DEMONSTRATIO MATHEMATICA
Vol. XXX No 2 1997

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AN EXISTENCE THEOREM FOR A PARABOLIC DIFFERENTIAL EQUATION IN $l^{\infty}(A)$ BASED ON THE TARSKI FIXED POINT THEOREM

1. Introduction

Let A be an ordinary nonempty set and let R denote the reals. With $l^{\infty}(A)$ we denote the Banach space of all bounded functions on A to R with the norm $\|(z^a)_{a\in A}\| = \sup_{a\in A}|z^a|$. In $l^{\infty}(A)$ there is an order given by $x \leq y \Leftrightarrow \{x^a \leq y^a \text{ for all } a \in A\}$. The object of this paper is to transform the parabolic differential equation

$$u_t(x,t) - u_{xx}(x,t) = f(x,t,u(x,t)),\ (x,t) \in R \times (0,\mathrm{T}],$$
$$u(x,0) = 0, \quad \text{for } x \in R,$$

into a fixed point problem in order to solve the latter with the Tarski fixed point theorem (see [5]). Its advantage is that there is no need to consider compact sets as it is the case in the Schauder fixed point theorem, so we are able to get an existence theorem in $l^{\infty}(A)$. This paper combines the main ideas of [3], where a real-valued semi-linear elliptic differential equation is solved with the Tarski fixed point theorem, and of [4], where the same elliptic differential equation is solved with the Lemmert fixed point theorem (see [2]) in any ordered Banach space.

2. The Cauchy Problem

If the function $f(x,t,z)$ doesn't depend on z, the problem has been already solved. It holds the following

THEOREM 1. *Let E be any real Banach space and let $g(x,t) : R\times[0, T] \to E$ be bounded, continuous and Lipschitz continuous in $x \in R$, uniformly with respect to t, i.e. there exists a constant L such that*

$$\|g(x,t) - g(y,t)\| \leq L|x - y|$$

for all $(x,t),(y,t) \in R \times [0,T]$. *Then the function*

$$u(x,t) := \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-\tau)}} e^{-\frac{(x-\xi)^2}{4(t-\tau)}} g(\xi,\tau)\, d\xi\, d\tau, \qquad (x,t) \in R \times [0,T],$$

is a solution of the Cauchy problem

$$u_t(x,t) - u_{xx}(x,t) = g(x,t), \quad (x,t) \in R \times (0,T],$$
$$u(x,0) = 0, \quad x \in R,$$

where the partial derivatives have to be considered refering to the norm.

See for example [1], p. 1–25, for the proof ([1]).

3. The existence theorem

Now we formulate the main result of the paper.

THEOREM 2. *Let* $f(x,t,z) : R \times [0,T] \times l^\infty(A) \to l^\infty(A)$ *be a function with the following properties:*

i) f *is continuous.*

ii) *There exists a constant* $L_1 > 0$ *such that for all*

$$(x,t,z),(y,t,z) \in R \times [0,T] \times l^\infty(A)$$

it holds

$$\|f(x,t,z) - f(y,t,z)\| \le L_1|x-y|.$$

iii) *There exists a constant* $L_2 > 0$ *such that for all* $(x,t,z_1),(x,t,z_2) \in R \times [0,T] \times l^\infty(A)$ *it holds*

$$\|f(x,t,z_1) - f(x,t,z_2)\| \le L_2\|z_1 - z_2\|.$$

iv) *For all* $z_1, z_2 \in l^\infty(A); (x,t) \in R \times [0,T]$ *it holds*

$$z_1 \le z_2 \Rightarrow f(x,t,z_1) \le f(x,t,z_2).$$

v) *There exists a constant* $M > 0$ *such that for all* $(x,t,z) \in R \times [0,T] \times l^\infty(A)$ *it holds*

$$\|f(x,t,z)\| \le M.$$

Then there exists a continuous function $u : R \times [0,T] \to l^\infty(A)$ *with*

$$u_t(x,t) - u_{xx}(x,t) = f(x,t,u(x,t)) \quad \text{for } (x,t) \in R \times (0,\mathrm{T}],$$
$$u(x,0) = 0 \quad \text{for } x \in R.$$

([1]) In [1] the theorem is only given for real-valued functions, but the theorem remains true if $g(x,t)$ has got its values in any real Banach space E, because one has just to substitute the norm for the absolute value.

Proof. We consider the following set of functions

$$\Omega := \{\omega : R \times [0,\mathrm{T}] \to l^{\infty}(A), -\Gamma \le \omega \le \Gamma,$$
$$\|\omega(x,t) - \omega(y,s)\| \le \mathrm{L}_3|x-y| + \mathrm{L}_4|t-s| \quad \text{for } (x,t),(y,s) \in R \times [0,\mathrm{T}]\},$$

with

$$\Gamma = (\Gamma^a)_{a\in A} \quad \text{with } \Gamma^a = \mathrm{TM}, \qquad \mathrm{L}_3 = 2\mathrm{M}\sqrt{\mathrm{T}} \qquad \text{and}$$
$$\mathrm{L}_4 = \mathrm{M} + (\mathrm{L}_1 + \mathrm{L}_2\mathrm{L}_3)2\sqrt{\mathrm{T}/\pi}.$$

If $\omega \in \Omega$, we define the map

$$\Phi(\omega)(x,t) := \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-\tau)}} e^{-\frac{(x-\xi)^2}{4(t-\tau)}} f(\xi,\tau,\omega(\xi,\tau))\, d\xi\, d\tau$$

$(x,t) \in R \times [0,\mathrm{T}]$. Now we have

$$\|f(x,t,\omega(x,t)) - f(y,t,\omega(y,t))\| \le (\mathrm{L}_1 + \mathrm{L}_2\mathrm{L}_3)|x-y|,$$

for $(x,t),(y,t) \in R \times [0,\mathrm{T}]$, so we can conclude according to Theorem 1 that

$$\Phi(\omega)_t(x,t) - \Phi(\omega)_{xx}(x,t) = f(x,t,\omega(x,t)) \quad \text{for } (x,t) \in R \times (0,\mathrm{T}],$$
$$\Phi(\omega)(x,0) = 0 \qquad \text{for } x \in R$$

holds. Now if there exists an $u \in \Omega$ with $u = \Phi(u)$, the theorem is proved. In order to show that Φ possesses a fixed point, we apply the Tarski fixed point theorem (see [5]). That is we have to show that: a) $(\Omega, \le)$ is a complete lattice and b) Φ is an increasing function on Ω to Ω.

It is easy to verify a), so it is omitted here.

For b): It's clear that Φ is increasing due to iv). Now let $\omega \in \Omega$. First we show that $\|\Phi(\omega)(x,t) - \Phi(\omega)(x,s)\| \le \mathrm{L}_4|t-s|$ holds for $x \in R$ and $t,s \in [0,\mathrm{T}]$. We assume w.l.o.g. $s < t$. It holds

$$\begin{aligned} \mathrm{D} &:= \|\Phi(\omega)(x,t) - \Phi(\omega)(x,s)\| \\ &\le \Big\| \int_0^s \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} f(\xi,\tau,\omega(\xi,\tau)) \\ &\quad \times \left[\frac{1}{\sqrt{4(t-\tau)}} e^{-\frac{(x-\xi)^2}{4(t-\tau)}} - \frac{1}{\sqrt{4(s-\tau)}} e^{-\frac{(x-\xi)^2}{4(s-\tau)}} \right] d\xi\, d\tau \Big\| \\ &\quad + \Big\| \int_s^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi 4(t-\tau)}} e^{-\frac{(x-\xi)^2}{4(t-\tau)}} f(\xi,\tau,\omega(\xi,\tau))\, d\xi\, d\tau \Big\| \\ &=: \mathrm{J}_1 + \mathrm{J}_2. \end{aligned}$$

Now we get

$$\mathrm{J}_2 \le \mathrm{M}(t-s) = \mathrm{M}|t-s|,$$

and

$$
\begin{aligned}
J_1 &\le \int_0^s \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-\alpha^2} \| f(x - 2\alpha\sqrt{t-\tau}, \tau, \omega(x - 2\alpha\sqrt{t-\tau}, \tau)) \\
&\quad - f(x - 2\alpha\sqrt{s-\tau}, \tau, \omega(x - 2\alpha\sqrt{s-\tau}, \tau)) \| \, d\alpha \, d\tau \\
&\le \frac{1}{\sqrt{\pi}} (L_1 + L_2 L_3) \int_0^s (\sqrt{t-\tau} - \sqrt{s-\tau}) \int_{-\infty}^{\infty} e^{-\alpha^2} 2|\alpha| \, d\alpha \, d\tau \\
&= \frac{1}{\sqrt{\pi}} (L_1 + L_2 L_3) \frac{2}{3} (t^{\frac{3}{2}} - s^{\frac{3}{2}} - (t-s)^{\frac{3}{2}}) 2 \\
&\le \frac{2}{\sqrt{\pi}} (L_1 + L_2 L_3) \sqrt{T} |t - s|.
\end{aligned}
$$

So we get $D \le L_4 |t - s|$.

Now let $a \in A$. Let w.l.o.g. $x, y \in R$, $t \in (0, T]$. With the mean value theorem we get

$$
|\Phi(\omega)^a(x,t) - \Phi(\omega)^a(y,t)| = |\Phi(\omega)^a_x(\Theta, t)||x - y| \le L_3 |x - y|
$$

with Θ between x and y. Since L_3 doesn't depend on a, we can conclude $\|\Phi(\omega)(x,t) - \Phi(\omega)(y,t)\| \le L_3 |x - y|$. So we have for $(x,t), (y,s) \in R \times [0, T]$, $\|\Phi(\omega)(x,t) - \Phi(\omega)(y,s)\| \le L_3 |x - y| + L_4 |t - s|$. Finally, it holds that $-\Gamma \le \Phi(\omega) \le \Gamma$ for $(x,t) \in R \times [0, T]$. This completes the proof.

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Received June 17, 1996.