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ON THE SIMPLICITY OF AN AXIOM SYSTEM FOR PLANE EUCLIDEAN GEOMETRY

The author has provided in [2] an axiom system in Tarski's [7] first order language L_{BD}[1] for plane Euclidean geometry coordinatized by Euclidean ordered fields, a theory we shall denote by $\mathcal{E}_2'$. All the axioms were statements which, when written in prenex form, contained at most 5 variables. It was stated in [2] that there is no axiom system for $\mathcal{E}_2'$, all of whose axioms have, when written in prenex form, fewer than 5 variables. The proof given in [2] for this fact is flawed. The aim of this note is to provide a valid proof of it.

Let, as in [2], $\mathcal{T} := Cn(\{\varphi \mid \varphi \in \mathcal{E}_2' \cap \mathrm{L}_4, \varphi \text{ is written in prenex form}\})$, where L_4 stands for the language that contains the same symbols as L_{BD}, except that there are not countably many, but only 4 individual variables.

In the proof given in [2], we stated that $\mathcal{T} \subset Cn(\mathcal{T}_1' \cup S)$. This is false, since the circle axiom may also be expressed by a 4-variable sentence, namely $(\forall abc)(\exists d)\,[B(abc) \rightarrow da \equiv db \wedge ad \equiv ac]$, therefore Szczerba's [6] model of independence for the Pasch axiom is not a model of $\mathcal{T}$. The result is nevertheless true, i. e. $\mathcal{T} \neq \mathcal{E}_2'$, that is the simplicity degree of $\mathcal{E}_2'$ is indeed 5. The idea of the proof is to show that $\mathcal{T}$ is a subtheory of a certain plane geometry in which the congruence relation is not transitive.

The model for this geometry with a non-transitive congruence relation is the plane over the field of real numbers, with the usual affine and order structures, but with a congruence relation that is strictly included in the usual congruence relation of the Cartesian plane over the reals.

Let $\mathcal{C}_2(\mathbb{R}) := \langle \mathbb{R} \times \mathbb{R}, \mathrm{B}_{\mathbb{R}}, \equiv_{\mathbb{R}} \rangle$ be the Cartesian plane over the real numbers, i. e. B and D will have the usual interpretation of "Betweenness"

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[1] Here B is a ternary and D a quaternary relation, with $B(abc)$ to be read as 'point b lies between points a and c' and $D(abcd)$ to be read as 'the segment ab is congruent to the segment cd'; the individual variables of L_{BD} are to be interpreted as 'points'.

and "Equidistance" (we wrote $\equiv_{\mathbb{R}}$ instead of $D_{\mathbb{R}}$ for improved readability). Let $\mathbf{L}_{\mathbb{R}}(\mathbf{abc})$ stand for "$\mathbf{a}, \mathbf{b}, \mathbf{c}$ are three collinear points in the standard Euclidean plane".

Let $\mathfrak{M} = \langle \mathbb{R} \times \mathbb{R}, \mathbf{B}_{\mathbb{R}}, \equiv_{\mathfrak{M}} \rangle$, where $\mathbf{a}_1\mathbf{a}_2 \equiv_{\mathfrak{M}} \mathbf{a}_3\mathbf{a}_4$ iff $\mathbf{a}_1\mathbf{a}_2 \equiv_{\mathbb{R}} \mathbf{a}_3\mathbf{a}_4$ and one of the following is true

(i) $\mathbf{a}_i\mathbf{a}_j$ is parallel to $\mathbf{a}_k\mathbf{a}_l$, for some $\{i, j, k, l\} = \{1, 2, 3, 4\}$;

(ii) $\mathbf{L}_{\mathbb{R}}(\mathbf{a}_i\mathbf{a}_j\mathbf{a}_k)$ for some i, j, k with $i \neq j \wedge j \neq k \wedge k \neq i$ and $i, j, k \in \{1, 2, 3, 4\}$;

(iii) $\mathbf{a}_i\mathbf{a}_j \equiv_{\mathbb{R}} \mathbf{a}_i\mathbf{a}_k$ for some i, j, k with $i \neq j \wedge j \neq k \wedge k \neq i$ and $i, j, k \in \{1, 2, 3, 4\}$;

(iv) the measure of one of the angles between $\mathbf{a}_1\mathbf{a}_2$ and $\mathbf{a}_3\mathbf{a}_4$ is $\frac{\pi}{n}$ for some $n \in \mathbb{N} \setminus \{0\}$.

Let $\mathcal{C} = Th_{\mathrm{L}_{BD}}(\mathfrak{M}) \cap \mathcal{E}_2'$.

A *quantifier type* (or *q-type*) is defined in [1, p. 402] to mean a finite sequence of the letters A and E. The *complement* σ^* of a q-type σ is the q-type obtained from σ by switching A and E. A q-type σ_1 is *simpler* than a q-type σ_2 if the former is a subsequence of the latter.

We now define the relation "the formula φ has q-type σ" as a least relation such that: (i) if φ has q-type σ and σ is simpler than σ', then φ has q-type σ'; (ii) quantifier-free formulas have empty q-type; (iii) if φ has q-type σ, then $\neg\varphi$ has q-type σ^*, $(\forall x)\varphi$ has q-type $A\sigma$, and $(\exists x)\varphi$ has q-type $E\sigma$; (iv) if φ and ψ have q-type σ, then $\varphi \vee \psi$ has q-type σ.

Let $\mathcal{T}_0 := Cn(\{\varphi \mid \varphi \in \mathcal{E}_2', \varphi$ has a q-type of length $4\})$. We obviously have $\mathcal{T} \subseteq \mathcal{T}_0$. We shall prove that $\mathcal{T} \neq \mathcal{E}_2'$ by first proving the stronger result $\mathcal{T}_0 \neq \mathcal{E}_2'$, which follows from

THEOREM. $\mathcal{T}_0 \subseteq \mathcal{C}$.

Proof. In order to prove that there is no sentence $\sigma \in \mathcal{T}_0 \setminus \mathcal{C}$ we shall use the model-theoretic method of Ehrenfeucht-Fraïssé games, as described in [1].

The method given there allows us to prove that a certain sentence $\sigma \in \mathcal{E}_2' \setminus \mathcal{C}$ is not equivalent (with respect to $\mathcal{C}$) to a sentence having a q-type of length at most 4. For each q-type of length 4 the game method allows us to prove that *no* sentence $\sigma \in \mathcal{E}_2' \setminus \mathcal{C}$ is equivalent to one of that particular q-type.

Let σ be any sentence in $\mathcal{E}_2' \setminus \mathcal{C}$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two models of $\mathcal{C}$, such that $\mathfrak{A} \models \sigma$, but $\mathfrak{B} \not\models \sigma$ (for example, let $\mathfrak{A}$ be $\mathfrak{C}_2(\mathbb{R})$ and let $\mathfrak{B}$ be the plane $\mathfrak{M}$ used to define $\mathcal{C}$). The Ehrenfeucht-Fraïssé game, used to prove that σ is not $\mathcal{C}$-equivalent to a sentence having a certain q-type of length 4, can be described as follows:

In this game, there are two players, I and II, that alternate in making choices from the two models $u(\mathfrak{A})$ and $u(\mathfrak{B})$, (it depends on the prefix which set a player is supposed to chose from at the n^{th} move; a universal quantifier in the n^{th} position forces I to choose from $u(\mathfrak{B})$, an existential one forces I to choose from $u(\mathfrak{A})$)). The choice of I at the n^{th} move will be denoted by $\mathbf{x}_n$, the choice of II at the n^{th} move by $\mathbf{y}_n$. Let $\{\mathbf{a}_n\} = \{\mathbf{x}_n, \mathbf{y}_n\} \cap u(\mathfrak{A})$ and $\{\mathbf{b}_n\} = \{\mathbf{x}_n, \mathbf{y}_n\} \cap u(\mathfrak{B})$. Player II wins the game, which in our case consists of 4 moves, if at the end of the game the function f, defined by $f(\mathbf{a}_n) = \mathbf{b}_n$ is a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. The fact that σ is not $\mathcal{C}$-equivalent to a sentence of q-type of length 4 is equivalent to the existence of a winning strategy for II in the corresponding game.

Let $\mathfrak{A}$ be $\mathfrak{C}_2(\mathbb{R})$ and let $\mathfrak{B}$ be $\mathfrak{M}$. The winning strategy for player II is: Choose for the first three moves points with coordinates identical to those chosen by I. By abuse of language we shall denote these first three moves by the same letters.

In the fourth move, if

(i) II has to chose from $u(\mathfrak{B})$,

(ii) the first three points chosen are not collinear,

(iii) I has made the fourth choice such that the distance from $\mathbf{x}_4$ to $\mathbf{v}$, one of the vertices of the triangle Δ formed by the first three ('common' choices), is congruent (in the standard plane) to $\mathbf{ab}$, that side of the triangle which does not pass through $\mathbf{v}$, but the distance from $\mathbf{x}_4$ to any other vertex of the triangle is not congruent to any other side of Δ, then II chooses $\mathbf{y}_4$ such that

(1) $\mathbf{vy}_4$ is congruent to $\mathbf{ab}$ (in the standard plane),

(2) the angle between $\mathbf{vy}_4$ and $\mathbf{ab}$ is $\frac{\pi}{n}$ for some $n \in \mathbb{N} \setminus \{0\}$,

(3) none of the distances $\mathbf{y}_4\mathbf{a}$ or $\mathbf{y}_4\mathbf{b}$ is equal to any of the sides of the triangle Δ.

Otherwise, i. e. if in the fourth move we are not in the situation described by (i), (ii) and (iii), choose the fourth point to have the same coordinates as the point chosen by player I. ∎

COROLLARY. *$\mathcal{T} \subsetneq \mathcal{C}$ and therefore $\mathcal{T} \neq \mathcal{E}_2'$.*

The problem whether there is a theory, synonymous with $\mathcal{E}_2'$, with simplicity degree lower than 5, which was left open in [2], has been answered in [3] and [4], where we give an axiom system for a theory synonymous with $\mathcal{E}_2'$ all of whose axioms are statements about at most 4 points. However, the language in which it is expressed contains function symbols. The question whether there is a theory synonymous with $\mathcal{E}_2'$, expressed in a language without function symbols, whose simplicity degree is 4 is still open. The fact that there is no theory, synonymous with $\mathcal{E}_2'$, with simplicity degree lower than 4, follows from [5].

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