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THE KOLMOGOROV-SINAJ THEOREM ON GENERATORS FOR FUZZY DYNAMICAL SYSTEMS

1. Introduction

In this paper, we shall work with a fuzzy generalization of the notion of dynamical system $(X, \mathcal{S}, P, T)$ from the Kolmogorov classical model of probability theory, with so-called fuzzy dynamical systems. In the classical case, given a probability space $(X, \mathcal{S}, P)$ and a measure - preserving transformation $T : X \to X$ Kolmogorov and Sinaj constructed an invariant $h(T)$ such that h coincides on isomorphic dynamical systems. The invariant $h(T)$ is called (see [4, 15]) the entropy of the dynamical system $(X, \mathcal{S}, P, T)$.

The notion of fuzzy dynamical system and its entropy have been introduced by the second author in [5]. Fuzzy dynamical systems include the classical systems, on the other hand enable us to study more general situations, for example Markov's operators. The classification of fuzzy dynamical systems is given in [6]. In the paper [8], it is shown that two isomorphic fuzzy dynamical systems have the same entropy.

Probably one of the most important results of the theory of invariant measures for practical purposes is the Kolmogorov–Sinaj theorem stating that $h(T) = h(T, \mathcal{A})$, whenever $\mathcal{A}$ is a partition generating the given σ-algebra $\mathcal{S}$. A fuzzy analogy of this theorem is proved in [7], see also [8]. The results of Piasecki (Theorem 9 in [11]), inspired us to prove the above theorem still in another simple way.

We note that some other approaches to the problem of a fuzzy generalization of Kolmogorov-Sinaj's entropy can be found, for example, in [1, 3, 12, 13, 14]. There, some other connectives have been used to define the set operations for fuzzy sets.

Key words and phrases: Fuzzy probability space; fuzzy dynamical system; entropy; generators.

2. Basic definitions and notations

Let us recall some definitions and basic facts which will be used throughout this note.

By a fuzzy probability space [10] we mean a triplet (X, M, m), where X is a non-empty set, M is a fuzzy σ-algebra (i.e. $M \subset \langle 0,1\rangle^X$ such that:

(i) $1 \in M$, $\frac{1}{2} \notin M$;

(ii) if $a \in M$, then $a' := 1 - a \in M$;

(iii) if $a_n \in M$, $n = 1, 2, \ldots$,

then $\bigvee_{n=1}^{\infty} a_n \in M$) and the mapping $m : M \to \langle 0, \infty)$ fulfils the following conditions:

(iv) $m(a \vee a') = 1$ for every $a \in M$;

(v) if $\{a_n\}_{n=1}^{\infty} \subset M$ such that $a_i \leqq 1 - a_j$ (pointwisely) whenever $i \neq j$, then $m(\bigvee_{n=1}^{\infty} a_n) = \sum_{n=1}^{\infty} m(a_n)$.

The symbols $\bigvee_n a_n := \sup_n a_n$ and $\bigwedge_n a_n := \inf_n a_n$ denote the fuzzy union and the fuzzy intersection of a sequence $\{a_n\}_n \subset M$, respectively, in the sense of Zadeh [16]. Each mapping $m : M \to \langle 0, \infty)$ having the properties (iv) and (v) is called in the terminology of Piasecki a fuzzy P-measure; the system M is called a soft-fuzzy σ-algebra [10]. The presented fuzzy P-measure fulfils all properties analogous to the properties of classical probability in the crisp case [10].

A fuzzy partition (of the space (X, M, m)) is a finite collection $\mathcal{A} = \{a_1, \ldots, a_n\}$ of members of M such that $m(\bigvee_{i=1}^{n} a_i) = 1$ and $a_i \leqq 1 - a_j$ whenever $i \neq j$. K. Piasecki has formulated in [10] the Bayes formula for these partitions. The entropy of these partitions is defined and studied by the second author in [5, 7]. We define the entropy of any fuzzy partition $\mathcal{A} = \{a_1, \ldots, a_n\}$ by Shannon's formula:

$$H_m(\mathcal{A}) = \sum_{i=1}^{n} F(m(a_i)),$$

where

$$F : \langle 0, \infty) \to R, \quad F(x) = \begin{cases} -x \log x & \text{if } x > 0 \\ 0 & \text{if } x = 0. \end{cases}$$

If $\mathcal{A} = \{a_1, \ldots, a_n\}$, $\mathcal{B} = \{b_1, \ldots, b_k\}$ are two fuzzy partitions, we define the conditional entropy as follows:

$$H_m(\mathcal{B}|\mathcal{A}) = \sum_{i=1}^{n} \sum_{j=1}^{k} m(a_i) \cdot F(\mathring{m}(b_j|a_i)),$$

where

$$\mathring{m}(g_j|f_i) = \begin{cases} \dfrac{m(g_j \wedge f_i)}{m(f_i)} & \text{if } m(f_i) > 0 \\ 0 & \text{if } m(f_i) = 0. \end{cases}$$

By a fuzzy dynamical system [5, 7] we mean a quadruple $(X, M, m, \mathcal{U})$, where (X, M, m) is a fuzzy probability space and $\mathcal{U} : M \to M$ is an m-preserving σ-homomorphism (i.e. $\mathcal{U}(a') = 1 - \mathcal{U}(a), \mathcal{U}(\bigvee_{n=1}^{\infty} a_n) = \bigvee_{n=1}^{\infty} \mathcal{U}(a_n)$ and $m(\mathcal{U}(a)) = m(a)$ for every $a \in M$ and any sequence $\{a_n\}_{n=1}^{\infty} \subset M$).

In the set $\mathcal{P}$ of all fuzzy partitions of the space (X, M, m) the operation $\vee$ is defined via $\mathcal{A} \vee \mathcal{B} := \{a \wedge b;\ a \in \mathcal{A},\ b \in \mathcal{B}\}$. We shall say that $\mathcal{B}$ is a refinement of $\mathcal{A}$ (and we write $\mathcal{A} \leqq \mathcal{B}$) iff for every $b \in \mathcal{B}$ there exists $a \in \mathcal{A}$ such that $b \leqq a$. Since $\mathcal{A} \leqq \mathcal{A} \vee \mathcal{B}$ and $\mathcal{B} \leqq \mathcal{A} \vee \mathcal{B}$, the symbol $\mathcal{A} \vee \mathcal{B}$ should be read as a common refinement of $\mathcal{A}$ and $\mathcal{B}$. We define the entropy h_m of fuzzy dynamical system $(X, M, m, \mathcal{U})$ as follows: $h_m(\mathcal{U}) = \sup\{h_m(\mathcal{U}, \mathcal{A});\ \mathcal{A} \in \mathcal{P}\}$, where $h_m(\mathcal{U}, \mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} H_m(\bigvee_{i=0}^{n-1} \mathcal{U}^i \mathcal{A})$.

Evidently, $\mathcal{U}\mathcal{A} := \{\mathcal{U}(f); f \in \mathcal{A}\}$ is also a fuzzy partition. In [7] it is proved that the above entropy has all properties analogous to the properties of entropy in the crisp case.

(2.1) If $\mathcal{A}, \mathcal{B} \in \mathcal{P}$, $\mathcal{A} \leqq \mathcal{B}$, then $H_m(\mathcal{A}) \leqq H_m(\mathcal{B})$.

(2.2) $H_m(\mathcal{B} \vee \mathcal{C}|\mathcal{A}) = H_m(\mathcal{C}|\mathcal{A} \vee \mathcal{B}) + H_m(\mathcal{B}|\mathcal{A})$ for every $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{P}$.

(2.3) Let $\mathcal{A}, \mathcal{B} \in \mathcal{P}$, $\mathcal{A} \leqq \mathcal{B}$. Then for each $\mathcal{C} \in \mathcal{P}$ $H_m(\mathcal{A}|\mathcal{C}) \leqq H_m(\mathcal{B}|\mathcal{C})$ and $H_m(\mathcal{C}|\mathcal{A}) \geqq H_m(\mathcal{C}|\mathcal{B})$.

(2.4) $H_m(\mathcal{B} \vee \mathcal{C}|\mathcal{A}) \leqq H_m(\mathcal{B}|\mathcal{A}) + H_m(\mathcal{C}|\mathcal{A})$ for each $\mathcal{A}, \mathcal{B}, \mathcal{C} \in P$.

(2.5) $h_m(\mathcal{U}, \mathcal{A}) = h_m(\mathcal{U}, \bigvee_{i=0}^{k} \mathcal{U}^i \mathcal{A})$ for every $\mathcal{A} \in \mathcal{P}$ and for every natural number k.

(2.6) $h_m(\mathcal{U}, \mathcal{B}) \leqq h_m(\mathcal{U}, \mathcal{A}) + H_m(\mathcal{B}|\mathcal{A})$ for every $\mathcal{A}, \mathcal{B} \in \mathcal{P}$.

3. The Kolmogorov-Sinaj theorem on generators

In this section we shall present a new proof of Kolmogorov-Sinaj's theorem on generators for fuzzy dynamical systems. The main tool is a representation of a fuzzy σ-algebra M by a Boolean σ-algebra. In [7], the theorem on generators was proved by means of representation of M by a Boolean σ-algebra $[M] := \{[f]; f \in M\}$, where $[f] = \{g \in M; m(f \wedge g') = m(f' \wedge g) = 0\}$ for any $f \in M$. Here, we shall use another type of representation. It is based on a Piasecki construction.

Let a fuzzy probability space (X, M, m) be given. In accordance with Piasecki [11] (see also [2]), we denote by $K(M)$ the system of subset $A \subset X$ for which there exists a fuzzy subset $a \in M$ such that

$$\left\{a > \frac{1}{2}\right\} \subset A \subset \left\{a \geqq \frac{1}{2}\right\}. \tag{3.1}$$

Of course, here

$$\left\{a > \frac{1}{2}\right\} = \left\{x \in X;\ a(x) > \frac{1}{2}\right\};$$

analogously

$$\left\{a \geqq \frac{1}{2}\right\} = \left\{x \in X;\ a(x) \geqq \frac{1}{2}\right\}.$$

From the next theorem it follows that any fuzzy probability space (X, M, m) determines a probability space in the classical sense.

3.1. THEOREM ([11]). *Let a fuzzy probability space (X, M, m) be given. $K(M)$ is a σ-algebra of subsets of the set X. The mapping $P_m : K(M) \to \langle 0, 1\rangle$ defined by the equality $P_m(A) = m(a)$ for all $A \in K(M)$, where A and $a \in M$ fulfil* (3.1), *is a probability measure on $K(M)$ satisfying the condition $P_m(\{a = \frac{1}{2}\}) = 0$ for any $a \in M$.*

3.2. Remark. Let $(X, \mathcal{S}, P)$ be a probability space in the sense of classical probability theory. Put $M = \{\chi_A;\ A \in \mathcal{S}\}$, where χ_A is the characteristic function of the set $A \in \mathcal{S}$. If we define the mapping $m : M \to \langle 0, 1\rangle$ via $m(\chi_a) = P(A)$, then the triplet (X, M, m) is a fuzzy probability space. We shall say that the system (X, M, m) is induced by the probability space $(X, \mathcal{S}, P)$. It is easy to see that in this case there holds $K(M) = \mathcal{S}$. Moreover, we have

$$P_m(A) = m(\chi_A) = P(A) \quad \text{for every } A \in K(M).$$

3.3. Notation. If $a \in M$, $A \in K(M)$ and $\{a > \frac{1}{2}\} \subset A \subset \{a \geqq \frac{1}{2}\}$, then we write $a \sim A$.

3.4. PROPOSITION. *To every fuzzy partition $\mathcal{A} = \{a_1, \ldots, a_k\} \subset M$ there exists a set partition $\mathbf{A} = \{A_1, \ldots, A_k, L\} \subset K(M)$ such that $a_i \sim A_i$ $(i = 1, \ldots, k)$ and $P_m(L) = 0$.*

Proof. Choose $\mathcal{B}_i \in K(M)$ such that $a_i \sim B_i$. Since $a_i + a_j \leqq 1$ $(i \neq j)$, we have $a_i \wedge a_j \leqq \frac{1}{2}$, hence $P_m(B_i \cap B_j) = m(a_i \wedge a_j) = 0$ $(i \neq j)$. Put $L = \bigcup_{i \neq j}(B_i \cap B_j)$, $A_i = B_i \setminus L$, $i = 1, \dots, k$. Then $a_i \sim A_i$, $i = 1, \dots, k$, $P_m(L) = 0$.

Hence $\mathbf{A} = \{A_1, \dots, A_k, L\}$ satisfies all the conditions stated above.

3.5. Notation. If $\mathcal{A} = \{a_1, \dots, a_k\}$ is a fuzzy partition and $\mathbf{A}$ a set partition such that $\mathbf{A} = \{A_1, \dots, A_k, L_1, \dots, L_t\}$, where $a_i \sim A_i$ $(i = 1, \dots, k)$ and $P_m(L_j) = 0$ $(j = 1, \dots, t)$, then we write $\mathcal{A} \sim \mathbf{A}$.

3.6. PROPOSITION. *Let $\mathcal{A}, \mathcal{C}$ be fuzzy partitions, $\mathbf{A}, \mathbb{C}$ set partitions, $\mathcal{A} \sim \mathbf{A}$, $\mathcal{C} \sim \mathbb{C}$. Then $H_m(\mathcal{A}) = H(\mathbf{A})$, $H_m(\mathcal{A}|\mathcal{C}) = H(\mathbf{A}|\mathbb{C})$.*

Proof. Let $\mathcal{A} = \{a_1, \dots, a_k\}$, $\mathbf{A} = \{A_1, \dots, A_k, L\}$, $a_i \sim A_i$ $(i = 1, \dots, k)$, $P_m(L) = 0$. Then

$$H(\mathbf{A}) = \sum_{i=1}^{k} F(P_m(A_i)) + F(P(L)) = \sum_{i=1}^{k} F(m(a_i)) = H_m(\mathcal{A}).$$

The second assertion can be proved similarly.

3.7. PROPOSITION. *If $\mathcal{A}, \mathcal{B}$ are fuzzy partitions and $\mathbf{A}, \mathbb{B}$ set partitions such that $\mathcal{A} \sim \mathbf{A}$, $\mathcal{B} \sim \mathbb{B}$, then $\mathcal{A} \vee \mathcal{B} \sim \mathbf{A} \vee \mathbb{B}$. Moreover, if $\mathcal{C}$ is such a fuzzy partition that $\mathcal{A} \leqq \mathcal{C}$, then there exists a set partition $\mathbb{C}$ such that $\mathcal{C} \sim \mathbb{C}$ and $\mathbf{A} \leqq \mathbb{C}$.*

Proof. Let $\mathcal{A} = \{a_1, \dots, a_k\}$, $\mathcal{B} = \{b_1, \dots, b_n\}$, $\mathbf{A} = \{A_1, \dots, A_k, L_1\}$, $\mathbb{B} = \{B_1, \dots, B_n, L_2\}$, where $a_i \sim A_i$ $(i = 1, \dots, k)$, $b_j \sim B_j$ $(j = 1, \dots, n)$, $P_m(L_1) = P_m(L_2) = 0$. Then

$$\begin{aligned} \mathcal{A} \vee \mathcal{B} &= \{a_i \wedge b_j;\ i = 1, \dots, k,\ j = 1, \dots, n\}, \\ \mathbf{A} \vee \mathbb{B} &= \{A_i \cap B_j,\ i = 1, \dots, k,\ j = 1, \dots, n\} \\ &\quad \cup \Big(\Big(\bigcup_{i=1}^{k} A_i\Big) \cap L_2\Big) \cup \Big(L_1 \cap \Big(\bigcup_{j=1}^{n} B_j\Big)\Big). \end{aligned}$$

If we put $L = ((\bigcup_{i=1}^{k} A_i) \cap L_2) \cup (L_1 \cap (\bigcup_{j=1}^{n} B_j))$, then $P_m(L) = 0$. Moreover, $a_i \wedge b_j \sim A_i \cap B_j$ for all i, j. Therefore $\mathcal{A} \vee \mathcal{B} \sim \mathbf{A} \vee \mathbb{B}$. Choose now $\mathbf{D}$ arbitrary such that $\mathcal{C} \sim \mathbf{D}$, $\mathbf{D} = \{D_1, \dots, D_n, K\}$, $\mathcal{C} = \{c_1, \dots, c_n\}$. For every c_j there exists a_i such that $c_j \leqq a_i$. Therefore

$$D_j \subset \left\{c_j \geqq \frac{1}{2}\right\} \subset \left\{a_i \geqq \frac{1}{2}\right\} = \left\{a_i > \frac{1}{2}\right\} \cup \left\{a_i = \frac{1}{2}\right\} \subset A_i \cup N_{ij},$$

where $P_m(N_{ij}) = 0$. Put $C_j = D_j \cap A_i$ $(j = 1, \ldots, n)$ and

$$\mathbb{C} = \{D_1, \ldots, D_n\} \cup \{N_{ij}\}_{i,j} \cup \{K\}.$$

Then $\mathbf{A} \leqq \mathbb{C}$. Since $\{c_j > \frac{1}{2}\} \subset D_j \cap A_i \subset \{c_j \geqq \frac{1}{2}\}$, hence $c_j \sim C_j$ and therefore $\mathcal{A} \sim \mathbb{C}$.

3.8. PROPOSITION. *Let $(\mathcal{C}_n)_{n=1}^{\infty}$ be a sequence of fuzzy partitions such that $\sigma(\bigcup_{n=1}^{\infty} \mathcal{C}_n) = M$. Then for every fuzzy partition $\mathcal{A}$*

$$\lim_{n \to \infty} H_m(\mathcal{A}|\mathcal{C}_n) = 0.$$

Proof. Put $\mathbb{C}_n = \{A; \exists a \in \mathcal{C}_n : a \sim A\}$, $\mathbf{S} = \sigma(\bigcup_{n=1}^{\infty} \mathbb{C}_n)$. Of course, $\mathbf{S} \subset K(M)$.

Let us denote $N = \{a; \exists A \in \mathbf{S} : a \sim A\}$. Since $a \in \mathcal{C}_n$ implies that there exists $A \in \mathbb{C}_n$ such that $a \sim A$, we have $\mathcal{C}_n \subset N$. Moreover, N is a fuzzy σ-algebra, so that $\sigma(\bigcup_{n=1}^{\infty} \mathcal{C}_n) \subset N$, i.e. $M \subset N$. Therefore $N = M$. Let $\mathcal{A} = \{a_1, \ldots, a_k\}$ be any fuzzy partition. Since $\mathcal{A} \subset N$, there exist $B_i \in \mathbf{S}$, $a_i \sim B_i$, $i = 1, \ldots, k$. Put $A_1 = B_1$, $A_2 = B_2|B_1$, $A_3 = B_3|(B_1 \cup B_2)$ etc. Then $a_i \sim A_i$, $A_i \in \mathbf{S}$, $i = 1, \ldots, k$ and

$$P_m\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} P_m(A_i) = \sum_{i=1}^{k} m(a_i) = 1.$$

Hence $\mathcal{A} \sim \mathbf{A} = \{A_1, \ldots, A_k\} \subset \mathbf{S} = \sigma(\bigcup_{n=1}^{\infty} \mathbb{C}_n)$. Take into account the quadruple $(X, \mathbf{S}, P_m, T)$, where T is the identity mapping on X. Applying Lemma 16.46 of [9] for the dynamical system $(X, \mathbf{S}, P_m, T)$, we have $\lim_{n \to \infty} H(\mathbf{A}|\mathbb{C}_n) = 0$.

Since $H_m(\mathcal{A}|\mathcal{C}_n) = H(\mathbf{A}|\mathbb{C}_n)$, we obtain $\lim_{n \to \infty} H_m(\mathcal{A}|\mathcal{C}_n) = 0$.

3.9. THEOREM. *Let $\mathcal{C}$ be a generator of fuzzy dynamical system $(X, M, m, \mathcal{U})$, i.e. $\mathcal{C}$ be such a fuzzy partition that $\sigma(\bigcup_{n=1}^{\infty} \mathcal{C}_n) = M$, where $\mathcal{C}_n = \bigvee_{i=0}^{n} \mathcal{U}^i\mathcal{C}$; $n = 1, 2, \ldots$. Then $h_m(\mathcal{U}) = h_m(\mathcal{U}, \mathcal{C})$.*

Proof. Since $h_m(\mathcal{U})$ is the supremum, we have to prove that $h_m(\mathcal{U},\mathcal{A}) \leqq h_m(\mathcal{U},\mathcal{C})$ for every fuzzy partition $\mathcal{A}$. By the preceding Proposition $\lim_{n\to\infty} H_m(\mathcal{A}|\mathcal{C}_n) = 0$. From (2.6) it follows the inequality

$$h_m(\mathcal{U},\mathcal{A}) \leqq h_m(\mathcal{U},\mathcal{C}_n) + H_m(\mathcal{A}|\mathcal{C}_n).$$

Now, by (2.5) we obtain

$$h_m(\mathcal{U},\mathcal{C}_n) = h_m(\mathcal{U},\mathcal{C}), \qquad \text{for } n = 1,2,\ldots,$$

so that $h_m(\mathcal{U},\mathcal{A}) \leqq h_m(\mathcal{U},\mathcal{C}) + H_m(\mathcal{A}|\mathcal{C}_n)$ for $n = 1,2,\ldots$. This implies the inequality

$$h_m(\mathcal{U},\mathcal{A}) \leqq h_m(\mathcal{U},\mathcal{C}) + \lim_{n\to\infty} H_m(\mathcal{A}|\mathcal{C}_n) = h_m(\mathcal{U},\mathcal{C}),$$

which ends the proof.

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