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# ON CLASSIFICATION OF THE LINEAR LAGRANGIAN AND ISOTROPIC SUBSPACES

## 0. Introduction

Let $M$ be a manifold, and $\omega$ be a 2-form on $M$. The pair $(M, \omega)$ is called a *symplectic manifold* if $\omega$ is closed, i.e. $d\omega = 0$ and nondegenerate [9].

The simple and representative model of a symplectic manifold is the cotangent bundle $T^*\mathbf{R}^n$, endowed with the canonical 2-form $\omega = d\Theta$, where 1-form $\Theta$ is the Liouville form on $T^*\mathbf{R}^n$ defined by

$$\langle u, \Theta \rangle = \langle T\pi_{R^n}(u), \tau_{T^*R^n}(u) \rangle \quad \text{here } u \in T(T^*\mathbf{R}^n),$$

the mapping $T\pi_{R^n}$ is the tangent mapping of $\pi_{R^n} : T^*\mathbf{R}^n \to \mathbf{R}^n$ and $\tau_{T^*R^n} : T(T^*\mathbf{R}^n) \to T^*\mathbf{R}^n$ is the tangent bundle projection [5].

If $(p, q)$ are coordinates of the bundle $T^*\mathbf{R}^n$, i.e. $q_1, \ldots, q_n$ are coordinates of the base and $p_1, \ldots, p_n$ coordinates of the fibres, then $\Theta$ and $\omega$ have the local Darboux form [9]

$$\Theta = \Sigma_{i=1}^n p_i dq_i, \quad \omega = \Sigma_{i=1}^n dp_i \wedge dq_i.$$

Let $M$ be a $2n$-dimensional manifold. A submanifold of the symplectic space $(M, \omega)$ is called *isotropic* if the restriction of the symplectic form to its tangent space at each point is zero form. The dimension of an isotropic submanifold is less then or equals $n$. If the dimension equals $n$, then this submanifold is called *Lagrangian*. Every isotropic (and Lagrangian) submanifold can be locally represented by its *generating family* ([3], [5]). Isotropic submanifolds are often called *the null-submanifolds*. They play very important role in many branches in mathematics and physics e.g. in diffraction theory, in classical and quantum mechanics etc. ([3], [4], [5], [7]).

In §1 we give a description in an algebraic language of the Grassmannian consisting of $k$-dimensional linear isotropic subspaces of a $2n$-dimensional

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symplectic space. In the next sections we obtain a precise formula for generating function of linear isotropic and Lagrangian subspaces of the cotangent bundle of the $n$-dimensional real space. In §2 we find a matrix representation of an arbitrary linear Lagrangian subspace of the cotangent bundle $T^*\mathbf{R}^n$. In §3 we perform a similar construction for a general isotropic case. In §4 we apply our methods from §1 and §2 to calculate the generating families for isotropic and Lagrangian subspaces.

For example, we consider a 2-dimensional isotropic subspace in $T^*\mathbf{R}^3$

$$I = \{(p_1, p_2, p_3, q_1, q_2, q_3) : \\ p_1 = t,\ q_2 = s,\ p_2 = p_3 = 0,\ q_1 = q_3 = 0\ t, s \in \mathbf{R}\}.$$

We see, that the subspace $I$ projected to the base of the bundle $T^*R^3$ is the line $\{(q_1, q_2, q_3) : q_1 = 0,\ q_2 = s,\ q_3 = 0,\ s \in R\} \subset R^3$, its generating function $S : \mathbf{R}^3 \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ has the form

$$S(q_1, q_2, q_3, \lambda, \beta) \\ = \frac{1}{2}(q_1, q_2, q_3, \lambda, \beta) \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ \lambda \\ \beta \end{pmatrix} = \lambda q_1 + \beta q_3.$$

## 1. Isotropic Grassmannian

Let $M^n$ be a smooth manifold. The cotangent bundle $T^*M^n$ is endowed with the canonical symplectic structure (see Introduction). The first step to classify isotropic submanifolds of the cotangent bundle $T^*M^n$ is to describe linear isotropic subspaces of $T^*\mathbf{R}^n$. We identify $T^*\mathbf{R}^n$ with $\mathbb{C}^n$, then the symplectic structure $\omega$ coincide with the imaginary part of the standard Hermitian scalar product $H$ (see [1]). This identification is the following

$$(q_1, \ldots, q_n, p_1, \ldots, p_n) \mapsto (q_1 + ip_1, \ldots, q_n + ip_n),$$

where $q_i$ are coordinates of the base of the bundle, and $p_i$ coordinates of the fibres.

Let $U(n)$ be the unitary group and let $O(n)$ be the orthogonal group considered as a subgroup of $U(n)$. We denote by $\Lambda_n$ the set of all Lagrangian subspaces in $T^*\mathbf{R}^n$. This set is called the Lagrange Grassmannian and it is smooth manifold. $\Lambda_n$ is homeomorphic to the quotient space $U(n)/O(n)$. The dimension of the Lagrange Grassmannian equals $\frac{1}{2}n(n+1)$ (see [2], [8]).

Analogously we can define the set of all $k$-dimensional isotropic subspaces in $2n$-dimensional symplectic vector spaces $T^*\mathbf{R}^n$. We denote it by $\mathcal{I}_k^{2n}$.

PROPOSITION 1.1. *The isotropic Grassmannian $\mathcal{I}_k^{2n}$ is homeomorphic to the quotient space*

$$U(n)/(O(k) \oplus U(n-k))$$

*and* $\dim \mathcal{I}_k^{2n} = 2nk - \frac{1}{2}k(3k-1)$.

Proof. We should show, that: 1) the unitary group $U(n)$ acts on $\mathbf{C}^n$ by symplectic homomorphism, 2) this action is transitive on $\mathcal{I}_k^{2n}$ , i.e. we can obtain an arbitrary subspace $I \in \mathcal{I}_k^{2n}$ from the fixed subspace $I_0 \in \mathcal{I}_k^{2n}$, 3) the stabilizer of a certain element $I_0 \in \mathcal{I}_k^{2n}$ is equal to $O(k) \oplus U(n-k)$ .

Ad. 1. By definition the group $U(n)$ consists of all transformations of $\mathbf{C}^n$ which preserve the Hermitian scalar product $H$. The symplectic form $\omega$ equals the imaginary part of $H$, thus the form $\omega$ is preserved by action of $U(n)$.

Ad. 2. We fix the isotropic subspace $I_0 = \mathrm{span}_R\{\frac{\partial}{\partial q_1}, \dots \frac{\partial}{\partial q_k}\}$.

Let $I$ be an arbitrary subspace from $\mathcal{I}_k^{2n}$, we construct a symplectomorphism $\varphi : (\mathbf{R}^{2n}, \omega) \longrightarrow (\mathbf{R}^{2n}, \omega)$ such, that $\varphi(I_0) = I$. We want to find matrix $A \in U(n)$ which represents this mapping.

The Hermitian scalar product $H$ restricted to the isotropic subspace $I$ is the real-valued function, because $\omega_{|I} = 0$ and $\omega = imH$. The form $H$ restricted to $I$ is the symmetric positive bilinear form therefore, there exists an orthonormal basis of $I$, $\{\alpha_1, \dots \alpha_k\}$ such, that $I = \mathrm{span}_R\{\alpha_1, \dots \alpha_k\}$ .

A natural way to construct the mapping $\varphi$ is to define it on the basis

$$\varphi\left(\frac{\partial}{\partial q_i}\right) := \alpha_i, \quad i = 1 \dots k.$$

Then the desired matrix $A$ has the following form

$$A = [\alpha_1, \dots \alpha_k, \beta_{k+1}, \dots \beta_n]$$

where $\beta_{k+1}, \dots \beta_n$ are such elements, that $\alpha_1, \dots \alpha_k, \beta_{k+1}, \dots \beta_n$ form an orthonormal basis of $\mathbf{C}^n$.

Ad.3. Let $I_0 = \mathrm{span}_R\{\frac{\partial}{\partial q_1}, \dots \frac{\partial}{\partial q_k}\}$. We want to find all matrices $A \in U(n)$ which represent the mapping $\varphi$ preserving the isotropic subspace $I_0$. Let $\varphi(\frac{\partial}{\partial q_i}) = \Sigma_{j=1}^k a_{ij} \frac{\partial}{\partial q_j} \in I_0,$, where $a_{ij} \in \mathbf{R}$ and $1 < i < k$. The matrix $A$ must consist of the block belonging to $O(k)$ which represents the transformation $I_0$ into $I$ and the complementary block from $U(n-k)$ and two zero blocks, i.e.

$$A \in \begin{pmatrix} O(k) & 0 \\ 0 & U(n-k) \end{pmatrix}. \blacksquare$$

We may calculate the dimension of the isotropic Grassmannian from the following two facts:

1) the group $U(n)$ is fibred over the real sphere $S^{2n-1}$ with the fibre $U(n-1)$,

2) the group $O(n)$ is fibred over the real sphere $S^{n-1}$ with the fibre $O(n-1)$.

Then $\dim U(n) = n^2$, $\dim O(n) = \frac{(n-1)n}{2}$.

Remark 1.2 We can't define in the isotropic Grassmannian the analog of the Maslov form as in [8], because $H^1(\Lambda_n, \mathbf{Z}) = \mathbf{Z}$, but $H^1(\mathcal{I}_k^{2n}, \mathbf{Z}) = 0$. Instead we have $H^1(\mathcal{I}_k^{2n}, \mathbf{Z}_2) = \mathbf{Z}_2$.

## 2. Generating families for Lagrangian subspaces in $\mathbb{R}^{2n}$

Every Lagrangian submanifold $L$ can be locally generated by some generating family $F$ (so-called Morse family [9]), i.e.

$$L = \left\{(p,q) \in T^*\mathbb{R}^n : \exists \lambda \in \mathbb{R}^k \frac{\partial F}{\partial \lambda}(q,\lambda) = 0, \quad \frac{\partial F}{\partial q}(q,\lambda) = p\right\},$$

where

$$F : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}, \quad F : (q,\lambda) \mapsto F(q,\lambda), \quad \operatorname{rank}\left(\frac{\partial^2 F}{\partial \lambda^2}, \frac{\partial^2 F}{\partial \lambda \partial q}\right) = k;$$

(see [6]). We will find a global generating family for a linear Lagrangian subspace.

### 2.1. Transversal case

If $L$ is transversal to the fibres of the cotangent bundle $T^*\mathbb{R}^n$, then $L$ is the graph of its generating function $F : \mathbb{R}^n \to \mathbb{R}$ (see [6]).

We fix a linear subspace $L \in \Lambda_n$. The subspace $L$ can be obtained from $L_0 = \operatorname{span}_R\{\frac{\partial}{\partial q_1}, \ldots, \frac{\partial}{\partial q_n}\}$by the action of the group $U(n)$, i.e. there exists $C = B + iA \in U(n)$ such, that

$$L = CL_0 = \{(p,q) : AX = p, \ BX = q, \ X \in L_0\},$$

where $A, B$ are real matrices. In transversal case $B$ is nonsingular. We can easily calculate, that $p = AB^{-1}q$.

LEMMA 2.1.1. *The relation between the coordinates $(p,q)$ on $L$ is given by the symmetric matrix $AB^{-1}$.*

Proof. From $\bar{C}^T \cdot C = I_n$ we see, that $B^T A$ is symmetric, then

$$AB^{-1} = (B^{-1})^T (B^T A) B^{-1}$$

is a symmetric matrix too. ∎

PROPOSITION 2.1.2. *The quadratic form*

$$S(q) = \frac{1}{2} q^T A B^{-1} q$$

*is a generating function for* $L$ .

Proof. From the definition of generating families we see, that $\frac{\partial S}{\partial q}(q) := AB^{-1}q = p$. If $q^T H q$ is a quadratic form, then $\frac{\partial}{\partial q}(q^T H q) = (H + H^T)q = 2Hq$, because H is a symmetric matrix. Then we obtain the Lagrangian subspace

$$L = \{(p,q) : p = AB^{-1}q\}. \blacksquare$$

**2.2. Nontransversal case**

If $L$ is not transversal to the fibres of $T^*\mathbb{R}^n$, then its generating family $S$ must have additional argument $\lambda \in \mathbb{R}^k$. In fact the function $S(q,\lambda)$ describes the Lagrangian subspace $\tilde{L}$ in $T^*\mathbb{R}^{n+k}$ such, that $\tilde{L}$ is transversal to the fibres of the bundle $T^*\mathbb{R}^{n+k}$, and the symplectic reduction at $\tilde{L}$ gives $L$, i.e.

$$\Pi : T^*\mathbb{R}^{n+k} \to T^*\mathbb{R}^n,$$

$$\Pi : ((p_1, \ldots p_n, p_{n+1}, \ldots p_{n+k}), (q_1, \ldots q_n, \lambda_1, \ldots \lambda_k)) \mapsto (p_1, \ldots p_n, q_1, \ldots q_n),$$

$$\Pi : \tilde{L} \cap \{p_{n+1} = \ldots = p_{n+k} = 0\} \mapsto L.$$

As in the section 2.1. we fix $L \in \Lambda_n$, then there exists $C \in U(n)$ such, that $L = CL_0 = \{(p,q) : p = AX,\ q = BX,\ X \in L_0\}$, $C = B + iA$. In nontransversal case $B$ is singular. We illustrate our construction on the diagram

$$\begin{array}{ccccccccc}
\mathbb{R}^n & \simeq & L_0 & \xrightarrow{B+Ai} & L & \subset & T^*\mathbb{R}^n & \xrightarrow{\pi_L} & \mathbb{R}^n \\
 & & & & \star\uparrow\Pi & & \uparrow\Pi & & \\
\mathbb{R}^{n+k} & \simeq & \tilde{L}_0 & \xrightarrow{\tilde{B}+\tilde{A}i} & \tilde{L} & \subset & T^*\mathbb{R}^{n+k} & \xrightarrow{\pi_L} & \mathbb{R}^{n+k}
\end{array}$$

where $\tilde{L}_0 = \mathrm{span}_R\{\frac{\partial}{\partial q_1}, \ldots, \frac{\partial}{\partial q_{n+k}}\}$, and

$$\tilde{L} = \{(\tilde{p},\tilde{q}) : \tilde{p} = \tilde{A}\tilde{X},\ \tilde{q} = \tilde{B}\tilde{X},\ \tilde{X} \in \tilde{L}_0\},$$

where $\tilde{L} = (\tilde{B} + \tilde{A}i)\tilde{L}_0$, and

$$(\tilde{p},\tilde{q}) = ((p_1, \ldots, p_n, p_{n+1}, \ldots, p_{n+k}), (q_1, \ldots, q_n, \lambda_1, \ldots, \lambda_k)) \in T^*\mathbb{R}^{n+k},$$

and $\star \uparrow_\Pi$ notes the symplectic reduction.

LEMMA 2.2.1. *Let* $\mathrm{rank}(B) = n - k$, *then there exists a real matrix* $W$ *of dimension* $n \times k$ *such, that its columns together with the columns of* $B$ *span* $\mathbb{R}^n$, *the matrix*

$$\tilde{B} = \begin{pmatrix} B & W \\ W^T A & 0 \end{pmatrix}$$

*is invertible, and* $\tilde{B}^T\tilde{A}$ *is symmetric, where*

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & E_k \end{pmatrix},$$

*and* $E_k$ *is the identity matrix.*

Proof. 1) We take any k vectors $w_1, \dots w_k \in \mathbf{R}^n$ such, that

$$\mathrm{span}_R\{b_1, \dots, b_n, w_1, \dots, w_k\} \simeq \mathbf{R}^n,$$

where $b_i$ $i = 1, \dots, n$ are columns of the matrix $B$.

We define

$$\tilde{B} = \begin{pmatrix} B & w_1, \dots, w_k \\ F & 0 \end{pmatrix}$$

and we look for a matrix $F$ of dimension $k \times n$ such, that $\tilde{B}^T\tilde{A}$ is symmetric. From the symmetry we calculate $F = W^T A$.

2) We prove, that $\tilde{B}$ is invertible.

a) For simplicity we assume, that $k = 1$, i.e. $\mathrm{rank}(B) = n - 1$.

We can find an orthogonal matrix $Y \in O(n)$ which transforms the matrix $B$ to the form $BY = (B_0\ 0) = \hat{B}$, where $\mathrm{rank}(B_0) = n - 1$. We define $\hat{C} = CY = (B + iA)Y = \hat{B} + i\hat{A} \in U(n)$, where

$$\hat{C} = \{\hat{b}_1 + i\hat{a}_1, \dots, \hat{b}_{n-1} + i\hat{a}_{n-1}, i\hat{a}_n\}.$$

We must also change the matrix $\tilde{B} = \begin{pmatrix} B & w \\ w^T A & 0 \end{pmatrix}$; the vector $w$ is linearly independent with the columns of the matrix $B$ if and only if, it is independent with the columns of $\hat{B}$

$$\tilde{\hat{B}} = \tilde{B}\begin{pmatrix} Y & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \hat{B} & w \\ w^T\hat{A} & 0 \end{pmatrix}.$$

The vector $w^T\hat{A}$ consists of the scalar products of the vector $w$ and the columns of the matrix $\hat{A}$. We easily calculate, that

$$\det \tilde{\hat{B}} = \det \begin{pmatrix} B_0 & 0 & w \\ \dots & \langle w, \hat{a}_n \rangle & 0 \end{pmatrix} = \pm\langle w, \hat{a}_n \rangle \det(B_0, w) = 0 \equiv w \perp \hat{a}_n,$$

where the symbol $\langle\ ,\ \rangle$ notes the scalar product. We suppose, that $w \in \hat{a}_n^{\perp}$. From the form of $\hat{C}$ we know, that $\hat{a}_n^{\perp} = \mathrm{span}_R\{\hat{b}_1, \dots \hat{b}_{n-1}\}$. Thus $w \in \mathrm{span}_R\{\hat{b}_1, \dots \hat{b}_{n-1}\}$, and we obtain the contradiction.

b) We assume, that $\mathrm{rank}(B) = n - k$, $1 < k < n$. There exists an orthogonal matrix $X \in O(n)$ which transforms the matrix $C$ into $\hat{C} =$

$CX \in U(n)$ in such a way, that $\hat{B} = BX = (\, B_0 \;\; 0 \;\; \ldots \;\; 0\,)$, where $\mathrm{rank}(B_0) = n - k$ and the last k columns of $\hat{B}$ are null

$$\hat{C} = \{\hat{b}_1 + i\hat{a}_1, \ldots, \hat{b}_{n-k} + i\hat{a}_{n-k}, i\hat{a}_{n-k+1}, \ldots i\hat{a}_n\} \in U(n).$$

The matrix $\tilde{B} = \begin{pmatrix} B & w \\ w^T A & 0 \end{pmatrix}$ is transformed into $\tilde{\hat{B}}$

$$\tilde{\hat{B}} = \tilde{B}\begin{pmatrix} X & 0 \\ 0 & E_k \end{pmatrix} = \begin{pmatrix} \hat{B} & W \\ W^T \hat{A} & 0 \end{pmatrix},$$

where $E_k$ is the identity matrix, and the matrix $W$ is not changed. From the form of $\tilde{\hat{B}}$ we calculate, that $\det(\tilde{\hat{B}}) = \pm \det P \cdot \det(B_0, W) = 0 \equiv \det P = 0$, where the matrix $P$ consists of the last k columns of the matrix $W^T \hat{A}$, its dimension is $k \times k$, and its terms are $\langle w_i, \hat{a}_j\rangle$ $i = 1, \ldots k, j = n - k + 1, \ldots n$. From the form of $\hat{C}$ we deduce, that $\hat{a}_j \perp \hat{b}_i$ $i = 1, \ldots n-k, j = n-k+1, \ldots n$, i.e. $\mathrm{span}_R\{\hat{b}_1, \ldots \hat{b}_{n-k}\}^\perp = \mathrm{span}_R\{\hat{a}_{n-k+1}, \ldots \hat{a}_n\}$.

We must transform the matrix $W$. Let $\chi$ be the projection to $\mathrm{span}_R\{\hat{b}_1, \ldots \hat{b}_{n-k}\}^\perp$, we define $v_i = \chi(w_i)$ $i = n-k+1, \ldots n$, then $\langle w_i, \hat{a}_j\rangle = \langle v_i, \hat{a}_j\rangle$ $i = 1, \ldots k$, $j = n - k + 1, \ldots n$. The vectors $v_i$ $i = 1, \ldots k$ form a basis of $\{\hat{b}_1, \ldots \hat{b}_{n-k}\}^\perp$. We can represent vectors $\hat{a}_j$ in the basis $\{v_1, \ldots v_k\}$, then $P = V^T \cdot \hat{A}_k$, where $V = (v_1 \ldots v_k)$, $\hat{A}_k = (\hat{a}_{n-k+1} \ldots \hat{a}_n)$. We see, that $\det P = \det V \cdot \det \hat{A}_k \neq 0$. ■

The subspace $\tilde{L} = (\tilde{B} + \tilde{A}i)\tilde{L}_0 \subset T^*\mathbf{R}^{n+k}$ is Lagrangian because the matrix $\tilde{B}^T \tilde{A}$ is symmetric. It is transversal to the fibres of the bundle, then from 2.1.2. we have the form of the generating function for $\tilde{L}$.

PROPOSITION 2.2.2. *The generating function $S$ for the Lagrangian subspace $L$ is given by the symmetric matrix $\tilde{A}\tilde{B}^{-1}$ and has the following form*

$$S = \frac{1}{2}\tilde{q}^T \tilde{A}\tilde{B}^{-1}\tilde{q}, \quad \tilde{q} = \begin{pmatrix} q \\ \lambda \end{pmatrix} \in \mathbf{R}^{n+k}.$$

Proof. We have

$$\tilde{L} = \{(\tilde{p}, \tilde{q}) : \tilde{p} = \tilde{A}\tilde{X},\ \tilde{q} = \tilde{B}\tilde{X},\ \tilde{X} \in \tilde{L}_0\}.$$

We calculate, that

$$[p_1, \ldots p_n, p_{n+1}, \ldots p_{n+k}]^T = \begin{pmatrix} A & 0 \\ 0 & E_k \end{pmatrix} \begin{pmatrix} x_1 \\ \ldots \\ x_n \\ x_{n+1} \\ \ldots \\ x_{n+k} \end{pmatrix} = \begin{pmatrix} A\begin{pmatrix} x_1 \\ \ldots \\ x_n \end{pmatrix} \\ x_{n+1} \\ \ldots \\ x_{n+k} \end{pmatrix},$$

$$[q_1,\dots q_n,\lambda_1,\dots,\lambda_k]^T = \begin{pmatrix} B & W \\ W^T A & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \dots \\ x_n \\ x_{n+1} \\ \dots \\ x_{n+k} \end{pmatrix} = \begin{pmatrix} K \\ W^T A \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix} \end{pmatrix},$$

where $K = B \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix} + W \begin{pmatrix} x_{n+1} \\ \dots \\ x_{n+k} \end{pmatrix}$.

We obtain the subspace $L$ from $\tilde{L}$ in the following way

$$\Pi : \tilde{L} \cap \{p_{n+1} = \dots = p_{n+k} = 0\} \mapsto L,$$
$$\Pi : ((p_1,\dots p_n,0,\dots 0),(q_1,\dots q_n,\lambda_1,\dots \lambda_k)) \mapsto (p_1,\dots p_n,q_1,\dots q_n).$$

We see, that in our case $[p_{n+1},\dots p_{n+k}] = [x_{n+1},\dots x_{n+k}]$.

From 2.1.2. we know, that the function $S$ generates the Lagrangian subspace $\tilde{L} \subset T^*\mathbb{R}^{n+k}$, i.e.

$$\frac{dS}{d\tilde{q}}(\tilde{q}) = \tilde{p}, \quad \tilde{p} = [p_1,\dots p_n,p_{n+1},\dots p_{n+k}]^T,$$
$$\tilde{q} = [q_1,\dots q_n,\lambda_1,\dots,\lambda_{n+k}]^T.$$

We write this formula in the following form

$$\frac{\partial S}{\partial q_i}(q,\lambda) = p_i \quad i = 1,\dots n, \quad \frac{\partial S}{\partial \lambda_j}(q,\lambda) = p_{n+j} \quad j = 1,\dots k.$$

We obtain $L$ from $\tilde{L}$ by the symplectic reduction, i.e.

$$\Pi : \tilde{L} \cap \{p_{n+1} = \dots = p_{n+k} = 0\} \mapsto L,$$

thus $\frac{\partial S}{\partial \lambda_j}(q,\lambda) = 0$, $j = 1,\dots k$. We see, that the function $S$ generates the following subspace

$$L = \left\{(p,q) \in T^*\mathbb{R}^n : \exists\lambda \in \mathbb{R}^k, \quad \frac{\partial S}{\partial \lambda}(q,\lambda) = 0, \quad \frac{\partial S}{\partial q}(q,\lambda) = p\right\}. \ \blacksquare$$

Remark 2.2.3. The generating function for the Lagrangian subspace has the following form $S(q,\lambda) = f(q) + \Sigma_{i=1}^k \lambda_i \cdot g_i(q)$, where $f(q)$ is the quadratic function and $g_i(q)$ are linear functions (see. [2]).

From the form of matrices $\tilde{A}$ and $\tilde{B}$, we see that the symmetric matrix $\tilde{A}\tilde{B}^{-1}$ has the block form, and the block corresponding to the products $\lambda_i\lambda_j$, $i,j = 1,\dots k$ equals 0, therefore the generating function $S(q,\lambda) = \frac{1}{2}(q,\lambda)\tilde{A}\tilde{B}^{-1}\begin{pmatrix} q \\ \lambda \end{pmatrix}$ is the linear function of $\lambda$.

## 3. Generating families for isotropic subspaces in $\mathbf{R}^{2n}$

In analogous way as for the Lagrangian submanifolds, to each germ of an immersed isotropic submanifold $(I,0) \subset T^*\mathbf{R}^n$ there exists a germ of $I$-Morse family

$$S : (\mathbf{R}^n \times \mathbf{R}^l \times \mathbf{R}^k, 0) \to \mathbf{R}$$

such, that $(I,0)$ may be written in the form

$$I = \Big\{ (p,q) \in T^*\mathbf{R}^n : \exists \lambda \in \mathbf{R}^k,\ \frac{\partial S}{\partial q}(q,0,\lambda) = p, \quad \frac{\partial S}{\partial \lambda}(q,0,\lambda) = \frac{\partial S}{\partial \beta}(q,0,\lambda) = 0 \Big\}$$

(see [5]).

### 3.1. Generic case

Let $I$ be an isotropic subspace of $T^*\mathbf{R}^n$ such, that $\dim I = n - l$, and the projection of $I$ to the base of the bundle has the maximal dimension, i.e. $I$ is in the generic position.

Any isotropic subspace can be included into a Lagrangian one (see [5], [10]).

If the subspace $I$ is in generic position, then we can find the Lagrangian subspace $L$ such, that $I$ is included into $L$, and $L$ is transversal to the fibres of $T^*\mathbf{R}^n$. We may use results from 2.1 and add the conditions which allow us to obtain $I$ from $L$, i.e. $\frac{\partial S}{\partial \beta}(q,0,\lambda) = 0$.

We can obtain $I$ from $I_0 = \text{span}_R\{\frac{\partial}{\partial q_1}, \dots \frac{\partial}{\partial q_{n-l}}\}$ in a way $I = DI_0$, where $D \in U(n)$, $D = B + iA$ and $B$ is nonsingular. The subspace $I_0$ is included into $L_0 = \text{span}_R\{\frac{\partial}{\partial q_1}, \dots \frac{\partial}{\partial q_n}\}$ which is transversal to the fibres. The Lagrangian subspace $L$ obtained from $L_0$ by action of the matrix $D$, i.e. $L = DL_0$ is transversal to the fibres too.

We illustrate our construction on the diagram

$$\begin{array}{ccccccccc} \mathbf{R}^{n-l} & \simeq & I_0 & \xrightarrow{B+Ai} & I & \subset & T^*\mathbf{R}^n & \xrightarrow{\pi_I} & \mathbf{R}^n \\ & & \cap & & \cap & & & & \\ \mathbf{R}^n & \simeq & L_0 & \xrightarrow{B+Ai} & L & \subset & T^*\mathbf{R}^n & \xrightarrow{\pi_L} & \mathbf{R}^n \end{array}$$

From 2.1. we know, that the generating function for $L$ has the form $F(q) = \frac{1}{2}q^T AB^{-1} q$ where $D = B + iA$. The generating function for $I$ must be represented by a matrix of dimension $(n+l) \times (n+l)$. We must formulate the condition which allow us to obtain $I$ from $L$.

$$L = \{(p,q) : [p_1, \dots p_n]^T = A[x_1, \dots x_n]^T,\ [q_1, \dots q_n]^T = B[x_1, \dots x_n]^T\},$$

$$I = \{(p,q) : [p_1, \dots, p_n]^T = A[x_1, \dots x_{n-l}, 0 \dots, 0]^T,$$
$$[q_1, \dots, q_n]^T = B[x_1, \dots, x_{n-l}, 0, \dots, 0]^T\},$$

where $x_i$, $i = 1, \dots n$, are coordinates of $I_0$ or $L_0$. The matrix $B$ is nonsingular, thus $B^{-1}[q_1, \dots q_n]^T = [x_1, \dots x_{n-l}, 0, \dots 0]^T$. The matrix $B^{-1}$ consists of two blocks $B^{-1} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$, where $C_1$ has the dimension $(n-l) \times n$ and $C_2$ dimension $l \times n$. The equation above decomposes:

$$C_1 \begin{pmatrix} q_1 \\ \dots \\ q_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \dots \\ x_{n-l} \end{pmatrix}, \quad C_2 \begin{pmatrix} q_1 \\ \dots \\ q_n \end{pmatrix} = \begin{pmatrix} 0 \\ \dots \\ 0 \end{pmatrix}.$$

The matrix $C$ transforms $I$ into $I_0$. The condition $C_2 q = 0$ consists of $l$ equations, and corresponds to the relations $\frac{\partial S}{\partial \beta}(q, 0, \lambda) = 0$ from the definition of generating families for an isotropic germ.

PROPOSITION 3.1.1. *The generating function for an isotropic subspace in $T^*\mathbb{R}^n$ in a generic position to the fibres of the bundle has the following form*

$$S(q,\beta) = \frac{1}{2}[q,\beta]M[q,\beta]^T \quad \textit{where } S : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}$$

*and*

$$M = \begin{pmatrix} AB^{-1} & C_2^T \\ C_2 & 0 \end{pmatrix}.$$

Proof. We have

$$dS(q,0) = M[q,0]^T = \begin{pmatrix} AB^{-1} & C_2^T \\ C_2 & 0 \end{pmatrix} \begin{pmatrix} q \\ 0 \end{pmatrix} = \begin{pmatrix} AB^{-1}q \\ C_2 q \end{pmatrix} = \begin{pmatrix} p \\ 0 \end{pmatrix}$$

$$\frac{\partial S}{\partial q}(q,0) = AB^{-1}q = p, \quad \frac{\partial S}{\partial \beta}(q,0) = C_2 q = 0. \ \blacksquare$$

### 3.2. Nongeneric case

Let $I$ be an isotropic subspace in $T^*\mathbb{R}^n$ of dimension $n-l$, and we assume, that $\dim \pi_I(I) = n - l - k$, where $\pi_I$ is the projection to the base of the bundle.

We can obtain $I$ from $I_0$, i.e. $I = DI_0$, $D \in U(n)$, $D = B + iA$. $B$ is singular. We can include $I$ in a Lagrangian subspace $L$ such, that $\dim \pi_L(L) = n - k$, i.e. $\text{rank}(B) = n - k$. Let $L = DL_0$ where $L_0 = \text{span}_R\{\frac{\partial}{\partial q_1}, \dots \frac{\partial}{\partial q_n}\}$. Then we may use results from 2.2 and write the generating family $F : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$ for $L$ in the form $F(q,\lambda) = \frac{1}{2}[q,\lambda]\tilde{A}\tilde{B}^{-1}[q,\lambda]^T$, where $\tilde{A}$ and $\tilde{B}$ are matrices of dimension $(n+k)\times(n+k)$. We know, that the function $F$ describes the Lagrangian subspace $\tilde{L} \in T^*\mathbb{R}^{n+k}$. The subspace $L$ is obtained from $\tilde{L}$ by the symplectic reduction $\Pi$ (see 2.2.).

We illustrate our construction on the diagram

$$\begin{array}{ccccccccc}
\mathbf{R}^{n-l} & \simeq & I_0 & \xrightarrow{B+Ai} & I & \subset & T^*\mathbf{R}^n & \xrightarrow{\pi_I} & \mathbf{R}^n \\
 & & \cap & & \cap & & & & \\
\mathbf{R}^n & \simeq & L_0 & \xrightarrow{B+Ai} & L & \subset & T^*\mathbf{R}^n & \xrightarrow{\pi_L} & \mathbf{R}^n \\
 & & & & \uparrow{\scriptstyle *} & & \uparrow{\scriptstyle \Pi} & & \\
\mathbf{R}^{n+k} & \simeq & \tilde{L}_0 & \xrightarrow{\tilde{B}+\tilde{A}i} & \tilde{L} & \subset & T^*\mathbf{R}^{n+k} & \xrightarrow{\pi_L} & \mathbf{R}^{n+k}
\end{array}$$

In fact $I$ is obtained from $\tilde{I} \subset \tilde{L}$ by the symplectic reduction, where

$$\tilde{I} = \{\tilde{q} + \tilde{p}i = (\tilde{B} + \tilde{A}i)X, \quad X = [x_1, \ldots, x_{n-l}, 0, \ldots, 0, x_{n+1}, \ldots x_{n+k}]\}.$$

The matrix representing the generating family for $I$ must have the dimension $(n + k + l) \times (n + k + l)$.

Using the description of the matrix $B^{-1}$ from 3.1, we write the matrix $\tilde{B}^{-1}$ in the block form

$$\tilde{B}^{-1} = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix},$$

where the matrices $C_1, C_2, C_3$ have the following dimensions $(n - l) \times (n + k)$, $l \times (n + k)$, $k \times (n + k)$. $C_1$ transforms $I$ into $I_0$, $C_2$ gives conditions corresponding to the equation

$$\frac{\partial S}{\partial \beta}(q, 0, \lambda) = 0$$

in the definition of the generating family of the isotropic germ

$$C_2 \begin{pmatrix} q \\ \lambda \end{pmatrix} = 0.$$

$C_3$ is due to the nontransversality of $L$ including $I$.

PROPOSITION 3.2.1. *The generating family for the isotropic subspace in $T^*\mathbf{R}^n$ in nongeneric position to the fibres of the bundle has the following form*

$$S(q, \lambda, \beta) = \frac{1}{2}[q, \lambda, \beta] M [q, \lambda, \beta]^T$$

*where $M$ is the symmetric matrix having the block form*

$$M = \begin{pmatrix} \tilde{A}\tilde{B}^{-1} & C_2^T \\ C_2 & 0 \end{pmatrix}.$$

Proof. The subspace $I$ is obtained from $\tilde{I}$ by the symplectic reduction, where

$$\tilde{I} = \{(\tilde{p},\tilde{q}) : \ \tilde{q} = \tilde{B}X, \ \tilde{p} = \tilde{A}X, \ X = [x_1,\dots x_{n-l},0,\dots 0,x_{n+1},\dots x_{n+k}]\}$$
$$= \{(\tilde{p},\tilde{q}) : \ \tilde{p} = \tilde{A}\tilde{B}^{-1}\tilde{q}, \ 0 = C_2\tilde{q}\}.$$

From the form of $\tilde{I}$, we have the description

$$I = \{(p,q) : \exists\lambda \in \mathbb{R}^k \ \ (p,0) = \tilde{A}\tilde{B}^{-1}[q,\lambda]^T, \ 0 = C_2[q,\lambda]^T\}.$$

We calculate

$$\frac{\partial S}{\partial \tilde{q}}(\tilde{q},0) = \tilde{A}\tilde{B}^{-1}\tilde{q}, \quad \frac{\partial S}{\partial \beta}(\tilde{q},0) = C_2\tilde{q}; \quad \tilde{q} = (q,\lambda).$$

Thus

$$I = \{(p,q) \in T^*\mathbb{R}^n :$$
$$\exists\lambda \in \mathbb{R}^k \frac{\partial S}{\partial q}(q,\lambda,0) = p, \ \frac{\partial S}{\partial \lambda}(q,\lambda,0) = \frac{\partial S}{\partial \beta}(q,\lambda,0) = 0\}. \ \blacksquare$$

## 4. Examples

We calculate the generating family for the isotropic subspace $I$ in $T^*\mathbb{R}^2$ which is not in generic position to the fibres.

Let $I$ be 1-dimensional subspace in $T^*\mathbb{R}^2$, and $I$ projected to the base of the bundle is the point $(0,0)$. We can fix $a,b \in \mathbb{R}$ such, that $a^2+b^2=1$, and

$$I = \{(p_1,p_2,q_1,q_2) \in T^*\mathbb{R}^2 : bp_1 - ap_2 = 0, \ q_1 = q_2 = 0\},$$

where $q_1,q_2$ are the coordinates of the base, and $p_1,p_2$ of the fibres.

The subspace $I$ is spanned by one vector

$$\alpha_1 = [q_1 + ip_1, q_2 + ip_2] = [ia, ib] \in \mathbb{C}^2.$$

We can include $I$ in such Lagrangian subspace $L$, that its projection to the base of the bundle is 1-dimensional. This Lagrangian subspace $L$ is generated by two vectors $\alpha_1$ and $\alpha_2$, where $\alpha_2 = [-b,a] \in \mathbb{R}^2$. We choose $\alpha_2$ such, that $\alpha_1 \perp \alpha_2$ and $|\alpha_2| = 1$.

Then from Proposition 2.2.2 the matrices $\tilde{A},\tilde{B}$ have the following form

$$\tilde{A} = \begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} 0 & -b & a \\ 0 & a & b \\ 1 & 0 & 0 \end{pmatrix}$$

and the matrix

$$\tilde{A}\tilde{B}^{-1} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ a & b & 0 \end{pmatrix}$$

represents the generating function $F$ for $L$,

$$F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$$
$$F(q_1, q_2, \lambda) = \frac{1}{2}[q_1, q_2, \lambda] \cdot \tilde{A}\tilde{B}^{-1} \cdot [q_1, q_2, \lambda]^T = \lambda a q_1 + \lambda b q_2.$$

In fact the function $F$ describes the Lagrangian subspace $\tilde{L} \subset T^*\mathbb{R}^3$ which is transversal to the fibres of the cotangent bundle $T^*\mathbb{R}^3$,

$$\tilde{L} = \{(p_1, p_2, p_3, q_1, q_2, \lambda) : p_1 = a\lambda,\ p_2 = b\lambda,\ p_3 = aq_1 + bq_2\}.$$

To obtain the Lagrangian subspace $L \subset T^*\mathbb{R}^2$, we intersect the subspace $\tilde{L}$ with the set $\{p_3 = 0\}$ and we project it to $T^*\mathbb{R}^2$,

$$L = \{(p_1, p_2, q_1, q_2) : \exists \lambda \in \mathbb{R} \quad p_1 = a\lambda,\ p_2 = b\lambda,\ aq_1 + bq_2 = 0\}.$$

From Proposition 3.2.1 we know, that the matrix representing the generating function $S : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for $I$, has the form

$$M = \begin{pmatrix} \tilde{A}\tilde{B}^{-1} & C_2^T \\ C_2 & 0 \end{pmatrix}$$

where $C_2 = [-b, a, 0]$ is the second row of $\tilde{B}^{-1}$.

We calculate, that

$$S(q_1, q_2, \lambda, \beta) = \frac{1}{2}(q_1, q_2, \lambda, \beta) \begin{pmatrix} 0 & 0 & a & -b \\ 0 & 0 & b & a \\ a & b & 0 & 0 \\ -b & a & 0 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \lambda \\ \beta \end{pmatrix}$$
$$= \lambda a q_1 - \beta b q_1 + \lambda b q_2 + \beta a q_2.$$

From the definition of the generating function we know, that the subspace $I$ is generated in the following way $I = \{(p, q) \in T^*\mathbb{R}^n : \exists \lambda \in \mathbb{R}^k$ such that $\frac{\partial S}{\partial q}(q, \lambda, 0) = p$, $\frac{\partial S}{\partial \lambda}(q, \lambda, 0) = \frac{\partial S}{\partial \beta}(q, \lambda, 0) = 0\}$. We calculate, that

$$\frac{\partial S}{\partial q_1}(q_1, q_2, \lambda, 0) = \lambda a = p_1, \qquad \frac{\partial S}{\partial q_2}(q_1, q_2, \lambda, 0) = \lambda b = p_2,$$
$$\frac{\partial S}{\partial \beta}(q_1, q_2, \lambda, 0) = aq_2 - bq_1 = 0, \quad \frac{\partial S}{\partial \lambda}(q_1, q_2, \lambda, 0) = aq_1 + bq_2 = 0.$$

We obtain the following description of the isotropic subspace $I$

$$I = \{(p_1, p_2, q_1, q_2) : \exists \lambda \in \mathbb{R} \quad p_1 = a\lambda,\ p_2 = b\lambda,\ q_1 = q_2 = 0, \}.$$

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