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ON DIRECTIONAL $\mathcal{I}_1$- DENSITY POINTS

Let $\Re^k$ denote the k-dimensional Euclidean space $(k = 1, 2)$, $\mathcal{N}$ – the set of positive integers and $\Re_+$ – the set of positive real numbers.

The ball centred at a point p and with radius $r > 0$ will be denoted by $K(p, r)$.

We introduce the following notations:

$\mathcal{S}_k$ – the σ-field of subset of $\Re^k$ having the Baire property,

$\mathcal{I}_k$ – the σ-ideal of subset of $\Re^k$ of the first category.

We shall say that a set $A \subset \Re^k$ is $\mathcal{S}_k$ – measurable if and only if $A \in \mathcal{S}_k$.

For $A \in \mathcal{S}_1$, we shall denote by $\phi(A)$ the set of all $\mathcal{I}_1$-density points of A [6]. It is known [6] that the mapping $\phi\ : \mathcal{S}_1 \to 2^{\Re^1}$ is a lower density operator.

If a plane set A is contained in a line, then we use linear $\mathcal{I}_1$ - density points of the set $A \in \mathcal{S}_1$.

Let $L_\theta(L_\theta(x, y))$ denote the line passing through the point $(0,0)$ (respectively, the point (x, y)) and forming an angle θ with the ox – axis for $\theta \in [0, \pi)$.

We denote by $A \triangle B$ the symmetric difference of A and B; if $A, B \in \mathcal{S}_k$, then $A \sim B$ means that $A \triangle B \in \mathcal{I}_k$, k=1,2.

Set $\theta \in [0, \pi)$. For $M \subset \Re^2$ we put

$$S_\theta(M) = \{(x, y) \in \Re^2 : \ \exists_{r \in \Re_+} M \cap L_\theta(x, y) \cap K((x, y), r) \in \mathcal{S}_1\}.$$

For each $M \subset \Re^2$, we define:

1. If $\theta \in [0, \pi)$, then

$$\Phi_\theta(M) = \{(x, y) \in S_\theta(M) : \ (x, y) \in \phi(M \cap L_\theta(x, y))\}.$$

2. If $\theta_1, \theta_2 \in [0, \pi)$, then

$$\Phi_{\theta_1,\theta_2}(M) = \Phi_{\theta_1}(M) \cap \Phi_{\theta_2}(M).$$

3. $\Phi(M) = \{(x,y) \in \Re^2 : (x,y) \in \Phi_\theta(M)$ in $\mathcal{I}_1$ – almost every direction $\theta \in [0,\pi)\}$.

We shall say that a point $(x,y) \in \Re^2$ is an $\mathcal{I}_1$-density point of M in the direction θ if and only if $(x,y) \in \Phi_\theta(M)$.

DEFINITION 1. We shall say that $\Psi : \mathcal{S}_2 \to 2^{\Re^2}$ is a lower density operator if and only if

I. $\Psi(A) \sim A$,

II. $A \sim B \Rightarrow \Psi(A) = \Psi(B)$,

III. $\Psi(\emptyset) = \emptyset$, $\Psi(\Re^2) = \Re^2$,

IV. $\Psi(A \cap B) = \Psi(A) \cap \Psi(B)$.

DEFINITION 2. We shall say that a function $f : \Re^2 \to \Re$ is continuous at $(x,y) \in \Re^2$ with respect to the operator $\Psi \in \{\Phi_\theta, \Phi_{\theta_1,\theta_2}, \Phi\}$ if and only if, for each open set $G \subset \Re$,

$$f^{-1}(G) \subset \Psi(f^{-1}(G)).$$

From the Kuratowski-Ulam theorem [5] we have the following

PROPOSITION 3. *If $E \in \mathcal{I}_2$, then*

$$\{x \in \Re^1 : \ E \cap L_\theta(x,0) \notin \mathcal{I}_1 \ \} \in \mathcal{I}_1.$$

THEOREM 4. *For any $\theta \in [0,\pi)$ and $A, B \in \mathcal{S}_2$ we have*

I. $\Phi_\theta(A) \sim A$,

II. $A \sim B \Rightarrow \Phi_\theta(A) \sim \Phi_\theta(B)$,

III. $\Phi_\theta(\emptyset) = \emptyset$, $\Phi_\theta(\Re^2) = \Re^2$,

IV. $\Phi_\theta(A \cap B) = \Phi_\theta(A) \cap \Phi_\theta(B)$.

Proof. Let $\theta \in [0,\pi)$. First, we shall prove I. Let $A \subset \Re^2$ and $A \in \mathcal{S}_2$. Then, there exist sets $P_1, P_2 \in \mathcal{I}_2$ and an open set $G \subset \Re^2$ such that $A = (G \setminus P_1) \cup P_2$, $P_1 \subset G$ and $P_2 \cap G = \emptyset$. Let

$$E_1 = \{x \in \Re^1 : \ P_1 \cap L_\theta(x,0) \notin \mathcal{I}_1\}.$$

By Proposition 3 we have $E_1 \in \mathcal{I}_1$.

Denote by

$$W_1(\theta) = \bigcup_{x \in E_1} L_\theta(x,0).$$

Thus $W_1(\theta) \in \mathcal{I}_2$, what easily follows from the fact that $E_1 \times \Re \in \mathcal{I}_2$. We shall prove that

$$G \setminus W_1(\theta) \subset \Phi_\theta(A). \tag{1}$$

Let $(x_0, y_0) \in G \setminus W_1(\theta)$. Then there exists $r \in \Re_+$ such that the ball $K = K((x_0, y_0), r) \subset G$ and, by the definition of the set E_1,

$$L_\theta(\widehat{x}, 0) \cap P_1 \in \mathcal{I}_1$$

where $(\widehat{x}, 0)$ is the unique point for which $(x_0, y_0) \in L_\theta(\widehat{x}, 0)$.

Thus

$$L_\theta(\widehat{x}, 0) \cap K \subset L_\theta(\widehat{x}, 0) \cap G$$

and the set $(L_\theta(\widehat{x}, 0) \cap K) \setminus P_1$ is the residual set on the interval $L_\theta(\widehat{x}, 0) \cap K$. Therefore

$$(L_\theta(\widehat{x}, 0) \cap K) \setminus P_1 \subset (L_\theta(\widehat{x}, 0) \cap G) \setminus P_1 = L_\theta(\widehat{x}, 0) \cap (G \setminus P_1).$$

But

$$(L_\theta(\widehat{x}, 0) \cap K) \setminus P_1 \subset L_\theta(\widehat{x}, 0) \cap (G \setminus P_1) \cap K$$

and the set $L_\theta(\widehat{x}, 0) \cap (G \setminus P_1) \cap K$ is the residual set on the interval $L_\theta(\widehat{x}, 0) \cap K$. Thus $L_\theta(\widehat{x}, 0) \cap A \cap K$ is also a residual set on the interval $L_\theta(\widehat{x}, 0) \cap K$. Consequently, $(x_0, y_0) \in \Phi_\theta(A)$. Hence we have (1).

Now, we denote by

$$E_2 = \{x \in \Re^1 : \ P_2 \cap L_\theta(x, 0) \notin \mathcal{I}_1\}.$$

From Proposition 3 it follows that $E_2 \in \mathcal{I}_1$. Similary as above, we put

$$W_2(\theta) = \bigcup_{x \in E_2} L_\theta(x, 0).$$

Thus $W_2(\theta) \in \mathcal{I}_2$ since $E_2 \times \Re \in \mathcal{I}_2$.

We shall prove that

$$\Phi_\theta(A) \subset \overline{G} \cup W_2(\theta). \tag{2}$$

Let $(x_0, y_0) \notin \overline{G} \cup W_2(\theta)$. Hence there exists a ball $K((x_0, y_0), r)$ such that $K \cap G = \emptyset$ and $(x_0, y_0) \notin W_2(\theta)$.

Let $L_\theta(\widehat{\widehat{x}}, 0)$ be the unique line for which $(x_0, y_0) \in L_\theta(\widehat{\widehat{x}}, 0)$. Therefore, by the definitions of K and E_2, we have that

$$L_\theta(\widehat{\widehat{x}}, 0) \cap K \cap (G \setminus P_1) = \emptyset$$

and

$$L_\theta(\widehat{\widehat{x}}, 0)) \cap P_2 \in \mathcal{I}_1.$$

Thus the set $L_\theta(\widehat{\widehat{x}}, 0)) \cap A$ is of the first category on the interval $L_\theta(\widehat{\widehat{x}}, 0)) \cap K$. Hence $(x_0, y_0) \notin \Phi_\theta(A)$. Consequently, the condition (2) holds.

By virtue of inclusions (1) and (2), we have that $\Phi_\theta(A) \sim G \sim A$ and, therefore, the proof of I is completed.

We observe that condition II easily follows from I and the proof of III is obvious.

Clearly, condition IV follows from the analogous condition IV for $\mathcal{I}_1$ – density points on the line $\Re^1$, [6]. The proof of the theorem is completed.

Let $W \notin \mathcal{S}_1$. We put, for each $(x,y) \in \Re^2$, $f(x,y) = \chi_W(x)$ (the characteristic function of W). It is easy to see that the function has not the Baire property and f is continuous function with respect to the operator $\Phi_{\frac{\pi}{2}}$. We observe that the family $\{A \in \mathcal{S}_2 : A \subset \Phi_{\frac{\pi}{2}}(A)\}$ is not a topology in $\Re^2$.

From Theorem 4 we derive

THEOREM 5. *For any $\theta_1, \theta_2 \in [0,\pi)$ and $A, B \in \mathcal{S}_2$ we have*

I. $\Phi_{\theta_1,\theta_2}(A) \sim A$,

II. $A \sim B \Rightarrow \Phi_{\theta_1,\theta_2}(A) \sim \Phi_{\theta_1,\theta_2}(B)$,

III. $\Phi_{\theta_1,\theta_2}(\emptyset) = \emptyset$, $\Phi_{\theta_1,\theta_2}(\Re^2) = \Re^2$,

IV. $\Phi_{\theta_1,\theta_2}(A \cap B) = \Phi_{\theta_1,\theta_2}(A) \cap \Phi_{\theta_1,\theta_2}(B)$.

In a similar way as in [1, Th 2.1, p.134] we can prove the following

COROLLARY 6. *Let $\theta_1, \theta_2 \in [0,\pi)$. Then the family*

$$\tau_{\theta_1,\theta_2} = \{A \in \mathcal{S}_2 : A \subset \Phi_{\theta_1,\theta_2}(A)\}$$

is a toplogy in $\Re^2$.

THEOREM 7. *Let $\theta_1, \theta_2 \in [0,\pi)$ and $\theta_1 \neq \theta_2$. If a function $f : \Re^2 \to \Re$ is continuous with respect to the operator Φ_{θ_1,θ_2}, at each point $(x,y) \in \Re^2$, then f is of the second class of Baire.*

Proof. We may assume that $\theta_1 = 0$ and $\theta_2 \neq 0$. We define a transformation $G : \Re^2 \to \Re^2$ as follows:

if $(x,y) \in \Re^2$ and $(x',y') = G(x,y)$ then $x' = x + y \sin\,\theta_2$ and $y' = y \cos\,\theta_2$.

By the continuity of G we see that the function $f(G(x,y))$ is continuous with respect to the operator $\Phi_{0,\frac{\pi}{2}}$. By the theorem from [3], we know that $f \circ G$ is of the second class of Baire in $\Re^2$. Since G^{-1} exists and it is a continuous function, we have that $f = f \circ G \circ G^{-1}$ is of the second class of Baire, too.

By [1, Th. 3.2, p. 146] we see that the above result is the best possible.

COROLLARY 8. *Let $\theta_1, \theta_2 \in [0,\pi)$ be such that $\theta_1 \neq \theta_2$. A function $f : \Re^2 \to \Re$ is continuous with respect to the operator Φ_{θ_1,θ_2} if and only if it is continuous with respect to the topology τ_{θ_1,θ_2}.*

We shall now give the generalization of the Kuratowski-Ulam theorem for the polar-coordinates. We shall also give the proof, because we cannot identify its source.

THEOREM 9. *For any $P \in \mathcal{I}_2$ and $(x,y) \in \Re^2$ there exists $\Theta \subset [0,\pi)$ such that $[0,\pi) \setminus \Theta \in \mathcal{I}_1$ and, for each $\theta \in \Theta$, $P \cap L_\theta(x,y) \in \mathcal{I}_1$.*

Proof. Let $P \in \mathcal{I}_2$ and $(x, y) \in \Re^2$.

First we assume that P is a closed, nowhere dense set. For each $\theta \in [0, \pi)$, we denote by

$$P_\theta = \{r \in \Re : (x + r\cos\theta, y + r\sin\theta) \in P\}.$$

Suppose that $A = \{(r, \theta) : r \in P_\theta\}$ is not a nowhere dense subset of $\Re \times [0, \pi)$. Since A is a closed set in $\Re \times [0, \pi)$ we have that there exists an open rectangle $(r_1, r_2) \times (\theta_1, \theta_2) \subset A$. Then

$$\bigcup_{\theta \in (\theta_1, \theta_2)} L_\theta(x, y) \cap (K((x, y), r_2) \setminus K((x, y), r_1)) \subset P,$$

a contradiction. Therefore A is nowhere dense set and by the classical Kuratowski-Ulam theorem there exists $\Theta \subset [0, \pi)$ such that $[0, \pi) \setminus \Theta \in \mathcal{I}_1$ and, for each $\theta \in \Theta$, $P_\theta \in \mathcal{I}_1$. Hence $P \cap L_\theta(x, y) \in \mathcal{I}_1$, for each $\theta \in \Theta$.

Now we assume that $P \in \mathcal{I}_2$. Then there exists a sequence of closed nowhere dense sets $\{P_n\}_{n \in \mathcal{N}}$ such that $P \subset \bigcup_{n \in \mathcal{N}} P_n$. By the first part of the proof we have that, for each $n \in \mathcal{N}$, there exists $\Theta_n \subset [0, \pi)$ such that $[0, \pi) \setminus \Theta_n \in \mathcal{I}_1$ and, for each $\theta \in \Theta_n$, $P_n \cap L_\theta(x, y) \in \mathcal{I}_1$. Put $\Theta = \bigcap_{n \in \mathcal{N}} \Theta_n$. Then $[0, \pi) \setminus \Theta \in \mathcal{I}_1$ and, for each $\theta \in \Theta$, $P \cap L_\theta \subset \bigcup_{n \in \mathcal{N}} P_n \cap L_\theta \in \mathcal{I}_1$.

PROPOSITION 10. *If $A, B \in \mathcal{S}_2$ and $A \sim B$ then $\Phi(A) = \Phi(B)$.*

Proof. Assume that $A, B \in \mathcal{S}_2$ and $A \sim B$. Let $(x, y) \in \Phi(A)$. Then there exists $\Theta_1 \subset [0, \pi)$ such that $[0, \pi) \setminus \Theta_1 \in \mathcal{I}_1$ and, for each $\theta \in \Theta_1$, $(x, y) \in \Phi_\theta(A)$. By Theorem 9, we can pick $\Theta_2 \subset [0, \pi)$ such that $[0, \pi) \setminus \Theta_2 \in \mathcal{I}_1$ and, for each $\theta \in \Theta_2$, $L_\theta(x, y) \cap (\Re^2 \setminus (A \setminus B))$ is a residual subset of $L_\theta(x, y)$. Put $\Theta = \Theta_1 \cap \Theta_2$. Then, for each $\theta \in \Theta$, $(x, y) \in \Phi_\theta(A \cap B)$. Thus $\Phi(A) = \Phi(A \cap B)$.

In a similar way we show that $\Phi(B) = \Phi(A \cap B)$. Therefore $\Phi(A) = \Phi(B)$.

PROPOSITION 11. *Let $M \in \mathcal{S}_2$. Then $\Phi(M) \sim M$.*

Proof. Let $M \in \mathcal{S}_2$. Then $M = (F \setminus P_1) \cup P_2$, where $P_1, P_2 \in \mathcal{I}_2$, $P_1 \subset F$, $P_2 \cap F = \emptyset$, and F is a closed set. By Proposition 10, we see that $\Phi(M) = \Phi(F)$. Thus, by the fact that $int(F) \subset \Phi(F) \subset F$, we have the following relations

$$M \setminus \Phi(F) \subset P_2 \cup (F \setminus \Phi(F)) \in \mathcal{I}_2$$

and

$$\Phi(F) \setminus M \subset \Phi(F) \setminus F = \emptyset \in \mathcal{I}_2.$$

Therefore $\Phi(M) \sim M$.

By Propositions 10 and 11 we have the following

THEOREM 12. *A mapping $\Phi : \mathcal{S}_2 \to 2^{\Re^2}$ is a lower density.*

Observe that a similar theorem for measure has been proved by A. M. Brückner and M. Rosenfeld in the paper [2].

COROLLARY 13. *A family* $\tau = \{A \in \mathcal{S}_2 : A \subset \Phi(A)\}$ *is a topology in* $\Re^2$.

The above topology was considered by E. Wagner-Bojakowska and W. Wilczyński in [7].

It is interesting to ask the question to which Baire class belong functions continuous in this topology.

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