \times [0,\infty), \end{aligned} \tag{10}$$

where $\partial B_1, \partial B_2, \partial B_3$ and $\partial B_1^c, \partial B_2^c, \partial B_3^c$ are subsets of ∂B and their complements with respect to ∂B, and n_i are the components of the unit outward normal to ∂B. Introducing (3) in (1) and (2), we obtain the following system

$$\begin{aligned} \varrho\ddot{u}_i &= A_{ijmn}\varepsilon_{mn,j} + B_{ijmn}\gamma_{mn,j} + D_{ij}(\theta_{,j} + \alpha\dot{\theta}_{,j}), \\ I_{ij}\ddot{\varphi}_i &= B_{mnij}\varepsilon_{mn,j} + C_{ijmn}\gamma_{mn,j} + E_{ij}(\theta_{,j} + \alpha\dot{\theta}_{,j}) \\ &\quad + \varepsilon_{ijk}(A_{jkmn}\varepsilon_{mn} + B_{jkmn}\gamma_{mn} + D_{jk}(\theta + \alpha\dot{\theta})), \end{aligned} \tag{11}$$

$$h\ddot{\theta} = -d\dot{\theta} + D_{ij}\dot{\varepsilon}_{ij} + E_{ij}\dot{\gamma}_{ij} + K_{ij}\theta_{,ij}, \quad (x,t) \in B \times [0,\infty). \tag{12}$$

Furthermore, we put the following initial conditions

$$\begin{aligned} &u_i(x,0) = u_i^0(x), \quad \dot{u}_i(x,0) = \dot{u}_i^0(x), \quad \varphi_i(x,0) = \varphi_i^0(x), \\ &\dot{\varphi}_i(x,0) = \dot{\varphi}_i^0(x), \quad \theta(x,0) = \theta^0(x), \quad \dot{\theta}(x,0) = \dot{\theta}^0(x). \end{aligned} \tag{13}$$

By a solution of the mixed initial boundary value problem of the micropolar thermoelasticity in the cylinder $\Omega_0 = B \times [0,\infty)$ we mean an ordered array (u_i, φ_i, θ) which satisfies the system (11), (12) for all $(x,t) \in \Omega_0$, the boundary conditions (10) and the intial conditions (13).

We observe that if *meas* $\partial B_1 = 0$ and *meas* $\partial B_2 = 0$, then there exists a family of rigid motions and null temperature which satisfy the equations (11), (12) and the boundary conditions (10). For this reason we decompose the initial data u_i^0, φ_i^0, $\dot{u}_i^0$, $\dot{\varphi}_i^0$, as follows

$$(14) \qquad u_i^0 = u_i^* + U_i^0, \quad \dot{u}_i^0 = \dot{u}_i^* + \dot{U}_i^0, \quad \varphi_i^0 = \varphi_i^* + \Phi_i^0, \quad \dot{\varphi}_i^0 = \dot{\varphi}_i^* + \dot{\Phi}_i^0,$$

where u_i^*, φ_i^*, $\dot{u}_i^*$, $\dot{\varphi}_i^*$ are so determined that U_i^0, Φ_i^0, $\dot{U}_i^*$, $\dot{\Phi}_i^*$ satisfy

$$(15) \qquad \begin{aligned} &\int_B \varrho U_i^0 dV = 0, \quad \int_B \varrho(\varepsilon_{ijk} x_j U_k^0 + \Phi_i^0) dV = 0, \\ &\int_B \varrho \dot{U}_i^0 dV = 0, \quad \int_B \varrho(\varepsilon_{ijk} x_j \dot{U}_k^0 + \dot{\Phi}_i^0) dV = 0. \end{aligned}$$

If *meas* $\partial B_1 = 0$ and *meas* $\partial B_2 \neq 0$, then we have the restriction

$$\int_B \varrho U_i^0 dV = 0, \quad \int_B \varrho \dot{U}_i^0 dV = 0.$$

Finally, if *meas* $\partial B_3 = 0$, then there exists a family of constant temperatures and null motion, which satisfy the equations (11), (12) and the boundary conditions (10). Therefore, we decompose the initial data $\theta^0, \dot{\theta}^0$ as follows

$$(16) \qquad \theta^0 = \theta^* + T^0, \quad \dot{\theta}^0 = \dot{\theta}^* + \dot{T}^0,$$

where θ^* and $\dot{\theta}^*$ are costants so determined that

$$(17) \qquad \int_B T^0 dV = 0, \quad \int_B \dot{T}^0 dV = 0.$$

3. Specific notations

We denote by $C^m(B)$ the class of scalar fields possessing derivatives up to the m-th order in B which are continuous on B. For $f \in C^m(B)$ we define the norm

$$\|f\|_{C^m(B)} \equiv \sum_{k=0}^{m} \sum_{i_1, i_2, \ldots, i_k} \max_B |f_{,i_1 \ldots i_k}|.$$

By $\mathbf{C}^m(B)$ we denote the class of vector fields with six components. For $\mathbf{w} \in \mathbf{C}^m(B)$ we define the norm

$$\|\mathbf{w}\|_{\mathbf{C}^m(B)} \equiv \sum_{i=1}^{6} \|w_i\|_{C^m(B)}.$$

By $W_m(B)$ we denote the Hilbert space obtained as the completion of $C^m(B)$ by means of the norm $\|.\|_{W_m(B)}$ induced by the inner product

$$(f,g)_{W_m(B)} \equiv \sum_{k=0}^{m} \int_B f_{,i_1\ldots i_k} g_{,i_1\ldots i_k} dV.$$

By $\mathbf{W}_m(B)$ we denote the completion of $\mathbf{C}^m(B)$ by means of the norm $\|.\|_{\mathbf{W}_m(B)}$ induced by the inner product

$$(w,\omega)_{\mathbf{W}_m(B)} \equiv \sum_{i=1}^{6} (w_i,\omega_i)_{W_m(B)}.$$

We will use as norm in Cartesian product of the normed spaces the sum of the norms of the factor spaces. Let us introduce the following notations
$\hat{C}^1(B) \equiv \{\chi \in C^1(B) : \chi = 0$ on ∂B_3 if *meas* $\partial B_3 = 0$, then $\int_B \chi dV = 0\}$;
$\hat{\mathbf{C}}^1(B) \equiv \{(v_i,\psi_i) \in \mathbf{C}^1(B)$: $v_i = 0$ on ∂B_1, $\psi_i = 0$ on ∂B_2;
if *meas* $\partial B_1 =$*meas* $\partial B_2 = 0$, then

$$\int_B \varrho v_i dV = 0, \quad \int_B \varrho(\varepsilon_{ijk} x_j v_k + \psi_i) dV = 0;$$

if *meas* $\partial B_1 = 0$ and *meas* $\partial B_2 \neq 0$, then $\int_B \varrho v_i dV = 0\}$;
$\hat{W}_1(B) \equiv$ the completion of $\hat{C}^1(B)$ by means of $\|.\|_{W_1(B)}$;
$\hat{\mathbf{W}}^1(B) \equiv$ the completion of $\hat{\mathbf{C}}^1(B)$ by means of $\|.\|_{\mathbf{W}_1(B)}$.
In these relations $W_m(B)$ represents the familiar Sobolev space, [1], and $\mathbf{W}_m(B) \equiv [W_m(B)]^6$. We note that hypothesis (8) assures that the following Korn's inequality, [4], holds for all $(v,\psi) \in \hat{\mathbf{W}}_1(B)$,

$$\begin{aligned}(18)\qquad & \int_B [A_{ijmn}\varepsilon_{ij}(v,\psi)\varepsilon_{mn}(v,\psi) + 2B_{ijmn}\varepsilon_{ij}(v,\psi)\gamma_{mn}(v,\psi) \\ & \qquad + C_{ijmn}\gamma_{ij}(v,\psi)\gamma_{mn}(v,\psi)]dV \\ & \qquad \geq m_1 \int_B (v_i v_i + v_{i,j} v_{i,j} + \psi_i \psi_i + \psi_{i,j}\psi_{i,j}) dV,\end{aligned}$$

where $m_1 > 0, m_1 =$ const. and $\varepsilon_{ij}(v,\psi) = v_{j,i} + \varepsilon_{jik}\psi_k$, $\gamma_{ij}(v,\psi) = \psi_{j,i}$. Under the hypothesis (8), for all $\chi \in \hat{W}_1(B)$ the following Poincare's inequality holds

$$(19)\qquad \int_B K_{ij}\chi_{,i}\chi_{,j} dV \geq m_2 \int_B \chi^2 dV, m_2 > 0.$$

If *meas* $\partial B_1 = 0$ and *meas* $\partial B_2 = 0$, then we shall find it is a convenient practice to decompose the solution (u_i, φ_i, θ) in the form

$$(20)\qquad u_i = u_i^* + t\dot{u}_i^* + \psi_i, \quad \varphi_i = \varphi_i^* + t\dot{\varphi}_i^* + \psi_i, \quad \theta = \chi,$$

where $((v_i,\psi_i),\chi) \in \hat{\mathbf{W}}_1(B)\times\hat{W}_1(B)$ represents the solution of (1), (2), (10) with the initial conditions $v_i = U_i^0$, $\dot{v}_i = \dot{U}_i^0$, $\varphi_i = \Phi_i^0$, $\dot{\varphi}_i = \dot{\Phi}_i^0$, $\chi = \theta^0$, $\dot{\chi} = \dot{\theta}^0$, on B, at $t = 0$.

Let us now consider that $\partial B_3 = 0$. Then we use the relations (16), (17) and (2) in order to decompose the solution $((u_i,\varphi_i),\theta)$ in the form

$$(21)\qquad u_i = v_i, \quad \varphi_i = \psi_i, \quad \theta = \theta^* + \frac{h}{d}[1-\exp(-\frac{dt}{h})]\dot{\theta}^* + \chi,$$

where $((v_i,\psi_i),\chi) \in \hat{\mathbf{W}}_1(B)\times\hat{W}_1(B)$ represents the solution of (1), (2), (10) with the initial conditions

$$v_i = u_i^0, \quad \dot{v}_i = \dot{u}_i^0, \quad \psi_i = \varphi_i^0, \quad \dot{\psi}_i = \dot{\varphi}_i^0, \quad \chi = T^0, \quad \dot{\chi} = \dot{T}^0, \text{ on } B, \text{ at } t = 0.$$

4. Some preliminary identities

In this section we shall establish some evolutionary integral identities which are essentially in proving the relations that express the asymptotic partition of energy. The first theorem presents a conservation law of total energy.

THEOREM 4.1. *Let $((u_i,\varphi_i),\theta)$ be a solution of the initial boundary value problem defined by* (11), (12), (10) *and* (13). *We assume that*

$$(u_i^0,\varphi_i^0) \in \mathbf{W}_1(B), (\dot{u}_i^0,\dot{\varphi}_i^0) \in \mathbf{W}_0(B), \theta^0 \in W_1(B), \dot{\theta}^0 \in W_0(B).$$

Then the following energy conservation law holds

$$(22)\quad E(t) \equiv \frac{1}{2}\int_B[\varrho\dot{u}_i(t)\dot{u}_i(t) + I_{ij}\dot{\varphi}_i(t)\dot{\varphi}_j(t) + A_{ijmn}\varepsilon_{ij}(t)\varepsilon_{mn}(t)$$
$$+ B_{ijmn}\varepsilon_{ij}(t)\gamma_{mn}(t) + C_{ijmn}\gamma_{ij}(t)\gamma_{mn}(t) + \alpha K_{ij}\theta_{,i}(t)\theta_{,j}(t)$$
$$+ d\theta^2(t) + \alpha h\dot{\theta}^2(t) + 2h\theta(t)\dot{\theta}(t)$$
$$+ \int_0^t\int_B[K_{ij}\theta_{,i}(s)\theta_{,j}(s) + (d\alpha - h)\dot{\theta}^2(s)]dVds = E(s),$$

for $t \in [0,\infty)$.

Proof. In view of equations (11) we get

$$(23)\qquad \frac{1}{2}\frac{d}{ds}[\varrho\dot{u}_i\dot{u}_i + I_{ij}\dot{\varphi}_i\dot{\varphi}_j] = (\dot{u}_i t_{ji} + \dot{\varphi}_i m_{ji})_{,j} - A_{jimn}\varepsilon_{mn}\dot{\varepsilon}_{ji}$$
$$-B_{ijmn}(\gamma_{mn}\dot{\varepsilon}_{ji} + \dot{\gamma}_{mn}\varepsilon_{ji}) - C_{ijmn}\gamma_{mn}\dot{\gamma}_{ji} - D_{ji}(\theta+\alpha\dot{\theta})\dot{\varepsilon}_{ji} - E_{ji}(\theta+\alpha\dot{\theta})\dot{\gamma}_{ji}.$$

On the other hand, by using the energy equation (12), we obtain

$$(24)\qquad D_{ji}(\theta+\alpha\dot{\theta})\dot{\varepsilon}_{ji}+E_{ji}(\theta+\alpha\dot{\theta})\dot{\gamma}_{ji}=-[K_{ij}\theta_{,j}(\theta+\alpha\dot{\theta})]_{,i}$$
$$+\frac{1}{2}\frac{d}{ds}[d\theta^2+\alpha K_{ij}\theta_{,i}\theta_{,j}+\alpha h\dot{\theta}^2+2\theta\dot{\theta}]+K_{ij}\theta_{,i}\theta_{,j}+(d\alpha-h)\dot{\theta}^2,$$

such that from (23) and (24), by integrating over $B\times(0,t)$ and by using the boundary conditions (10) and the initial conditions (13), we arrive at the desired result (22).

THEOREM 4.2. *Let $((u_i,\varphi_i),\theta)$ be a solution of the initial boundary value problem given by* (10)–(13). *We assume that*

$$(u_i^0,\varphi_i^0)\in\mathbf{W}_1(B),(\dot{u}_i^0,\dot{\varphi}_i^0)\in\mathbf{W}_0(B),\theta^0\in W_1(B),\dot{\theta}^0\in W_0(B).$$

Then the following identity holds

$$(25)\qquad 2\int_B[\varrho u_i(t)\dot{u}_i(t)+I_{ij}\varphi_i(t)\dot{\varphi}_j(t)]\,dV+2\int_B[(d\alpha-h)\theta^2(t)$$
$$+K_{ij}\left(\int_0^t\theta_{,i}(\xi)d\xi\right)\left(\int_0^t\theta_{,j}(\xi)d\xi\right)+2\alpha K_{ij}\theta_{,i}(t)\left(\int_0^t\theta_{,j}(\xi)d\xi\right)]dV$$
$$=2\int_0^t\int_B[\varrho\dot{u}_i(s)\dot{u}_i(s)+I_{ij}\dot{\varphi}_i(s)\dot{\varphi}_j(s)-A_{ijmn}\varepsilon_{ij}(s)\varepsilon_{mn}(s)$$
$$-2B_{ijmn}\varepsilon_{ij}(s)\gamma_{mn}(s)-C_{ijmn}\gamma_{ij}(s)\gamma_{mn}(s)-d\theta^2(s)-2h\theta(s)\dot{\theta}(s)$$
$$-\alpha h\dot{\theta}^2(s)-\alpha K_{ij}\theta_{,i}(s)\theta_{,j}(s)]dV\,ds+2\int_B[\varrho u_i^0\dot{u}_i^0+I_{ij}\varphi_i^0\dot{\varphi}_j^0]\,dV$$
$$+\int_B(d\alpha-h)(\theta^0)^2(t)dV-2\int_0^t\int_B(a-\varrho\eta^0)[\theta(s)+\alpha\dot{\theta}(s)]dV\,ds,$$

where $\varrho\eta^0=a+d\theta^0+h\dot{\theta}^0-D_{ij}\varepsilon_{ij}^0-E_{ij}\gamma_{ij}^0,\varepsilon_{ij}^0=u_{j,i}^0+\varepsilon_{jik}\varphi_k^0,\gamma_{ij}^0=\varphi_{j,i}^0.$

Proof. First, by using the equations (11), we obtain

$$(26)\qquad \frac{d}{ds}[\varrho u_i\dot{u}_i+I_{ij}\varphi_i\dot{\varphi}_j]=(u_it_{ji}+\varphi_im_{ji})_{,j}-A_{jimn}\varepsilon_{mn}\varepsilon_{ji}$$
$$-2B_{jimn}\gamma_{mn}\varepsilon_{ji}-C_{jimn}\gamma_{mn}\gamma_{ji}-D_{ji}(\theta+\alpha\dot{\theta})\varepsilon_{ji}$$
$$-E_{ji}(\theta+\alpha\dot{\theta})\gamma_{ji}+\varrho\dot{u}_i\dot{u}_i+I_{ij}\dot{\varphi}_i\dot{\varphi}_j.$$

On the other hand, by using the energy equation (12), we obtain

$$(27)\quad D_{ji}(\theta+\alpha\dot\theta)\varepsilon_{ji}+E_{ji}(\theta+\alpha\dot\theta)\gamma_{ji}=\alpha K_{ij}\left[\dot\theta_{,i}\int_0^s\theta_{,j}(\xi)d\xi+\theta_{,i}\theta_{,j}\right]$$

$$-\alpha K_{ij}\theta_{,i}\theta_{,j}+K_{ij}(\theta_{,i}+\alpha\dot\theta_{,i})\int_0^s\theta_{,j}(\xi)d\xi-\left[K_{ij}(\theta+\alpha\dot\theta)\int_0^s\theta_{,j}(\xi)d\xi\right]_{,i}$$

$$+\alpha K_{ij}\left[\dot\theta_{,i}\int_0^s\theta_{,j}(\xi)d\xi+\theta_{,i}\theta_{,j}\right]+(d\alpha-h)\theta\dot\theta$$

$$+K_{ij}\theta_{,i}\int_0^s\theta_{,j}(\xi)d\xi+d\theta^2+\alpha h\dot\theta^2+2h\theta\dot\theta+(a-\varrho\eta^0)(\theta+\alpha\dot\theta).$$

From (26) and (27) it results

$$(28)\quad \frac{d}{ds}[\varrho u_i\dot u_i+I_{ij}\varphi_i\dot\varphi_j]=(u_it_{ji}+\varphi_im_{ji})_{,j}$$

$$-A_{jimn}\varepsilon_{mn}\varepsilon_{ji}-2B_{jimn}\gamma_{mn}\varepsilon_{ji}-C_{jimn}\gamma_{mn}\gamma_{ji}$$

$$+\varrho\dot u_i\dot u_i+I_{ij}\dot\varphi_i\dot\varphi_j+\left[K_{ij}(\theta+\alpha\dot\theta)\int_0^s\theta_{,j}(\xi)d\xi\right]_{,i}$$

$$-(a-\varrho\eta^0)(\theta+\alpha\dot\theta)+\alpha K_{ij}\theta_{,i}\theta_{,j}-\alpha K_{ij}\left[\dot\theta_{,i}\int_0^s\theta_{,j}(\xi)d\xi+\theta_{,i}\theta_{,j}\right]$$

$$-K_{ij}\theta_{,i}\int_0^s\theta_{,j}(\xi)d\xi-d\theta^2-\alpha h\dot\theta^2-2h\theta\dot\theta-(d\alpha-h)\theta\dot\theta.$$

An integration of the identity (28) over $B\times(0,t)$, followed by the use of the boundary conditions (10), the initial conditions (13) and the symmetry relations (5), lead to the identity (25) and the proof of Theorem 4.2 is complete.

THEOREM 4.3. *Let $((u_i,\varphi_i),\theta)$ be a solution of the initial boundary value problem defined by* (10)–(13), *corresponding to the initial data*

$$(u_i^0,\varphi_i^0)\in\mathbf{W}_1(B),(\dot u_i^0,\dot\varphi_i^0)\in\mathbf{W}_0(B),\theta^0\in W_1(B),\dot\theta^0\in W_0(B).$$

Then the following identity holds

$$(29)\qquad 2\int_B[\varrho u_i(t)\dot u_i(t)+I_{ij}\varphi_i(t)\dot\varphi_j(t)]\,dV+\int_B[(d\alpha-h)\theta^2(t)$$

$$+K_{ij}\Big(\int_0^t\theta_{,i}(\xi)d\xi\Big)\Big(\int_0^t\theta_{,j}(\xi)d\xi\Big)+2\alpha K_{ij}\theta_{,i}(t)\Big(\int_0^t\theta_{,j}(\xi)d\xi\Big)]dV$$

$$= \int_B \varrho[\dot{u}_i^0 u_i(2t) + u_i^0 \dot{u}_i(2t)] + I_{ij}[\dot{\varphi}_i^0 \varphi_j(2t) + \varphi_i^0 \dot{\varphi}_j(2t)]dV$$

$$+ \int_B [(d\alpha - h)\theta^0\theta(2t) + \alpha K_{ij}\theta^0_{,i}\Big(\int_0^{2t} \theta_{,j}(\xi)d\xi\Big)]dV$$

$$+ \int_0^t \int_B (a - \varrho\eta^0)\Big(\theta(t+s) - \theta(t-s) + \alpha[\dot{\theta}(t+s) - \dot{\theta}(t-s)]\Big)dV ds.$$

P r o o f. Let $f_i(x,s)$ and $f_i(x,s)$ be twice continuously differentiable functions with respect to time variable s. It is easy to see that

$$\frac{d}{ds}[\varrho\Big(f_i(s)\dot{g}_i(s) - \dot{f}_i(s)g_i(s)\Big)] = \varrho[f_i(s)\ddot{g}_i(s) - \ddot{f}_i(s)g_i(s)],$$

such that, by integrating over $B \times (0,t)$, it results

$$\text{(30)} \quad \int_B \varrho[f_i(t)\dot{g}_i(t) - \dot{f}_i(t)g_i(t)]dV = \int_0^t \int_B \varrho[f_i(s)\ddot{g}_i(s) - \ddot{f}_i(s)g_i(s)]dV ds$$
$$+ \int_B \varrho[f_i(0)\dot{g}_i(0) - \dot{f}_i(0)g_i(0)]dV.$$

By setting $f_i(x,\tau) = u_i(x,t-\tau), g_i(x,\tau) = u_i(x,t+\tau), \tau \in [0,t], t \in (0,\infty)$, the relation (30) becomes

$$\text{(31)} \quad 2\int_B \varrho u_i(t)\dot{u}_i(t)dV = \int_B \varrho[u_i^0\dot{u}_i(2t) + \dot{u}_i^0 u_i(2t)]dV$$
$$+ \int_0^t \int_B \varrho[u_i(t+s)\ddot{u}_i(t-s) - u_i(t-s)\ddot{u}_i(t+s)]dV ds,$$

for $t \in (0,\infty)$. Similarly, for $t \in (0,\infty)$, we have

$$\text{(32)} \quad 2\int_B I_{ij}\varphi_i(t)\dot{\varphi}_j(t)dV = \int_B I_{ij}[\varphi_i^0\dot{\varphi}_j(2t) + \dot{\varphi}_i^0\varphi_j(2t)]dV$$
$$+ \int_0^t \int_B I_{ij}[\varphi_i(t+s)\ddot{\varphi}_j(t-s) - \varphi_i(t-s)\ddot{\varphi}_j(t+s)]dV ds.$$

We now eliminate the inertial terms in the last integrals in (31) and (32). By (5), (11), we get

$$\begin{aligned}(33)\quad &\varrho[u_i(t+s)\ddot{u}_i(t-s)-u_i(t-s)\ddot{u}_i(t+s)]+I_{ij}[\varphi_i(t+s)\ddot{\varphi}_j(t-s)\\ &-\varphi_i(t-s)\ddot{\varphi}_j(t+s)]=[u_i(t+s)t_{ji}(t-s)-u_i(t-s)t_{ji}(t+s)]_{,j}\\ &+[\varphi_i(t+s)m_{ji}(t-s)-\varphi_i(t-s)m_{ji}(t+s)]_{,j}\\ &+[D_{ji}\varepsilon_{ji}(t-s)+E_{ji}\gamma_{ji}(t-s)][\theta(t+s)+\alpha\dot{\theta}(t+s)]\\ &-[D_{ji}\varepsilon_{ji}(t+s)+E_{ji}\gamma_{ji}(t+s)][\theta(t-s)+\alpha\dot{\theta}(t-s)].\end{aligned}$$

On the other hand, in view of (12), we obtain

$$\begin{aligned}(34)\quad &[D_{ji}\varepsilon_{ji}(t-s)+E_{ji}\gamma_{ji}(t-s)][\theta(t+s)+\alpha\dot{\theta}(t+s)]\\ &-[D_{ji}\varepsilon_{ji}(t+s)+E_{ji}\gamma_{ji}(t+s)][\theta(t-s)+\alpha\dot{\theta}(t-s)]\\ &\times(a-\varrho\eta^0)[\theta(t-s)-\theta(t+s)+\alpha(\dot{\theta}(t-s)-\dot{\theta}(t+s))]\\ &+(d\alpha-h)[\theta(t-s)\dot{\theta}(t+s)-\theta(t+s)\dot{\theta}(t-s)]\\ &+K_{ij}[\theta_{,i}(t+s)\left(\int_0^{t-s}\theta_{,j}(\xi)d\xi\right)-\theta_{,i}(t-s)\left(\int_0^{t+s}\theta_{,j}(\xi)d\xi\right)]\\ &+\alpha K_{ij}[\dot{\theta}_{,i}(t+s)\left(\int_0^{t-s}\theta_{,j}(\xi)d\xi\right)-\theta_{,i}(t-s)\theta_{,j}(t+s)]\\ &+\alpha K_{ij}[\dot{\theta}_{,i}(t-s)\left(\int_0^{t+s}\theta_{,j}(\xi)d\xi\right)-\theta_{,i}(t+s)\theta_{,j}(t-s)]\\ &+\left(K_{ij}[\theta(t-s)+\alpha\dot{\theta}(t-s)]\int_0^{t+s}\theta_{,j}(\xi)d\xi\right)_{,i}\\ &-\left(K_{ij}[\theta(t+s)+\alpha\dot{\theta}(t+s)]\int_0^{t-s}\theta_{,j}(\xi)d\xi\right)_{,i}.\end{aligned}$$

We now substitute (34) into (33) and we use the boundary conditions (10) in order to obtain the following identity

$$\begin{aligned}(35)\quad &2\int_B[\varrho u_i(t)\dot{u}_i(t)+I_{ij}\varphi_i(t)\dot{\varphi}_j(t)]dV=\int_B[\varrho(u_i^0\dot{u}_i(2t)+\dot{u}_i^0u_i(2t))\\ &+I_{ij}(\varphi_i^0\dot{\varphi}_j(2t)+\dot{\varphi}_i^0\varphi_j(2t))]dV+\int_0^t\int_B(a-\varrho\eta^0)[\theta(t+s)-\theta(t-s)\\ &+\alpha(\dot{\theta}(t+s)-\dot{\theta}(t-s))]dV\,ds+\int_0^t\int_B[(d\alpha-h)\frac{d}{ds}(\theta(t+s)\theta(t-s))\end{aligned}$$

$$+\frac{d}{ds}\left(K_{ij}\int_0^{t+s}\theta_{,i}(\xi)d\xi\int_0^{t-s}\theta_{,j}(\xi)d\xi\right)+\alpha K_{ij}\theta_{,i}(t+s)\int_0^{t-s}\theta_{,j}(\xi)d\xi$$

$$+\alpha K_{ij}\theta_{,i}(t-s)\int_0^{t+s}\theta_{,j}(\xi)d\xi]dVds.$$

Using the initial conditions (13) in (35), we arrive at the desired result (29) and Theorem 4.3 is proved.

5. Asymptotic equipartition of energy

In this section we shall use the identities (22), (25) and (29) and under the hypotheses made in Section 2 we establish the asymptotic partition of total energy. First, we introduce the Cesaro means of various energies contained in (22). Thus, we define

$$\mathcal{K}_C(t)\equiv\frac{1}{2t}\int_0^t\int_B[\varrho\dot{u}_i(s)\dot{u}_i(s)+I_{ij}\dot{\varphi}_i(s)\dot{\varphi}_j(s)]dVds,$$

$$\mathcal{L}_C(t)\equiv\frac{1}{2t}\int_0^t\int_B[A_{ijmn}\varepsilon_{ij}(s)\varepsilon_{mn}(s)+2B_{ijmn}\varepsilon_{ij}(s)\gamma_{mn}(s)$$

$$+C_{ijmn}\gamma_{ij}(s)\gamma_{mn}(s)]dVds,\mathcal{P}_C(t)\equiv\frac{1}{2t}\int_0^t\int_B\alpha Kij\theta_{,i}(s)\theta_{,j}(s)dVds, \tag{36}$$

$$\mathcal{T}_C(t)\equiv\frac{1}{2t}\int_0^t\int_B d\theta^2(s)dVds,\quad \mathcal{T}_{KC}(t)\equiv\frac{1}{2t}\int_0^t\int_B\alpha h\dot{\theta}^2(s)dVds,$$

$$\mathcal{S}_C(t)\equiv\frac{1}{2t}\int_0^t\int_0^s\int_B[K_{ij}\theta_{,i}(\xi)+(d\alpha-h)\dot{\theta}^2(\xi)]dVd\xi ds.$$

We are now in a position to state and proof the main result of our study.

THEOREM 5.1 *We assume that the hypotheses from the Section 2 hold. Then, for all choices of initial data*

$$(u_i^0,\varphi_i^0)\in\mathbf{W}_1(B),\quad(\dot{u}_i^0,\dot{\varphi}_i^0)\in\mathbf{W}_0(B),\quad\theta^0\in W_1(B),\quad\dot{\theta}^0\in W_0(B),$$

we have

$$\lim_{t\to\infty}\mathcal{P}_C(t)=0,\quad\lim_{t\to\infty}\mathcal{T}_{KC}(t)=0. \tag{37}$$

Moreover, the following assertions hold

(i) *if meas* $\partial B_3\neq0$, *then*

$$\lim_{t\to\infty}\mathcal{T}_C(t)=0; \tag{38}$$

(ii) *if meas* $\partial B_2 = 0$, *then*

$$\lim_{t\to\infty} \mathcal{T}_C(t) = \frac{1}{2}\int_B \frac{1}{d}(d\theta^* + h\dot{\theta}^*)dV; \tag{39}$$

(iii) *if meas* $\partial B_1 \neq 0$ *and meas* $\partial B_2 \neq 0$, *then*

$$\lim_{t\to\infty} \mathcal{K}_C(t) = \lim_{t\to\infty} \mathcal{L}_C(t), \tag{40}$$

$$\lim_{t\to\infty} \mathcal{S}_C(t) = \mathcal{E}(0) - 2\lim_{t\to\infty} \mathcal{K}_C(t) = \mathcal{E}(0) - 2\lim_{t\to\infty} \mathcal{L}_C(t); \tag{41}$$

(*iv*) *if meas* $\partial B_1 = 0$ *and meas* $\partial B_2 = 0$, *then*

$$\lim_{t\to\infty} \mathcal{K}_C(t) = \lim_{t\to\infty} \mathcal{L}_C(t) + \frac{1}{2}\int_B [\varrho \dot{u}_i^* \dot{u}_i^* + I_{ij}\dot{\varphi}_i^*\dot{\varphi}_j^*]dV, \tag{42}$$

$$\begin{aligned}\lim_{t\to\infty} \mathcal{S}_C(t) &= \mathcal{E}(0) - 2\lim_{t\to\infty} \mathcal{K}_C(t) + \frac{1}{2}\int_B [\varrho \dot{u}_i^* \dot{u}_i^* + I_{ij}\dot{\varphi}_i^*\dot{\varphi}_j^*]dV \\ &= \mathcal{E}(0) - 2\lim_{t\to\infty} \mathcal{L}_C(t) - \frac{1}{2}\int_B [\varrho \dot{u}_i^* \dot{u}_i^* + I_{ij}\dot{\varphi}_i^*\dot{\varphi}_j^*]dV.\end{aligned} \tag{43}$$

Proof. We use the energy conservation law (22) and the hypotheses from the Section 2 in order to prove the relations (37). Thus, by the hypotheses (9), we have

$$\begin{aligned} & d\theta^2(t) + \alpha h\dot{\theta}^2(t) + 2h\theta(t)\dot{\theta}(t) \\ &= \frac{1}{d}\Big(d\theta(t) + h\dot{\theta}(t)\Big)^2 + \frac{h}{d}(d\alpha - h)\dot{\theta}^2(t) \\ &= \frac{h}{\alpha}\Big(\theta(t) + \alpha\dot{\theta}(t)\Big)^2 + \frac{1}{\alpha}(d\alpha - h)\theta^2(t) \geq 0. \end{aligned} \tag{44}$$

Now, by (36) and (22), we get

$$\mathcal{T}_{KC}(t) \leq \frac{1}{2t}h\alpha\Big(d\alpha - h\Big)^{-1}\mathcal{E}(0), \tag{45}$$

$$\mathcal{P}_C(t) \leq \frac{\alpha}{2t}\mathcal{E}(0). \tag{46}$$

Letting $t \to \infty$ into (45) and (46), we deduce the relations (37).

(i) Suppose that *meas* $\partial B_3 \neq 0$. It is easy to prove that $\theta \in \hat{W}_1(B)$. Then by using the Poincare's inequality (19) and the identity (22), we get

$$\int_0^t\int_B d\theta^2(s)dVds \leq \frac{d}{m_2}\int_0 t^0 \int_B K_{ij}\theta_{,i}(s)\theta_{,j}(s)dVds \leq \frac{d}{m_2}\mathcal{E}(0). \tag{47}$$

From the relation (47) and by (36), we obtain the conclusion (38).

(ii) We now suppose that *meas* $\partial B_3 = 0$. We use the decomposition (21) and the fact that $\chi \in \hat{W}_1(B)$ in order to obtain the following identity

$$(48)\qquad \int_B \theta^2(t)dV = \int_B (\theta^* + \frac{h}{d}\dot{\theta}^*)^2 dV + \int_B \chi^2(t)dV$$
$$-2\int_B \frac{h}{d}\Big(\theta^* + \frac{h}{d}\dot{\theta}^*\Big)\dot{\theta}^* \exp\Big(-\frac{dt}{h}\Big)dV + \int_B \frac{h^2}{d^2}(\dot{\theta}^*)^2 \exp\Big(-2\frac{dt}{h}\Big)dV.$$

In view of (48) and (36), we get

$$(49)\qquad \mathcal{T}_C(t) = \frac{1}{2}\int_B \frac{1}{d}(d\theta^* + h\dot{\theta}^*)^2 dV + \frac{1}{2t}\int_0^t\int_B d\chi^2(s)dVds$$
$$-\frac{1}{t}[1-\exp(-\frac{d}{h}t)]\int_B \frac{h^2}{d^2}\dot{\theta}^*(d\theta^* + h\dot{\theta}^*)dV$$
$$+\frac{1}{4t}\Big[1-\exp(-2\frac{d}{h}t)\Big]\int_B \frac{h^3}{d^2}(\dot{\theta}^*)^2 dV.$$

From the Poincare's inequality (19), the identity (22) and the fact that $\chi \in \hat{W}_1(B)$, we get

$$(50)\qquad \frac{1}{2t}\int_0^t\int_B d\chi^2(s)dVds \le \frac{d}{2tm_2}\int_0^t\int_B K_{ij}\chi_{,i}(s)\chi_{,j}(s)dVds$$
$$= \frac{d}{2tm_2}\int_0^t\int_B K_{ij}\theta_{,i}(s)\theta_{,j}(s)dVds \le \frac{d}{2tm_2}\mathcal{E}(0).$$

Letting $t \to \infty$ in (49) and taking into account (9*) and (50), we arrive to (40). We now use the relation (44), the energy conservation law (22) and the hypotheses from Section 2 in order to obtain the following estimates

$$(51)\qquad \int_B \theta^2(t)dV \le 2\alpha\frac{1}{d\alpha - h}\mathcal{E}(0), \qquad t \in [0,\infty),$$

$$(52)\qquad \int_B [\varrho\dot{u}_i(t)\dot{u}_i(t) + I_{ij}\dot{\varphi}_i(t)\dot{\varphi}_j(t)]dV \le 2\mathcal{E}(0), \qquad t \in [0,\infty),$$

$$(53)\qquad \int_0^t\int_B K_{ij}\theta_{,i}(\tau)\theta_{,j}(\tau)dVd\tau \le \mathcal{E}(0), \qquad t \in [0,\infty),$$

$$(54)\qquad \int_0^t\int_B \dot{\theta}^2(\tau)dV \le \frac{1}{d\alpha - h}\mathcal{E}(0), \qquad t \in [0,\infty).$$

On the other hand, the identities (25) and (29) imply

$$(55)\quad \frac{1}{2t}\int_0^t\int_B[\varrho\dot{u}_i(s)\dot{u}_i(s)+I_{ij}\dot{\varphi}_i(s)\dot{\varphi}_j(s)$$
$$-A_{ijmn}\varepsilon_{ij}(s)\varepsilon_{mn}(s)-2B_{ijmn}\varepsilon_{ij}(s)\gamma_{mn}(s)-C_{ijmn}\gamma_{ij}(s)\gamma_{mn}(s)]dVds$$
$$=\frac{1}{2t}\int_0^t\int_B[d\theta^2(s)+2h\theta(s)\dot{\theta}(s)-\alpha h\dot{\theta}^2(s)-\alpha K_{ij}\theta_{,i}(s)\theta_{,i}(s)]dVds$$
$$-\frac{1}{2t}\int_B[\varrho u_i^0\dot{u}_i^0+I_{ij}\varphi_i^0\dot{\varphi}_j^0]dV-\frac{1}{4t}\int_B(d\alpha-h)(\theta^0)^2dV$$
$$+\frac{1}{4t}\int_B[\varrho(\dot{u}_i^0u_i(2t)+u_i^0\dot{u}_i(2t))+I_{ij}(\dot{\varphi}_i^0\varphi_j(2t)+\varphi_i^0\dot{\varphi}_j(2t))]dV$$
$$+\frac{1}{2t}\int_0^t\int_B(a-\varrho\eta^0)(\theta(s)+\alpha\dot{\theta}(s))dVds+\frac{1}{4t}\int_B(d\alpha-h)\theta^0\theta(2t)dV$$
$$+\alpha K_{ij}\theta_{,i}^0\int_0^{2t}\theta_{,j}(\xi)d\xi+\frac{1}{4t}\int_0^t\int_B(a-\varrho\eta^0)[\theta(t+s)$$
$$-\theta(t-s)+\alpha\frac{d}{ds}(\theta(t+s)+\theta(t-s))]dVds.$$

In view of definitions (36) and the initial conditions (13), from (55) it results

$$(56)\quad \mathcal{K}_C(t)-\mathcal{L}_C(t)=\frac{1}{4t}\int_B[(d\alpha-h)(\theta^0+\alpha(a-\varrho\eta^0)][\theta(2t)-\theta^0]dV$$
$$+\frac{1}{4t}\int_0^{2t}\int_B\alpha K_{ij}\theta_{,i}^0\theta_{,j}(s)dVds+\frac{1}{t}\int_0^t\int_B h\theta(s)\dot{\theta}(s)dVds$$
$$-\frac{1}{2t}\int_B[\varrho u_i^0\dot{u}_i^0+I_{ij}\varphi_i^0\dot{\varphi}_j^0]dV+\mathcal{T}_{KC}(t)-\mathcal{P}_C(t)$$
$$+\frac{1}{4t}\int_B[\varrho(\dot{u}_i^0u_i(2t)+u_i^0\dot{u}_i(2t))+I_{ij}(\dot{\varphi}_i^0\varphi_j(2t)+\varphi_i^0\dot{\varphi}_j(2t))]dV$$
$$+\mathcal{T}_C(t)+\frac{1}{4t}\int_0^t\int_B(a-\varrho\eta^0)[\theta(t+s)+\theta(s)]dVds.$$

Now we shall use the Schwarz and Cauchy inequalities on the right-hand side of the identity (56). We use the relations (45)–(47), (51)–(54) and thus we get

$$(57)\quad \left| -\frac{1}{2t}\int_B [\varrho u_i^0 \dot{u}_i^0 + I_{ij}\varphi_i^0\dot{\varphi}_j^0]dV \right|$$
$$\leq \frac{1}{4t}\int_B [\varrho(u_i^0u_i^0 + \dot{u}_i^0\dot{u}_i^0) + I_{ij}(\varphi_i^0\varphi_j^0 + \dot{\varphi}_i^0\dot{\varphi}_j^0)]dV;$$
$$\left|\frac{1}{4t}\int_B [(d\alpha - h)(\theta^0 + \alpha(a - \varrho\eta^0))][\theta(2t) - \theta^0]dV\right|$$
$$\leq \frac{1}{8t}\int_B [[(d\alpha - h)(\theta^0 + \alpha(a-\varrho\eta^0))]^2 + 2(\theta^0)^2]dV + \frac{\alpha}{2t(d\alpha - h)}\mathcal{E}(0)$$
$$\left|\frac{1}{4t}\int_0^{2t}\int_B \alpha K_{ij}\theta_{,i}^0\theta_{,j}(s)dVds\right| \leq \frac{1}{4t}\left(\int_0^{2t}\int_B \alpha K_{ij}\theta_{,i}^0\theta_{,j}^0 dVds\right)^{\frac{1}{2}}$$
$$\times\left(\int_0^{2t}\int_B \alpha K_{ij}\theta_{,i}(s)\theta_{,j}(s)dVds\right)^{\frac{1}{2}} \leq \frac{1}{2}\left(\frac{\alpha}{2t}\mathcal{E}(0)\int_B \alpha K_{ij}\theta_{,i}^0\theta_{,j}^0 dVds\right)^{\frac{1}{2}}$$
$$\left|\frac{1}{t}\int_0^{t}\int_B h\theta(s)\dot{\theta}(s)dVds\right| \leq \frac{1}{t}\left(\int_0^{t}\int_B h\theta^2(s)dVds\right)^{\frac{1}{2}}$$
$$\times\frac{1}{t}\left(\int_0^{t}\int_B h\dot{\theta}^2(s)dVds\right)^{\frac{1}{2}} \leq \left(\frac{2\alpha}{t}\right)^{\frac{1}{2}}\frac{h}{d\alpha - h}\mathcal{E}(0);$$
$$\left|\frac{1}{4t}\int_B [\varrho(\dot{u}_i^0u_i(2t) + u_i^0\dot{u}_i(2t)) + I_{ij}(\dot{\varphi}_i^0\varphi_j(2t) + \varphi_i^0\dot{\varphi}_j(2t))]dV\right|$$
$$\leq \frac{1}{8t}\int_B [\varrho u_i^0u_i^0 + I_{ij}\varphi_i^0\varphi_j^0]dV + \frac{1}{4t}\mathcal{E}(0).$$

(iii) Assume that *meas* $\partial B_1 \neq 0$ and *meas* $\partial B_2 \neq 0$. Since $(u_i, \varphi_i) \in \hat{\mathbf{W}}_1(B)$, from (6), (18), (22) it results for $\tau \in [0, \infty)$

$$(58)\quad \int_B [\varrho u_i(\tau)u_i(\tau) + I_{ij}\varphi_i(\tau)\varphi_j(\tau)]dV \leq \frac{k}{m_1}\int_B [A_{ijmn}\varepsilon_{ij}(\tau)\varepsilon_{mn}(\tau)$$
$$+ 2B_{ijmn}\varepsilon_{ij}(\tau)\gamma_{mn}(\tau) + C_{ijmn}\gamma_{ij}(\tau)\gamma_{mn}(\tau)]dV \leq \frac{2k}{m_1}\mathcal{E}(0),$$

and we obtain

$$(59)\quad \left|\frac{1}{4t}\int_B [\varrho(\dot{u}_i^0u_i(2t) + u_i^0\dot{u}_i(2t)) + I_{ij}(\dot{\varphi}_i^0\varphi_j(2t) + \varphi_i^0\dot{\varphi}_j(2t))]dV\right|$$

$$\leq \frac{1}{8t}\int_B [\varrho \dot{u}_i^0 \dot{u}_i^0 + I_{ij}\dot{\varphi}_i^0\dot{\varphi}_j^0]dV + \frac{k}{4tm_1}\mathcal{E}(0).$$

If we suppose that *meas* $\partial B_3 \neq 0$, then we have

$$(60)\quad \left|\mathcal{T}_C(t) + \frac{1}{4t}\int_0^t\int_B (a-\varrho\eta^0)[\theta(t+s)+\theta(s)]dVds\right| \leq \mathcal{T}_C(t)$$

$$+\frac{1}{4t}\left(\int_0^t\int_B (a-\varrho\eta^0)^2 dVds\right)^{\frac{1}{2}}\left(\int_0^t\int_B [\theta(t+s)+\theta(s)]^2 dVds\right)^{\frac{1}{2}}$$

$$\leq \mathcal{T}_C(t) + \left(\frac{1}{2d}\int_B (a-\varrho\eta^0)^2 dV\right)^{\frac{1}{2}}[\mathcal{T}_C(2t)]^{\frac{1}{2}}.$$

Letting $t \to \infty$ in (56) and taking into account the estimates (57), (59), (60) and the relations (37), (38), we conclude that the relation (40) holds. Next we suppose that *meas* $\partial B_3 = 0$. If we use the decompositions (16), (21), the relations (17), (49) and the expresion of η^0 (from Theorem 4.2), we conclude that the following identity holds

$$(61)\quad \mathcal{T}_C(t) + \frac{1}{4t}\int_0^t\int_B (a-\varrho\eta^0)[\theta(t+s)+\theta(s)]dVds$$

$$= -\frac{1}{4t}\int_B \frac{h^2}{d^2}\dot{\theta}^*\left(d\theta^* + \frac{3}{2}h\dot{\theta}^*\right)\left[\exp\left(-2\frac{dt}{h}\right) - 1\right]dV$$

$$+\frac{1}{t}\int_B \frac{h^2}{d^2}\dot{\theta}^*(d\theta^* + h\dot{\theta}^*)\left[\exp\left(-\frac{dt}{h}\right) - 1\right]dV + \frac{1}{2t}\int_0^t\int_B d\chi^2(s)dVds$$

$$+\frac{1}{4t}\int_0^t\int_B [D_{ij}\varepsilon_{ij}^0 + E_{ij}\gamma_{ij}^0 - dT^0 - h\dot{T}^0][\chi(t+s)+\chi(s)]dVds.$$

Now, by using the Schwarz and Cauchy inequalities in (61) and taking into account the relation (51), we get

$$(62)\quad \lim_{t\to\infty}\left\{\mathcal{T}_C(t) + \frac{1}{4t}\int_0^t\int_B (a-\varrho\eta^0)[\theta(t+s)+\theta(s)]dVds\right\} = 0.$$

It is easy to see that the use of relations (37), (57), (59), (62) in (56) implies again the conclusion (40). Also, it is a simple matter to see that the relation (41) is obtained from (22) by taking the Cesaro mean and by using the relations (37), (38) and (40).

(iv) If *meas* $\partial B_1 = 0$ and *meas* $\partial B_2 = 0$, then we use the decomposition (20), the relations (14), (15) and the fact that $(u_i, \varphi_i) \in \hat{\mathbf{W}}_1(B)$ in order to obtain

$$(63)\quad \frac{1}{4t}\int_B [\varrho u_i^0 \dot{u}_i(2t) + I_{ij}\varphi_i^0\dot{\varphi}_j(2t)]dV = \frac{1}{4t}\int_B [\varrho u_i^* \dot{u}_i^* + I_{ij}\varphi_i^*\dot{\varphi}_j^*]dV$$
$$+\frac{1}{2}\int_B [\varrho \dot{u}_i^* \dot{u}_i^* + I_{ij}\dot{\varphi}_i^*\dot{\varphi}_j^*]dV + \frac{1}{4t}\int_B [\varrho \dot{U}_i^0 v_i(2t) + I_{ij}\dot{\Phi}_i^0\psi_j(2t)]dV.$$

Also, since $(v_i, \psi_i) \in \hat{\mathbf{W}}_1(B)$, the Korn inequality (18) leads to the relation

$$(64)\quad \frac{1}{4t}\int_B [\varrho v_i(\tau)v_i(\tau) + I_{ij}\psi_i(\tau)\psi_j(\tau)]dV \le \frac{k}{m_1}\int_B [A_{ijmn}\bar{\varepsilon}_{ij}(\tau)\bar{\varepsilon}_{mn}(\tau)$$
$$+ 2B_{ijmn}\bar{\varepsilon}_{ij}(\tau)\bar{\gamma}_{mn}(\tau) + C_{ijmn}\bar{\gamma}_{ij}(\tau)\bar{\gamma}_{mn}(\tau)]dV$$
$$= \frac{k}{m_1}\int_B [A_{ijmn}\varepsilon_{ij}(\tau)\varepsilon_{mn}(\tau) + 2B_{ijmn}\varepsilon_{ij}(\tau)\gamma_{mn}(\tau)$$
$$+ C_{ijmn}\gamma_{ij}(\tau)\gamma_{mn}(\tau)]dV \le \frac{2k}{m_1}\mathcal{E}(0), \tau \in [0,\infty),$$

where $\bar{\varepsilon}_{ij} = v_{j,i} + \varepsilon_{jik}\psi_k, \bar{\gamma}_{ij} = \psi_{j,i}$. Leting $t \to \infty$ in (56) and by means of relations (37), (57), (60), (63) and (64) we arrive to the conclusion (42).

Finally, the relation (43) is proved on the basis of (22) by taking the Cesaro mean and by using the relations (37), (38), (42), (57). The proof of Theorem 5.1 is complete.

At last we remark that the relations (40) and (42), restricted to the class of initial data for which $\dot{u}_i^* = \dot{\varphi}_i^* = 0$, prove the asymptotic equipartition in mean of the kinetic and strain energies.

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