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ALGEBRAIC PROPERTIES OF GENERALIZED RIGHT INVERTIBLE OPERATORS

0. Introduction

The theory of right invertible operators was started with works of D. Przeworska-Rolewicz [8]–[12] and then has been developed by M. Tasche [13]–[14], H. von Trotha [15], Z. Binderman [3] and many others (see [12]). The algebraic theory of generalized invertible operators was studied by P. M. Anselone and M. Z. Nashed [1], A. Ben-Israel and T. N. E. Greville [2], S. G. Caradus [4], M. Z. Nashed [5] and others (see [2]). However, the set of all generalized invertible operators is so large that, if we admit the axiom of choice, then every linear operator is generalized invertible [5]. Whereas, the generalized invertible operators do not satisfy desirable algebraic properties which the right invertible operators do (see [12]). For example, if a linear operator $V \in L(X)$ is generalized invertible and W is a generalized inverse of V, then there is not any general algorithm for constructing generalized inverses neither of V^n, $n \in \mathbb{N}$, nor of algebraic polynomials induced by V. Hence, there is the lack of effective methods to solve equations induced by algebraic polynomials with a generalized invertible operator.

In this paper, we introduce a so-called class of right invertible operators of degree $r \in \mathbb{N}$, and we denote its by $R_r(X)$. According to Definition 2, the set of all linear operators have been classified by means of the same invertible degree of operators (inclusion (2)). From classification (2) we can see that the set of all differential operators is exactly the first

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class. Then the paper mainly involves in studying second class $R_1(X)$ really containing not only all right invertible operators and some well-known operators in Analysis as Projections, Integral-Differential Operators (see Example 2 and Remark 4) but also a class of algebraic operators (see Examples 2-3 and Theorem 5). We obtain many important algebraic characterizations of operators in $R_1(X)$ like fundamental characterizations of operators in $R(X)$ (see Section 1). Especially, if $V \in R_1(X)$ and W is a generalized right inverse of V (such written $W \in \mathcal{R}_V^1$), then $V^n \in R_1(X)$ and $W^n \in \mathcal{R}_{V^n}^1$. Theorem 5 gives a sufficient and necessary criterion for an algebraic operator to be generalized right invertible and Theorem 6 indicates that the generalized right invertibility and the almost right invertibility are identical in the class of algebraic operators. Theorems 8, 9 generalize the Rolewicz and von Trotha theorems from class $R(X)$ to class $R_1(X)$. Lastly, we apply these results to solve corresponding equations induced by algebraic polynomials with a generalized right invertible operator (Section 4).

1. Generalized right invertible operators

Let X be a linear space over a field of scalars $\mathcal{F}$. Denote by $L(X)$ the set of all linear operators with domains and ranges in X and write

$$L_0(X) = \{A \in L(X) \colon \operatorname{dom} A = X\}.$$

The set of all right invertible operators in $L(X)$ will be denoted by $R(X)$. For a $D \in R(X)$ we denote by $\mathcal{R}_D$ the set of all right inverses of D, i.e.

$$\mathcal{R}_D = \{R \in L_0(X) \colon DR = I\}.$$

The theory of right invertible operators and its applications are presented in [12].

Definition 1, [7]. An operator $V \in L(X)$ is said to be generalized invertible (GI-operator), if there is a $W \in L(X)$ such that $\operatorname{Im} V \subset \operatorname{dom} W$, $\operatorname{Im} W \subset \operatorname{dom} V$ and $VWV = V$ on $\operatorname{dom} V$.

The set of all GI-operators in $L(X)$ will be denoted by $W(X)$. For a $V \in W(X)$ we denote by $\mathcal{W}_V$ the set of all generalized inverses in $L(X)$ of V.

Definition 2. An operator $V \in W(X)$ is said to be right invertible of degree $r \in \mathbb{N}$, if there is a $W \in \mathcal{W}_V$ such that

$$\operatorname{Im}(VW - I) \subset \ker V^r, \tag{1}$$

where we admit $V^0 = I$ for the case $r = 0$.

The set of all right invertible operators (in $L(X)$) of degree r will be denoted by $R_r(X)$.

Remark 1. (i) By Definitions 1 and 2, we have

$$R(X) = R_0(X) \subset R_1(X) \subset R_2(X) \subset \cdots \subset R_n(X) \subset W(X), \quad n = 0, 1, 2, \ldots; \tag{2}$$

(ii) $R(X)$ is the set of generalized right invertible operators of degree 0.

In this paper, we mainly deal with the set $R_1(X)$.

DEFINITION 3. Every $V \in R_1(X)$ is called a generalized right invertible operator (shortly: GR-invertible operator).

For a $V \in R_1(X)$ we denote by $\mathcal{R}_V^1$ the set of all generalized right inverses (shortly: GR-inverses) of V.

Remark 2. $V \in R_1(X)$ if and only if there exists a $W \in L(X)$ such that

$$VWV = V, \quad V^2W = V. \tag{3}$$

DEFINITION 4. If there is $W \in \mathcal{R}_V^1$ such that $\operatorname{Im} W \subset \ker(VW - I)$, then V is said to be almost right invertible and W is called an almost right inverse of V.

Denote by $R_{(1)}(X)$ the set of all almost right invertible operators and by $\mathcal{R}_V^{(1)}$ the set of all almost right inverses of $V \in R_{(1)}(X)$.

Remark 3. $V \in R_{(1)}(X)$ if and only if there exists a $W \in L(X)$ such that

$$VWV = V, \quad V^2W = V, \quad VW^2 = W. \tag{4}$$

In the sequel, the identities (3) and (4) will be used to check if the $V \in L(X)$ is generalized right invertible (or almost right invertible).

PROPOSITION 1. *Let $D \in R(X)$, $R \in \mathcal{R}_D$ and let $V = R^mD^n$, where $n \geq m$ and $n, m \in \mathbb{N}$. Then $V \in R_1(X)$. Moreover, if $n \geq 2m$, then there is an almost right inverse of V.*

Proof. Write $W_0 = R^{n-m}$, where we admit $R^0 = I$ for the case $n = m$. Since $R \in L_0(X)$, we conclude that $W_0 \in L_0(X)$. Using equalities $D^kR^k = I$, we find

$$V^2W_0 = R^mD^nR^mD^nR^{n-m} = R^mD^{n-m}D^m = R^mD^n = V \tag{5}$$

and

$$VW_0V = R^mD^nR^{n-m}R^mD^n = R^mD^n = V. \tag{6}$$

The identities (5), (6) together imply that $V \in R_1(X)$. On the other hand, for the case $n \geq 2m$ we have

$$VW_0^2 = R^m D^n R^{n-m} R^{n-m} = R^m D^m R^{n-m} = R^m D^m R^m R^{n-2m} = W_0.$$

Hence $V \in R_{(1)}(X)$ and $W_0 \in \mathcal{R}_V^{(1)}$. The proof is complete.

EXAMPLE 1. Let

$$X = \mathcal{C}([0,1],\mathcal{F}),\ D = \frac{d}{dt},$$

$$(Rx)(t) = \int_{t_0}^{t} x(s)ds,\ (Fx)(t) = x(t_0),\ t_0 \in [0,1].$$

Put $V = FD$. Then $V \not\equiv 0$ and $V^2 \equiv 0$. It is easy to see that $V \in W(X)$ by $R \in \mathcal{W}_V$. However, $V \notin R_r(X)$ for $r \in \{0,1\}$. Indeed, it is clear that $V \notin R(X)$, i.e. $V \notin R_0(X)$. If $V \in R_1(X)$ and $W \in \mathcal{R}_V$, then from $V^2W = V$ it follows $V \equiv 0$, which contradicts the condition $V \not\equiv 0$.

EXAMPLE 2. Consider a projection $P \in L_0(X)$, $P \not\equiv I$, i.e. $P^2 = P$. Evidently, $P \notin R(X)$. However, it is easy to see that $P \in R_{(1)}(X)$, by $P \in \mathcal{R}_P^{(1)}$.

Remark 4. If $m \neq 0$, then the operators V in Proposition 1 are not right invertible, since $V \in R_1(X) \setminus R(X)$. Moreover, there is a infinite set of integral-differential operators of the form $V = R^m D^n$ $(m, n \in \mathbb{N}, n \geq m > 0)$, which belongs to $R_1(X)$ (when this set does not belong to $R(X)$). Thus, we have the following inclusions

$$R(X) \subsetneq R_1(X) \subsetneq W(X). \tag{7}$$

PROPOSITION 2. *Let $V \in R_1(X)$ and $W \in \mathcal{R}_V^1$. Then for every $m, n \in \mathbb{N}$ we have*

$$V^n W^m = \begin{cases} V^{n-m} & \text{if } n > m \geq 1, \\ VW & \text{if } n = m \geq 2, \\ VW^{m-n+1} & \text{if } m > n \geq 1. \end{cases}$$

Proof. If $n = m \geq 2$, we find

$$\begin{aligned} V^m W^m &= V^{n-2}(V^2W)W^{n-1} = V^{n-1}W^{n-1} \\ &= V^2W^2 = (V^2W)W = VW. \end{aligned}$$

If $n > m \geq 1$, we have

$$V^n W^m = V^{n-m}(V^m W^m) = V^{n-m}(VW) = V^{n-m-1}(V^2W) = V^{n-m}.$$

Finally, for $m > n \geq 1$, we get

$$V^n W^m = (V^n W^n) W^{m-n} = (VW) W^{m-n} = VW^{m-n+1}.$$

The proof is complete.

PROPOSITION 3. *Let $V \in R_1(X)$ and $W \in \mathcal{R}_V^1$. Then $V^n \in R_1(X)$ for all $n \in \mathbb{N}$ and $W^n \in \mathcal{R}_{V^n}^1$, where we admit $V^0 = I$.*

Proof. The assumptions and Proposition 2 together imply the following equalities

$$V^n = (VWV)V^{n-1} = (VW)V^n = (V^n W^n)V^n = V^n W^n V^n,$$

i.e., $V^n \in W(X)$. On the other hand, also by Proposition 2, we have

$$V^n = V^{n+1}W = V^n(VW) = V^n(V^n W^n) = (V^n)^2 W^n.$$

Hence $V^n \in R_1(X)$ and $W^n \in \mathcal{R}_{V^n}^1$.

THEOREM 1. *Let $V \in R_1(X)$ and let $W_0 \in \mathcal{R}_V^1$. Then $W \in L(X)$ is a GR-inverse of V if and only if there is an $A \in L(X)$ such that* $\operatorname{Im} A \subset \ker V^2$ *and*

$$W = W_0 + A - W_0 V A V W_0. \tag{8}$$

Proof. Let W be of the form (8), where $\operatorname{Im} A \subset \ker V^2$ and $W_0 \in \mathcal{R}_V^1$. We have

$$VWV = V + VAV - VW_0VAVW_0V = V + VAV - VAV = V, \tag{9}$$

$$V^2 W = V^2 W_1 + V^2 A - V^2 W_1 V A V W_1 = V - V^2 A V A W_1 = V. \tag{10}$$

Equalities (9) and (10) together imply $W \in \mathcal{R}_V^1$.

Conversely, let $W_0, W \in \mathcal{R}_V^1$. Put: $A = W - W_0$. We find $V^2 A = V^2 W - V^2 W_0 = V - V = 0$, i.e. $\operatorname{Im} A \subset \ker V^2$. The equalities $VWV = V$ and $VW_0V = V$ together imply $V(W - W_0)V = 0$. Hence, we have

$$W = W_0 + (W - W_0) - W_0 V (W - W_0) V W_0 = W_0 + A - W_0 V A V W_0,$$

which gives the representation (8). Theorem is proved.

THEOREM 2. *Let be given $A, B \in L(X)$ such that* $\operatorname{Im} A \subset \operatorname{dom} B$, $\operatorname{Im} B \subset \operatorname{dom} A$. *Then $I + AB \in R_1(X)$ if and only if $I + BA \in R_1(X)$. Moreover, if $W_{AB} \in \mathcal{R}_{I+AB}^1$, then*

$$W = I - BW_{AB}A \in \mathcal{R}_{I+BA}^1. \tag{11}$$

Proof. Let $I + AB \in R_1(X)$ and $W_{AB} \in \mathcal{R}_{I+AB}^1$. Then $(I + AB)^2 W_{AB} = (I + AB)$ and W, defined by (11), is well-defined on X. From Theorem

10.3 in [7] it follows $I + BA \in W(X)$ and $W \in \mathcal{W}_{I+BA}$. On the other hand, we have the equalities

$$(I + BA)^2W = (I + BA)^2(I - BW_{AB}A) = (I + BA)^2 - (I + BA)^2BW_{AB}A$$
$$= (I + BA)^2 - B(I + AB)^2W_{AB}A = (I + BA)^2 - B(I + AB)A = I + BA$$

which imply that $W \in \mathcal{R}^1_{I+BA}$ and $I + BA \in R_1(X)$. The proof is complete.

PROPOSITION 4. *Let $V \in R_1(X)$ and $W_0 \in \mathcal{R}^1_V$. Write $W_1 = W_0VW_0$. Then $W_1 \in \mathcal{R}^1_V$ and $W_1VW_1 = W_1$.*

Proof. The assumptions $V^2W_0 = V$ and $VW_0V = V$ together imply following equalities

$$V^2W_1 = V^2W_0VW_0 = V(VW_0V)W_0 = V^2W_0 = V,$$
$$W_1VW_1 = W_0VW_0VW_0VW_0 = W_0(VW_0V)W_0VW_0$$
$$= W_0VW_0VW_0 = W_0VW_0 = V$$

which give $W_1 \in \mathcal{R}^1_V$ and $W_1VW_1 = W_1$.

In the sequel we write $\mathcal{R}^{(1,0)}_V = \{W \in \mathcal{R}^1_V: WVW = W\}$.

2. The generalized right invertibility of algebraic operators

Let $\mathcal{F} = \mathbf{C}$. We say that $A \in L_0(X)$ is algebraic if there exists a non-zero normed polynomial $P(t) = t^n + p_1t^{n-1} + \cdots + p_{n-1}t + p_n$ with coefficients in $\mathcal{F}$ such that $P(A) = 0$ on X. An algebraic operator A is called of order n, if there does not exist a normed polynomial $Q(t)$ of degree $m < n$ such that $Q(A) = 0$ on X. Such a minimal polynomial $P(t)$ is called characteristic polynomial of A and denoted by $P_A(t)$. The set of all algebraic operators in $L_0(X)$ will be denoted by $\mathcal{A}^0(X)$.

Let S be an algebraic operator in $L_0(X)$ with the characteristic polynomial of the form

$$P_S(t) = t^N + p_1t^{N-1} + \cdots + p_{N-1}t + p_N. \tag{12}$$

THEOREM 5. *Let S be an algebraic operator of order N in $L_0(X)$ with the characteristic polynomial $P_s(t)$ of the form (12). Then $S \in R_1(X)$ if and only if $|p_{N-1}| + |p_N| \neq 0$.*

Proof. Necessity. Let $S \in R_1(X)$ and $W \in \mathcal{R}^1_S$. Suppose that $|p_{N-1}| + |p_N| = 0$. It means that $p_{N-1} = p_N = 0$. From $P_S(S) = 0$ and $S^2W = S$ we have the following equalities

$$0 = P_S(S)W = (S^{N-2} + p_1S^{N-3} + \cdots + p_{N-2}I)S^2W$$
$$= (S^{N-2} + p_1S^{N-3} + \cdots + p_{N-2}I)S = S^{N-1} + p_1S^{N-2} + \cdots + p_{N-2}S,$$

which contradict the assumption that S is of order N.

Sufficiency. If $p_N \neq 0$, then S is invertible and it is right invertible and GR-invertible, simultaneously. So it is enough to deal with the case when $p_N = 0$ and $p_{N-1} \neq 0$, simultaneously.

(i) *The case* $N = 2$. Let $P_S(t) = t^2 - p_1 t$, $p_1 \neq 0$. It is clear that the operator $S = -p_1^{-1}S$ satisfies (3). Hence $S \in R_1(X)$ and $W = -p_1^{-1}S \in \mathcal{R}_S^1$.

(ii) *The case* $N \geq 3$. We set

$$W := p_{N-1}^{-2}\left(\sum_{k=0}^{N-2} p_{N-2}p_k S^{N-k-1} - \sum_{k=0}^{N-3} p_{N-1}p_k S^{N-k-2}\right), \tag{13}$$

where $p_0 = 1$.

We shall prove that $W \in \mathcal{R}_S^1$. Indeed, we have

$$\begin{aligned} SWS &= p_{N-1}^{-2}\left(\sum_{k=0}^{N-2} p_{N-2}p_k S^{N-k+1} - \sum_{k=0}^{N-3} p_{N-1}p_k S^{N-k}\right) \\ &= p_{N-1}^{-2}\left(p_{N-2}S\sum_{k=0}^{N-2} p_k S^{N-k} - p_{N-1}\sum_{k=0}^{N-3} p_k S^{N-k}\right) \\ &= p_{N-1}^{-2}\left(p_{N-2}S(P_S(S) - p_{N-1}S) - p_{N-1}\sum_{k=0}^{N-3} p_k S^{N-k}\right) \\ &= p_{N-1}^{-2}\left(-p_{N-1}p_{N-2}S^2 - p_{N-1}\sum_{k=0}^{N-3} p_k S^{N-k}\right) \\ &= -p_{N-1}^{-1}\left(P_S(S) - p_{N-1}S\right) = S. \end{aligned}$$

Hence $S \in W(X)$ and $W \in \mathcal{W}_S$. On the other hand, we also have $SW = WS$, which gives $S^2W = S$. The proof is complete.

THEOREM 6. *Let $S \in \mathcal{A}^0(X) \cap R_1(X)$. Then $S \in R_{(1)}(X)$ and there exists a unique almost right inverse of S.*

Proof. Let $P_S(t)$ be of the form (12). It follows from Theorem 5 that $|p_N| + |p_{N-1}| \neq 0$.

1. *The case* $p_N \neq 0$. It is clear that S is invertible. Hence $S \in R_{(1)}(X)$ and $S^{-1} \in \mathcal{R}_S^{(1)}$. Let $W \in \mathcal{R}_S^{(1)}$ be arbitrary. Since $S^2W = S$, we find $W = S^{-2}S^2W = S^{-2}S = S^{-1}$. So $W = S^{-1}$, i.e. S^{-1} is the unique almost right inverse of S.

2. *The case* $p_N = 0$ *and* $p_{N-1} \neq 0$, *simultaneously.*

(i) If $N = 1$, then it is trivial.

(ii) If $N = 2$, then it means that $S^2 = -p_1 S$. Evidently, the operator $W = p_1^{-2} S$ satisfies the identities $SWS = S$, $S^2 W = S$, $SW^2 = W$, so $S \in R_{(1)}(X)$ and $W = p_1^{-2} S \in \mathcal{R}_S^{(1)}$. Suppose that $W' \in \mathcal{R}_S^{(1)}$ is arbitrary. Then we find

$$\begin{aligned} W' &= SW'^2 = (-p_1^{-1} S^2) W'^2 = (-p_1^{-1})(S^2 W'^2) \\ &= (-p_1^{-1}) SW' = (-p_1^{-1})(-p_1^{-1} S^2) W' = p_1^{-2} S = W. \end{aligned}$$

(iii) Now let $N \geq 3$.

The existence. By the proof of Theorem 5, it is enough to check that the operator W defined by (13) satisfies the following identity $SW^2 = W$. Indeed, putting

$$W_* := p_{N-1}^{-2} \left(\sum_{k=0}^{N-2} p_{N-2} p_k S^{N-k-2} - \sum_{k=0}^{N-3} p_{N-1} p_k S^{N-k-3} \right),$$

we obtain $SW^2 = SWW = SW(SW_*) = (SWS)W_* = SW_* = W$. Thus, $S \in R_{(1)}(X)$ and $W \in \mathcal{R}_S^{(1)}$.

The uniqueness. Suppose that $W' \in \mathcal{R}_S^{(1)}$ is arbitrary. Prove that $W' = W$, where W is the operator defined by (13). Indeed, putting

$$P_*(S) = S^{N-2} + p_1 S^{N-3} + \ldots + p_{N-3} S + p_{N-2} I,$$

we have $S = -p_{N-1}^{-1} P_*(S) S^2$. Then, using Proposition 2, we find

$$\begin{aligned} W' &= SW'^2 = -p_{N-1}^{-1} P_*(S) S^2 W'^2 = -p_{N-1}^{-1} P_*(S) SW' \\ &= -p_{N-1}^{-1} P_*(S) \left(-p_{N-1}^{-1} P_*(S) S^2\right) W' \\ &= p_{N-1}^{-2} P_*(S) P_*(S) S^2 W' = p_{N-1}^{-2} P_*(S) P_*(S) S \\ &= p_{N-1}^{-2} \left(S^{N-2} + p_1 S^{N-3} + \ldots + p_{N-3} S + p_{N-2} I\right) P_*(S) S \\ &= p_{N-1}^{-2} \left[\left(S^{N-3} + p_1 S^{N-4} + \ldots + p_{N-3} I\right) P_*(S) S^2 + p_{N-2} P_*(S) S\right] \\ &= p_{N-1}^{-2} \left[\left(S^{N-3} + p_1 S^{N-4} + \ldots + p_{N-3} I\right)(-p_{N-1} S) + p_{N-2} P_*(S) S\right] \\ &= p_{N-1}^{-2} \left[p_{N-2} P_*(S) S - p_{N-1}(S^{N-2} + p_1 S^{N-3} + \ldots + p_{N-3} S)\right] = W. \end{aligned}$$

Thus $W' = W$. The proof is complete.

Remark 5. (i) It is well-known that the algebraic operator S is one-side invertible if and only if it is invertible (two-side). Theorem 6 indicates that the algebraic operator S is generalized right invertible (of degree 1) if and only if it is almost right invertible.

(ii) Suppose that S is the algebraic operator with $P_S(t)$ of the form (12). In particular, if $S \in R_1(X)$ but it is not invertible (corresponding to the case $p_N = 0$ and $p_{N-1} \neq 0$), then at least there are two generalized right inverses of S. Indeed, besides W, defined by (13), the operator

$$(14) \qquad T = -p_{N-1}^{-1}(S^{N-2} + p_1 S^{N-3} + \ldots + p_{N-3}S + p_{N-2}I)$$

is also generalized right inverse of S. In general, T is not almost right inverse of S. For example, consider the projection $P \in L_0(X)$ ($P \not\equiv I$). Then $P_P(t) = t^2 - t$. Operators W and T constructed by (13) and (14), respectively are both generalized right inverses of P, but I is not almost right inverse of P, since $PI^2 \not\equiv I$.

COROLLARY 2. *Let $S \in R_{(1)}(X)$ and $W \in \mathcal{R}_S^{(1)}$. Then $S \in \mathcal{A}^0(X)$ if and only if $W \in \mathcal{A}^0(X)$.*

Proof. Let $S \in \mathcal{A}^0(X)$. Then, by Theorem 6, W is uniquely determined by (13), since $W \in \mathcal{A}^0(X)$ (see [8]). Conversely, suppose that $W \in \mathcal{A}^0(X)$ and $P_W(t) = t^M + a_1 t^{M-1} + \cdots + a_M$. Then we find $S^{M+1}P_W(W) = 0$ and $S + a_1 S^2 + \cdots + a_M S^{M+1} = 0$, which gives $S \in \mathcal{A}^0(X)$.

EXAMPLE 3. Let $X = L_2(R^n)$. Consider Fourier's transform in X

$$(Fx)(t) = (2\pi)^{-n/2} \int_{R^n} \exp(-i(t,s))x(s)ds,$$

where $(t,s) = t_1 s_1 + t_2 s_2 + \ldots + t_n s_n$. We have $F \in L_0(X)$ and $F^4 = I$. Then F is invertible, since $F \in R_1(X)$. Furthermore, the following operators

$$(T_C x)(t) = (2\pi)^{-n/2} \int_{R^n} \cos(t,s)x(s)ds,$$

$$(T_S x)(t) = (2\pi)^{-n/2} \int_{R^n} \sin(t,s)x(s)ds$$

are algebraic with the characteristic polynomial $P_{T_C}(t) = P_{T_S}(t) = t^3 - t$ (see [10], p. 287). Thus, by Theorem 5, $T_C \in R_1(X)$ and $T_S \in R_1(X)$.

EXAMPLE 4. Let Γ be a closed Liapunov curve on the complex plane C. Consider the integral operator of Cauchy's type

$$(S\varphi)(t) = \frac{1}{\pi i} \int_{\Gamma} \frac{\varphi(\tau)}{\tau - t} d\tau.$$

If X is one of spaces $L_p(\Gamma)$, $H^\mu(\Gamma)$, then $S^2 = I$ (see [16]). It is easy to check that operators

$$P = \tfrac{1}{2}(I + S), \quad Q = \tfrac{1}{2}(I - S)$$

are the projections in X, since P, Q are the algebraic operators with characteristic polynomial $P_P(t) = P_Q(t) = t^2 - t$. Thus, $P, Q \in R_1(X)$. Moreover, consider operators

$$(S_{n,k}\varphi)(t) = \frac{1}{\pi i}\int_\Gamma \frac{t^{n-k-1}\tau^k}{\tau^n - t^n}\varphi(\tau)d\tau, \quad (M_n\varphi)(t) = \sum_{k=0}^{n-1} a_k S_{n,k},$$

where $a_k \in \mathcal{F}$, $k, n \in \mathbb{N}$, $0 \leq k \leq n$. We have (see [6], [7]) $S^3_{n,k} = S_{n,k}$, $M^3_n = M_n$. Hence, $S_{n,k}, M_n \in \mathcal{A}^0(X)$ with characteristic polynomial $P_{S_{n,k}}(t) = P_{M_n}(t) = t^3 - t$. Thus, by Theorem 5, $S_{n,k}$ and M_n are the generalized right invertible operators in X.

3. Volterra characterizations of GR-inverses

Let $A \in L_0(X)$. If the operator $I - \lambda A$ is invertible for all $\lambda \in \mathcal{F}$, then A is said to be a Volterra operator. The set of all Volterra operators in $L_0(X)$ will be denoted by $V(X)$. Let $\mathcal{F} = C$. Write

$$Q(t,s) := \sum_{k=0}^{N} q_k t^k s^{N-k} = \prod_{m=0}^{n}(t - t_m s)^{r_m}, \tag{15}$$

$$Q(t) := Q(t,1), \quad P(t) := t^M Q(t), \tag{16}$$

where $q_0, q_1, \ldots, q_{N-1} \in \mathcal{F}; q_N = 1; t_i \neq t_j$ for $i \neq j, r_0 + \cdots + r_n = N \geq 1$, $M \in \mathbb{N}$.

Recall the following result of D. Przeworska-Rolewicz and H. von Trotha.

THEOREM 7, [12]. *Let $D \in R(X)$ and let $R \in \mathcal{R}_D \cap V(X)$. Then $P(D) \in R(X)$ and $Q(I, R)$ is invertible. Moreover, $T := R^{N+M}(Q(I,R))^{-1} \in \mathcal{R}_{P(D)} \cap V(X)$. Conversely, if T is Volterra operator, then R is Volterra operator.*

In this Section, we generalize Theorem 7 for the case of GR-inverses.

THEOREM 8. *Let $V \in R_1(X), W \in \mathcal{R}^1_V \cap V(X)$ and let $Q(t,s), Q(t)$ and $P(t)$ be of the forms (15), (16). If $q_0 F^{(l)}_W \equiv 0$, then $Q := Q(I, W)$ is invertible and $P(V) \in R_1(X)$. Moreover,*

$$T = W^{N+M}Q^{-1} \in \mathcal{R}^1_{P(V)} \cap V(X). \tag{17}$$

Proof. The assumption $q_0 F^{(l)}_W \equiv 0$ means that $q_0 = 0$ or $F^{(l)}_W \equiv 0$.

(i) The case $F^{(l)}_W \equiv 0$. It follows that $V \in R(X)$. Thus, it is exactly the case of Theorem 7.

(ii) The case $q_0 = 0$. From $W \in V(X)$ it follows that the operator $I - \lambda W$ is invertible for all $\lambda \in \mathcal{F}$, since Q is invertible. By Proposition 2, we have $Q(V)W^N = VWQ$, i.e., $Q(V)W^N Q^{-1} = VW$. Hence, we find

$$\begin{aligned} P(V)TP(V) &= V^M Q(V) W^N W^M Q^{-1} P(V) = V^M VWQW^M Q^{-1} P(V) \\ &= V^{M+1} W^{M+1} P(V) = VWP(V) = VWV^M Q(V) \\ &= V^M Q(V) = P(V), \end{aligned}$$

which gives $P(V) \in W(X)$. On the other hand, we have

$$\begin{aligned} (P(V))^2 T &= V^{2M}(Q(V))^2 T = (Q(V))^2 V^{2M} W^{M+N} Q^{-1} \\ &= (Q(V))^2 V^M W^N Q^{-1} = V^M Q(V) Q(V) W^N Q^{-1} \\ &= V^M Q(V) VWQQ^{-1} = V^M Q(V) VW = V^M Q(V) = P(V). \end{aligned}$$

Hence $P(V) \in R_1(X)$. To finish the proof, we show that $T \in V(X)$. It is easy to see that for all $\lambda \in \mathcal{F}$ the following factorization yields

$$Q(I, W) - \lambda W^{N+M} = \prod_{j=0}^{q} (I - \gamma_j W)^{s_j}.$$

So that

$$I - \lambda T = (Q - \lambda W^{N+M}) Q^{-1} = \prod_{j=0}^{q} (I - \gamma_j W)^{s_j} Q^{-1}.$$

From that all operators $I - \gamma_j W$ are invertible, we conclude that $I - \lambda T$ is invertible for all $\lambda \in \mathcal{F}$. Theorem is proved.

THEOREM 9. *Let $Q(t, s)$ be of the form (15) and let $Q := Q(I, W)$. If Q is invertible, then T of the form (17) is a GR-inverse of $P(V)$. Moreover, if $T \in V(X)$, then $W \in V(X)$.*

Proof. Let $0 \neq s_0 \in \mathcal{F}$, and $\lambda \in \mathcal{F}$ be fixed. Put $\mu := Q(\lambda) s_0^{-M}$. Then $I - \mu T = (Q - \mu W^{M+N}) Q^{-1}$. Hence $H := Q - \mu W^{M+N}$ is invertible, provided that $T \in V(X)$. We set $P_\mu(t, s_0) := Q(t, s_0) - \mu s_0^{M+N}$. It is easy to check that $P_\mu(\lambda s_0, s_0) = 0$. It follows the factorization $P_\mu(t, s_0) = (t - \lambda s_0) Q_\mu(t, s_0)$, where $Q_\mu(t, s_0)$ is a certain polynomial. Since we have $H = (I - \mu W) Q_\mu(I, W)$ invertible, we conclude that $I - \lambda W$ is invertible for any $\lambda \in \mathcal{F}$, i.e., $W \in V(X)$. Theorem is proved.

4. Equations induced by polynomials with GR-invertible operators

Equations and initial problems induced by a generalized invertible operator were studied in [7]. In this Section we consider equations induced by polynomials with a GR-invertible operator.

THEOREM 10. *Let $Q(t,s)$ and $P(t)$ be of the form (15) and (16), respectively, and $q_0 = 0$. Let $V \in R_1(X)$ and $W \in V(X) \cap \mathcal{R}^1_V$. Then the equation*

$$P(V)x = y \tag{18}$$

has solutions if and only if $F^{(l)}_W y = 0$. If this condition is satisfied, then all solutions of (18) are given by the formula

$$x = W^{N+M}(Q(I,W))^{-1}y + z, \quad \textit{where } z \in \ker P(V). \tag{19}$$

Proof. By Theorem 8, $P(V) \in R_1(X)$ and $T := W^{N+M}Q^{-1} \in \mathcal{R}^1_{P(V)} \cap V(X)$, where $Q = Q(I,W)$. By Theorem 11.1 in [7], the equation (18) has solutions if and only if $(I - P(V)T)y = 0$, i.e.,

$$(I - P(V)W^{N+M}Q^{-1})y = 0. \tag{20}$$

On the other hand, we have

$$\begin{aligned} P(V)W^{N+M} &= V^M Q(V)W^{N+M} = V^M(Q(V)W^N)W^M \\ &= V^M VWQ(I,W)W^M = V^{M+1}W^{M+1}Q = VWQ. \end{aligned}$$

It follows $I - P(V)W^{N+M}Q^{-1} = I - VW$. Hence the identity (20) is of the form $(I - VW)y = 0$, i.e., $F^{(l)}_W y = 0$. It follows from Theorem 11.1 in [7] that, if this condition is satisfied, then all solutions of (18) are given by (19). The proof is complete.

Now we deal with the case $q_0 \neq 0$. Consider the following equation

$$Q(V)x = y, \tag{21}$$

where $y \in X$ is given, $Q(t)$ is of the form (16). Write $Q^*(t,s) = Q(t,s) - q_0 s^N$, $Q^*(t) = Q^*(t,1)$.

THEOREM 11. *Let $V \in R_1(X), W \in V(X) \cap \mathcal{R}^1_V$ and $y \in \operatorname{Im} Q(V)$. Then all solutions of (21) are given by*

$$x = Q^*(I,W)Q^{-1}\left[W^N\left(Q^*(I,W)\right)^{-1}y + z\right], \tag{22}$$

where $z \in \ker Q^*(V)$.

Proof. We have $Q(V) = Q^*(V) + q_0 I$. Hence, (21) may be written in the form

$$Q^*(V)x = y - q_0 x. \tag{23}$$

It follows from Theorem 8 that $Q^*(V) \in R_1(X)$ and $W^N[Q^*(I,W)]^{-1} \in \mathcal{R}^1_{Q^*(V)}$. By Theorem 10, the equation (23) is equivalent to $x = W^N[Q^*(I,W)]^{-1}[y - q_0 x] + z$, and then

$$\left[I + q_0 W^N[Q^*(I,W)]^{-1}\right] x = W^N[Q^*(I,W)]^{-1} y + z, \tag{24}$$

where $z \in \ker Q^*(V)$. Futhermore, we have

$$\begin{aligned} M &= \left[I + q_0 W^N[Q^*(I,W)]^{-1}\right] = \left[Q^*(I,W) + q_0 W^N\right][Q^*(I,W)]^{-1} \\ &= Q(I,W)[Q^*(I,W)]^{-1}. \end{aligned}$$

This implies that M is invertible. Thus, from (24) we get all solutions of (21) in the form (22). The theorem is proved.

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