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CLASSIFICATION OF FIVE-DIMENSIONAL LIE ALGEBRAS OF CLASS T_2

1. Introduction

The authors of the paper [6] have classified all the connected, complete and simply connected Riemannian manifolds (M, g) for $\dim M = 3, 4$, which admit a non-trivial homogeneous structure T of class T_2.

I shall generalize the results of the paper [6] to the five-dimensional case, i.e., for $\dim M = 5$. Because the solution to this problem is very large I present here separately the essential and independent part (F) of this solution, see the last formula (xvi).

I have come to the conclusion that it is necessary to summarize here the basic facts about the Riemannian manifolds (M, g) admitting a homogeneous structure T of class T_2 in the sense of F. Tricerri and L. Vanhecke and the classification method presented in the paper [6].

Ambrose and Singer [1] have proved a theorem that a connected, complete and simply connected Riemannian manifold (M, g) is homogeneous (i.e. it admits a transitive group G of isometries) if and only if there exists a tensor field T of type (1,2) such that

$$(\mathrm{AS})\quad \begin{cases} \text{(i) } g(T_XY, Z) + g(Y, T_XZ) = 0, \\ \text{(ii) } (\nabla_X R)_{YZ} = [T_X, R_{YZ}] - R_{T_XYZ} - R_{YT_XZ}, \\ \text{(iii) } (\nabla_X T)_Y = [T_X, T_Y] - T_{T_XY}, \quad \text{for all } X, Y, Z \in \mathfrak{X}(M). \end{cases}$$

Here ∇ and R denote the Levi-Civita connection and the Riemannian tensor field, respectively. A tensor field T satisfying the conditions (AS) on M is called a *homogeneous structure* on (M, g).

In [10] F. Tricerri and L. Vanhecke studied the decomposition of the space of all the (algebraic) tensors T satisfying the condition (AS) (i) in the *irreducible components* under the action of orthogonal group. In this way they found *three irreducible classes* of possible homogeneous structure denoted by T_1, T_2, T_3.

In ([10], pp17, 49, 56) the following theorems were proved:

Each connected, complete, simply connected Riemannian manifold (M,g) satisfying the conditions (AS) is reductive homogeneous Riemannian manifold of the form $(M,g) = G/H$ with respect to the decomposition $\mathfrak{g} = \mathfrak{m} \bigoplus \mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of G, where G is a group of isometries of (M,g), and $\mathrm{Ad}_G(H)\mathfrak{m} \subseteq \mathfrak{m}$. If the connected Riemannian manifold (M,g) admits a non-trivial homogeneous structure T of class T_1, then (M,g) is a space of constant curvature.

A connected, complete, simply connected Riemannian manifold (M,g) admits a homogeneous structure of class T_3 if and only if it is naturally reductive homogeneous Riemannian manifold.

The following definition is equivalent to that following ([10], p. 38):

DEFINITION. A homogeneous structure D on a Riemannian manifold (M,g) is said to be of class T_2 if the following two identities hold:

(iv) $$\underset{XYZ}{\mathfrak{S}}\, g(D_XY, Z) = 0, \quad (\mathfrak{S} \text{ is a cyclic sum}),$$

(v) $$\sum_{i=1}^{n} g(X, D_{e_i}e_i) = 0,$$

for any tangent vectors X, Y, Z and any orthonormal basis $\{e_1, \ldots e_n\}$ belonging to T_pM, $p \in M$.

In what follows, we shall describe our classification method ([10], pp. 4–6).

Let $\tilde{\nabla}$ be the canonical connection of a reductive homogeneous Riemannian manifold (M,g), $M = G/H$ and $\mathfrak{g} = \mathfrak{m} \bigoplus \mathfrak{h}$. Then the torsion tensor field $\tilde{T}$ and the curvature tensor field $\tilde{R}$ of $\tilde{\nabla}$ are parallel, and so is g, i.e., there are satisfied the relations:

(vi) $$\tilde{\nabla}\tilde{T} = \tilde{\nabla}\tilde{R} = \tilde{\nabla}g = 0.$$

We identify the subspace $\mathfrak{m} \hookrightarrow \mathfrak{g}$ with the tangent space T_0M at the origin $o \in M$ via the projection $\pi : G \longrightarrow G/H$.

From the parallelism of $\tilde{T}, \tilde{R}, g$ and from the supposition that the curvature tensor $\tilde{R}_{XY} : Z \longrightarrow \tilde{R}_{XY}Z$ acts as a derivation on the tensor algebra $\mathcal{T}(\mathfrak{m})$ of $\mathfrak{m} = T_0M$ we have the following algebraic conditions:

(vii) $$\tilde{R}_{XY} \cdot \tilde{R} = \tilde{R}_{XY} \cdot \tilde{T} = \tilde{R}_{XY} \cdot g = 0,$$

(viii) $$\tilde{R}_{XY} = -\tilde{R}_{YX}, \quad \tilde{T}_XY = -\tilde{T}_YX,$$

(ix) $$\underset{XYZ}{\mathfrak{S}}(\tilde{T}_{\tilde{T}_XY}Z - \tilde{R}_{XY}Z) = 0,$$

(x) $$\underset{XYZ}{\mathfrak{S}}\, \tilde{R}_{\tilde{T}_XYZ} = 0, \quad \text{for all } X, Y, Z \in \mathfrak{m}.$$

Now let $D = \nabla - \tilde{\nabla}$ be the difference tensor field (we know already that D is a homogeneous structure on (M, g)). Between D, g and $\tilde{T}$ there is satisfied the relation ([6], p. 4)

(xi) $2g(D_X Y, Z) = g(\tilde{T}_Y X, Z) + g(\tilde{T}_X Z, Y) + g(\tilde{T}_Y Z, X), X, Y, Z \in \mathfrak{X}(M)$.

From (xi) and (iv) - (v) it follows that the homogeneous structure D on (M, g) of class T_2 can be characterized by the conditions:

(xii) $$\underset{XYZ}{\mathfrak{S}}\, g(\tilde{T}_X Y, Z) = 0,$$

(xiii) $$\sum_{i=1}^{n} g(\tilde{T}_Z e_i, e_i) = 0,$$

for all $X, Y, Z \in \mathfrak{m}$ and any orthonormal basis $\{e_1, \ldots, e_n\} \subset \mathfrak{m}$. On the basis of the considerations represented by the formulas (vii) - (xiii) we get the following

PROPOSITION. *Let (M, g) be a connected, complete and simply connected Riemannian manifold admitting a non-trivial homogeneous structure D of the class T_2. Then (M, g) and D uniquely define a quadruplet $(V, g, \tilde{T}, \tilde{R})$, where V is a vector space of dimension n, g is a positive inner product and $\tilde{T} \neq 0, \tilde{R}$ are tensor fields of type* (1,2), (1,3), *respectively such that the conditions* (vii)–(xiii) *are satisfied.*

Conversely, let $(V, g, \tilde{T}, \tilde{R})$ be a quadruplet satisfying all the conditions mentioned in the thesis of the above Proposition. Then we use a construction of K. Nomizu ([10], p. 6) to attach to this quadruplet a Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Firstly, let $\mathfrak{h}$ be the subalgebra of the Lie algebra $\mathrm{End}(V)$ spanned by the "curvature operations" $\tilde{R}_{XY}$, $X, Y \in V$.

Secondly, we define $\mathfrak{g}$ as the direct sum of two vector spaces:

(xiv) $$\mathfrak{g} = V \bigoplus \mathfrak{h}.$$

Finally, we endowe the vector space $\mathfrak{g}$ with the following brackets:

(xv) $$\begin{cases} [X, Y] = (-\tilde{T}_X Y, -\tilde{R}_{XY}), & X, Y \in V, \\ [A, X] = A(X), & A \in \mathfrak{h},\ X \in V, \\ [A, B] = A \circ B - B \circ A, & A, B \in \mathfrak{h}. \end{cases}$$

Using the formulas (vii)–(xiii) one can easily check that the Jacobi identity holds. Thus, $\mathfrak{g}$ is a Lie algebra.

Now, let G be the simply connected Lie group whose Lie algebra is $\mathfrak{g}$, and let H be the connected Lie subgroup of G corresponding to $\mathfrak{h}$. If $H \subset G$ is closed, then we can define the homogeneous space $M = G/H$. Then the vector space V can be identified with the tangent space $T_0 M$ at the origin

$o \in M$ via the projection $\pi : G \longrightarrow G/H$. G/H is reductive with respect to the decomposition $\mathfrak{g} = V \bigoplus \mathfrak{h}$, because $[\mathfrak{h}, V] \subset V$ and H is connected.

From the conditions (vi) and (xv) it follows immediately that the canonical connection $\widetilde{\nabla}$ of G/H has the curvature tensor and the torsion tensor at the origin $o \in M = G/H$ equal to the prescribed $\widetilde{R}$ and $\widetilde{T}$, respectively. The inner product g on V yields on G/H a G-invariant Riemannian metrics, and thus G/H is a homogeneous Riemannian manifold.

The conditions (xii)–(xiii) mean that the homogeneous structure $D = \nabla - \widetilde{\nabla}$ is of class T_2. Thus, we have proved the inverse Proposition.

Because we shall work in the sequel with the canonical connection $\widetilde{\nabla}$ of a homogeneous space G/H and not with the Levi-Civita connection ∇, we shall always write simply T, R instead of $\widetilde{T}, \widetilde{R}$ in the quadruplet (V, g, T, R).

Now, as in the paper ([6], Propositions 1.3, 3.1) we consider a quadruplet (V, g, T, R), $\dim V = 5$.

Let us denote by $\mathfrak{k} = \{A \in \mathrm{End}(V) : A \cdot g = A \cdot T = 0\}$. Here, all $A \in \mathfrak{k}$ acting as a derivation on the tensor algebra $\mathcal{T}(V)$ are skew-symmetric, and all the curvature transformations R_{XY}, $X, Y \in V$, must belong to $\mathfrak{k}$.

We must consider the following possibilities:

(xvi)
- (A) V is $\mathfrak{k}$-irreducible.
- (B) There is a 4-dimensional $\mathfrak{k}$-irreducible subspace of V.
- (C) There is a 3-dimensional $\mathfrak{k}$-irreducible subspace of V.
- (D) There are two 2-dimensional $\mathfrak{k}$-irreducible subspaces $\mathrm{Span}(e_1, e_2)$, $\mathrm{Span}(e_3, e_4) \subset V$ and $\mathfrak{k}$ acts trivially on $\mathrm{Span}(e_5) \subset V$.
- (E) There is one 2-dimensional $\mathfrak{k}$-irreducible subspace $\mathrm{Span}(e_1, e_2) \subset V$, and $\mathfrak{k}$ acts trivially on $\mathrm{Span}(e_3, e_4, e_5) \subset V$.
- (F) $\mathfrak{k} = \{0\}$, and hence $R = 0$.

If the subspace $W \subset V$, $\dim W = 2, 3, 4$, is $\mathfrak{k}$-irreducible, then its orthogonal complement $W^{\perp}$ with respect to the scalar product g is still $\mathfrak{k}$-invariant. In what follows we shall consider in this paper the cases (F) and (C). It is necessary to underline that to the case (F) there are also reduced some special subcases of (C) and (D) (see the subcase (C_2) in [6], pp. 14, 17).

In the second section the author will carry out a classification of five-dimensional Lie algebras $\mathfrak{g} = (V, T)$ of class T_2, finding five classes of the Lie algebras (V, T) containing the real parameters. This is a generalization of

Proposition 3.6, [6] for five-dimensional case. In the third section the author shall determine the Lie algebras of derivations of the five-dimensional Lie algebras (V,T) found above. This is a generalization of the algebraic"part" of Proposition 3.7, [6] on the five-dimensional Lie algebras (V,T). In the fourth section we shall present an application of the results obtained in the previous section.

2. Classification of five-dimensional Lie-algebras of class T_2

We consider a triplet (V,T,g) satisfying the following five conditions:

(1) (V,T) is a five-dimensional Lie algebra with the Lie multiplication defined by the structure tensor $T(x,y)$ fulfilling the Jacobi identity,

(2) $[x,y]=T(x,y)$, for $x,y\in V$,

(3) g is an inner positive defined product on V,

(4) $\underset{XYZ}{\mathfrak{S}}\, g([x,y],z)=0$, for $x,y,z\in V$, ($\mathfrak{S}$ is a cyclic sum),

(5) $\sum_{i=1}^{5} g([z,e_i]\,,\,e_i)=0$, for $z\in V$,

for any orthonormal basis $\{e_1,e_2,e_3,e_4,e_5\}$ of V.

In the sequel we shall denote the Lie algebra (V,T) shortly by $\mathfrak{g}$.

The conditions (4) and (5) have exactly the same meaning as the conditions (6) and (7), respectively, in the paper [6], hence the Lie algebra $\mathfrak{g}$ is unimodular ([8], p. 318).

The considerations from ([6], p.18) can be generalized and thus we have the following

LEMMA 1. *If a finite dimensional Lie algebra $\mathfrak{g}$ has provided with a positive defined inner product g, and if there are satisfied the conditions (1)-(5), then the following relations are true:*

(6)
- *(a) for any ideal $\mathfrak{h}$ of $\mathfrak{g}$ the orthogonal complement $\mathfrak{h}^{\perp}$ is a subalgebra of $\mathfrak{g}$;*
- *(b) if $z\in\mathfrak{h}^{\perp}$, then ad_z is symmetric on $\mathfrak{h}$;*
- *(c) for the ideal $\mathfrak{h}\subset\mathfrak{g}$ the conditions (4)–(5) are satisfied, where the condition (4) is satisfied for every subalgebra $\mathfrak{h}\subset\mathfrak{g}$;*
- *(d) $\mathfrak{g}=\mathfrak{h}\bigoplus\mathfrak{h}^{\perp}$, (a direct sum of vector subspaces).*

Basing on the formulas (1)–(5) and using the notations:

$$[e_i,e_j]=T_{e_i}e_j=\sum_{k=1}^{5} t_{ij}{}^{k}\,e_k,\qquad i,j=1,\ldots,5, \tag{7}$$

after detailed calculations we obtain the following forms of the Lie brackets of sought Lie algebra $\mathfrak{g}$ which are characteristic for every orthonormal basis $\{e_1, \dots, e_5\}$ on $\mathfrak{g}$:

$$
(8)\quad \begin{cases}
[e_1,e_2] = t_{12}^1 e_1 + t_{12}^2 e_2 + t_{12}^3 e_3 + t_{12}^4 e_4 + t_{12}^5 e_5, \\
[e_1,e_3] = t_{13}^1 e_1 + t_{13}^2 e_2 + t_{13}^3 e_3 + t_{13}^4 e_4 + t_{13}^5 e_5, \\
[e_1,e_4] = t_{14}^1 e_1 + t_{14}^2 e_2 + t_{14}^3 e_3 + t_{14}^4 e_4 + t_{14}^5 e_5, \\
[e_1,e_5] = t_{15}^1 e_1 + t_{15}^2 e_2 + t_{15}^3 e_3 + t_{15}^4 e_4 + (-t_{12}^2 - t_{13}^3 - t_{14}^4) e_5, \\
[e_2,e_3] = (t_{13}^2 - t_{12}^3) e_1 + t_{23}^2 e_2 + t_{23}^3 e_3 + t_{23}^4 e_4 + t_{23}^5 e_5, \\
[e_2,e_4] = (t_{14}^2 - t_{12}^4) e_1 + t_{24}^2 e_2 + t_{24}^3 e_3 + t_{24}^4 e_4 + t_{24}^5 e_5, \\
[e_2,e_5] = (t_{15}^2 - t_{12}^5) e_1 + t_{25}^2 e_2 + t_{25}^3 e_3 + t_{25}^4 e_4 + (t_{12}^1 - t_{23}^3 - t_{24}^4) e_5, \\
[e_3,e_4] = (t_{14}^3 - t_{13}^4) e_1 + (t_{24}^3 - t_{23}^4) e_2 + t_{34}^3 e_3 + t_{34}^4 e_4 + t_{34}^5 e_5, \\
[e_3,e_5] = (t_{15}^3 - t_{13}^5) e_1 + (t_{25}^3 - t_{23}^5) e_2 + t_{35}^3 e_3 + t_{35}^4 e_4 + \\
\qquad\qquad (t_{13}^1 + t_{23}^2 - t_{34}^4) e_5, \\
[e_4,e_5] = (t_{15}^4 - t_{14}^5) e_1 + (t_{25}^4 - t_{24}^5) e_2 + (t_{35}^4 - t_{34}^5) e_3 + \\
\qquad\qquad (-t_{15}^1 - t_{25}^2 - t_{35}^3) e_4 + (t_{14}^1 + t_{24}^2 + t_{34}^3) e_5.
\end{cases}
$$

In the formulas (8) there are 35 essential parameters t_{ij}^k which must satisfy the 10 Jacobi identities. Hence, a further investigation of the Lie algebra $\mathfrak{g}$ immediately by the aid of the structure constants t_{ij}^k is very complicated. Also, we shall determine the corresponding Lie group G.

From (8) it follows immediately that $\operatorname{tr}(ad_{e_i}) = 0$, $i = 1, \dots, 5$, hence the Lie algebra $\mathfrak{g}$ is unimodular, as expected.

If all $t_{ij}^k = 0$, then $\mathfrak{g} = R^5$. Thus in the sequel we shall suppose that not all t_{ij}^k are equal to zero.

In order to simplify the further detailed calculations we shall describe some facts of the algebraic nature, [7].

It is well known that changing the basis of the Lie algebra $\mathfrak{g}$ in the following way (we adopt the Einstein's summation convention):

$$(9)\quad e_{i'} = A_{i'}^i e_i, \quad A = \left[A_{i'}^i\right], \quad \det(A) \neq 0, \quad i = 1, \dots, n, \; i' = 1', \dots, n',$$

then the structure constants t_{ij}^k defined by the change of the basis are transformed according to the formula

$$(10)\qquad t_{i'j'}^{k'} = A_{i'}^i A_{j'}^j A_k^{k'} t_{ij}^k, \qquad i,j,k = 1, \dots, n, \; i',j',k' = 1', \dots, n',$$

where $\left[A_k^{k'}\right] = A^{-1}$.

The structure constants are skew-symmetric: $t_{ij}^k = -t_{ji}^k$, and must satisfy the conditions (4)–(5), see for $n = 5$ the right hand side of the formula (8).

By the aid of the formula (7) an endomorphism $\mathfrak{g} \longrightarrow g'$ is defined (also for an arbitrary n), where $\mathfrak{g}' = [\mathfrak{g}, g]$ is a derived Lie algebra.

From (9) we receive easily an endomorphism $\mathfrak{g} \longrightarrow g'$ of the form

$$[e_{i'}, e_{j'}] = 2\, A^{[i}_{i'} A^{j]}_{j'}\, [e_i, e_j]\,, \tag{11}$$

where $i < j$, $i' < j'$, $i, j = 1, \ldots, n$, $i', j' = 1', \ldots, n'$, and $P = (i, j)$, $P' = (i', j')$ are ordered in the lexicographic order of the magnitude of pairs (i, j), (i', j'), respectively.

The sets $\{[e_i, e_j]\,,\ i < j\}$, and $\{[e_{i'}, e_{j'}]\,, i' < j'\}$ of the fundamental Lie brackets of the Lie algebra $\mathfrak{g}$ constitute separately the sets of generators of the derived Lie algebra $\mathfrak{g}'$.

Thus is true the following lemma

LEMMA 2. *From* (7)–(11), *and* ([7], pp. 7, 11) *we have the following relations*

$$(12)\quad \begin{cases} (a) & \text{for } D = \left[D^P_{P'}(A)\right] = \left[2A^{[i}_{i'} A^{j]}_{j'}\right], \quad P = 1, \ldots, \binom{n}{2}, \\ & \det(D) = [\det(A)]^{n-1} \neq 0, \qquad P' = 1', \ldots, \binom{n}{2}'; \\ (b) & \dim \mathfrak{g}' \text{ is invariant under the change of the basis on } \mathfrak{g}; \\ (c) & \dim \mathfrak{g}' = \operatorname{rank}(C), \text{ where } C \text{ is in general a matrix of the} \\ & \text{coefficients of the decomposition of the } [e_i, e_j] \text{ for an} \\ & \text{arbitrary } n, \text{ analogously as for } n = 5 \text{ in } (8). \end{cases}$$

In the sequel we shall apply Lemma 2 for $n = 5$ to the considerations and calculations.

Because $\dim \mathfrak{g} = 5$, then it follows from ([2], §§ 4, 5) that the sought Lie algebra $\mathfrak{g}$ is not simple.. Thus, we must consider the following cases:

$$(13)\quad \begin{cases} \text{(I)} & \text{There is a 4-dimensional ideal } \mathfrak{h} \subset \mathfrak{g}; \\ \text{(II)} & \text{There is a 3-dimensional ideal } \mathfrak{h} \subset \mathfrak{g}; \\ \text{(III)} & \text{There is a 2-dimensional ideal } \mathfrak{h} \subset \mathfrak{g}; \\ \text{(IV)} & \text{There is a 1-dimensional ideal } \mathfrak{h} \subset \mathfrak{g}. \end{cases}$$

In what follows we shall consider the above cases (i)–(iv) step by step. We shall consider only the orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\} \subset \mathfrak{g}$.

We shall prove the following

THEOREM 1. *If* $\mathfrak{g}$ *is a five-dimensional non-abelian Lie algebra provided with an inner scalar product such that there are fulfilled the conditions* (1)–(5), *then the Lie algebra* $\mathfrak{g}$, *and its corresponding Lie group* G *have one of the following forms*:

a) $\mathfrak{g} = \mathfrak{sl}(2, R) \bigoplus R^2$, $\quad G = SL(2, R) \times R^2$;

b) $\mathfrak{g} = e(1, 1) \bigoplus R^2$, $\quad G = E(1, 1) \times R^2$;

c) $\mathfrak{g} = R^3 \oplus R^2$ *with the multiplication table:*

$$(14)\quad \begin{cases} [e_i, e_4] = \alpha_i e_i, i = 1,2,3, \alpha_i \neq 0, \alpha_1 + \alpha_2 + \alpha_3 = 0, \\ [e_j, e_k] = 0 \text{ otherwise,} \\ G = G_0 \times R, \text{ where} \\ G = \left\{ \begin{bmatrix} e^{\alpha_1 t} & 0 & 0 & x \\ 0 & e^{\alpha_2 t} & 0 & y \\ 0 & 0 & e^{\alpha_3 t} & z \\ 0 & 0 & 0 & 1 \end{bmatrix}, (x,y,z,t) \in R^4(x,y,z,t) \right\}; \\ \text{see ([5], p. 10);} \end{cases}$$

d) $\mathfrak{g} = R^3 \oplus R^2$ *with the multiplication table:*

$$\begin{cases} [e_1, e_4] = \alpha e_1, \ [e_2, e_4] = \beta e_2, \ [e_3, e_4] = -(\alpha + \beta) e_3, \\ [e_2, e_5] = \gamma e_2, \ [e_3, e_5] = -\gamma e_3, \ \alpha, \gamma \neq 0, \ \beta\text{-arbitrary,} \\ [e_i, e_j] = 0 \text{ otherwise;} \\ G = \left\{ \begin{bmatrix} e^{\alpha u} & 0 & 0 & x \\ 0 & e^{\beta u + \gamma v} & 0 & y \\ 0 & 0 & e^{-(\alpha+\beta)u - \gamma v} & z \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{matrix} (x,y,z,u,v) \in \\ R^5(x,y,z,u,v) \end{matrix} \right\}; \\ \text{see ([5], p. 10, type 9));} \end{cases}$$

e) $\mathfrak{g} = R^4 \oplus R$ *with the multiplication table:*
$[e_i, e_5] = \alpha_i e_i, \ i = 1,2,3,4 \ \alpha_i \neq 0, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0,$
$[e_j, e_k] = 0$ *otherwise,*

$$G = \left\{ \begin{bmatrix} e^{\alpha_1 t} & 0 & 0 & 0 & x \\ 0 & e^{\alpha_2 t} & 0 & 0 & y \\ 0 & 0 & e^{\alpha_3 t} & 0 & z \\ 0 & 0 & 0 & e^{\alpha_4 t} & u \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{matrix} (x,y,z,u,t) \in \\ R^5(x,y,z,u,t) \end{matrix} \right\};$$

see ([5], p. 6, type 2)).

In the subcases $a), b)$ we have to deal with the orthogonal direct sum of two ideals, and in the subcase c), d), e) — with the orthogonal semidirect sum of two abelian ideals of the Lie algebra.

The Lie groups G_0, and G are isomorphic to the Lie groups of inner automotorphisms corresponding to the suitable Lie algebras whose adjoint representations are faithful.

P r o o f in the case (I). We shall receive all the results $(14)a$ - $(14)e$. Let $\{e_1, e_2, e_3, e_4, e_5\}$ be an orthonormal basis of $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$, (a direct sum of two vector subspaces) such that $\mathfrak{h} = \mathrm{Span}(e_1, e_2, e_3, e_4)$, and $\mathfrak{h}^\perp = \mathrm{Span}(e_5)$. Because the ideal $\mathfrak{h}$ fulfills the condition $(6)c$, hence from ([6], Proposition

3.6), and (I) in (13) we get the following four subcases for $\mathfrak{h}$:

$$(15)\quad \begin{cases} Ia) & \mathfrak{h} = \mathfrak{sl}(2,R) \oplus R, \text{ (an orthogonal direct sum of two ideals),} \\ & \text{where for } \mathfrak{sl}(2,R): [e_1,e_2] = \lambda_3 e_3,\ [e_2,e_3] = \lambda_1 e_1, \\ & \qquad [e_3,e_1] = \lambda_2 e_2,\ \lambda_1,\lambda_2 > 0, \\ & \qquad \lambda_3 = -\lambda_1 - \lambda_2; \\ Ib) & \mathfrak{h} = \mathfrak{e}(1,1) \oplus R, \text{ (an orthogonal direct sum of two ideals),} \\ & \text{where for } \mathfrak{e}(1,1): [e_1,e_2] = 0,\ [e_2,e_3] = \lambda e_1, \\ & \qquad [e_3,e_1] = -\lambda e_2,\ \lambda \neq 0; \\ Ic) & \mathfrak{h} = R^3 \oplus R, \text{ (an orhogonal semidirect sum of two} \\ & \qquad \text{abelian ideals);} \\ Id) & \mathfrak{h} = R^4. \end{cases}$$

In the subcase $Ic)$ one can choose an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{h}$ such that $R^3 = \mathrm{Span}(e_1, e_2, e_3)$, $R = \mathrm{Span}(e_4)$, and $[e_i, e_4] = \alpha_i e_i$, $i = 1,2,3$, $\alpha_1 + \alpha_2 + \alpha_3 = 0$, where not all α_i are equal to zero. If one of the coefficients α_i is equal to zero, then the subcase $Ic)$ is reduced to the subcase $Ib)$. Hence, we suppose for the future in the subcase $Ic)$ that all $\alpha_i \neq 0$, $i = 1,2,3$.

In the first subcase $(15)Ia$ we receive the result $(14)a$.

From (8) it follows that the sought Lie algebra $\mathfrak{g}$ has the following Lie brackets:

$$(16)\quad \begin{cases} [e_1,e_2] = \lambda_3 e_3, \quad [e_2,e_3] = \lambda_1 e_1, [e_3,e_1] = \lambda_2 e_2, \\ [e_1,e_4] = [e_2,e_4] = [e_3,e_4] = 0, \\ [e_1,e_5] = t^1_{15} e_1 + t^2_{15} e_2 + t^3_{15} e_3 + t^4_{15} e_4, \\ [e_2,e_5] = t^2_{15} e_1 + t^2_{25} e_2 + t^3_{25} e_3 + t^4_{25} e_4, \\ [e_3,e_5] = t^3_{15} e_1 + t^3_{25} e_2 + t^3_{35} e_3 + t^4_{35} e_4, \\ [e_4,e_5] = t^4_{15} e_1 + t^4_{25} e_2 + t^4_{35} e_3 + t^4_{15} e_4 + (-t^1_{15} - t^2_{25} - t^3_{35}) e_4. \end{cases}$$

Now, substituting the Lie brackets (16) into the Jacobi identity we get that all $t^k_{ij} = 0$, hence we have received the result $(14)a$.

In the second subcase $(15)Ib$ we shall obtain the results $(14)b, (14)d$.

In the subcase Ib we receive the Lie brackets of the form (16) with $\lambda_3 = 0, \lambda_2 = -\lambda_1 = -\lambda \neq 0$. Using first the Jacobi identity and next introducing a new orthonormal basis of the form:

$$(17)\quad \begin{cases} e'_1 = \frac{1}{2}(e_1 + e_2), \quad e'_2 = \frac{1}{2}(e_1 - e_2), \quad e'_4 = e_4, \\ e'_3 = \frac{1}{\alpha}(\lambda e_3 + \mu e_5), \quad e'_5 = \frac{1}{\alpha}(-\mu e_3 + \lambda e_5), \\ \text{where} \quad \alpha = \sqrt{\lambda^2 + \mu^2} \neq 0, \quad \mu = t^1_{15} + t^2_{15}, \end{cases}$$

then the brackets (16) transform into the following ones (we omit the prime $'$)

$$(18)\qquad \begin{cases} [e_1,e_3]=\alpha e_1, \quad [e_2,e_3]=\beta e_2, \quad [e_4,e_3]=-(\alpha+\beta)e_4, \\ [e_2,e_5]=\lambda e_2, \quad [e_4,e_5]=-\lambda e_4, \quad [e_i,e_j]=0 \text{ otherwise}, \\ \text{where} \quad \beta=\dfrac{-\lambda^2+\mu\nu}{\alpha}, \ \gamma=\dfrac{\lambda(\mu+\nu)}{\alpha}. \end{cases}$$

The sought Lie algebra $\mathfrak{g}$ has the form $\mathfrak{g}=\mathrm{Span}(e_1,e_2,e_4)\oplus\mathrm{Span}(e_3,e_5)=R^3\oplus R^2$.

From Lemma 2 and (18) we get the inequality

$$(19)\qquad 2\le \dim\mathfrak{g}'\le 3.$$

If $\dim\mathfrak{g}'=2$, then $\gamma=0$, and $\beta=-\alpha$. Hence, the Lie algebra $\mathfrak{g}$ represented by the Lie brackets (18) is of type (14)b.

If $\dim\mathfrak{g}'=3$, then $\gamma\neq 0$. Using the matrix representation of ad_{e_i}, $i=1,2,3,4,5$, and calculating them on the basis of (18), we see that the adjoint representation ad: $\mathfrak{g}\longrightarrow \mathrm{ad}\mathfrak{g}$ is faithful.

In connection with this we present here certain general information, see [4], Chap. II, §.5, p. 126, and next.

The Lie group of inner automorphisms $\mathrm{Int}(\mathfrak{g})$ is Lie group corresponding to the Lie algebra $\mathfrak{g}$, represented by formula (18). To every $ad_{e_k}, k=1,2,3,4,5$ presented in the matrix form there corresponds a vector field X_k as an infinitesimal linear transformation on $R^5(x^1,x^2,x^3,x^4,x^5)$:

$$(20)\qquad X_k=\sum_{i,j=1}^{5} a^i_j x^j\frac{\partial}{\partial x^i}, \quad \text{for } [a^i_j]=-ad_{e_k}, \quad k=1,\dots 5.$$

Thus, for the Lie algebra $\mathfrak{g}$ presented by (18) we receive the following vector fields respectively

$$(21)\qquad \begin{cases} X_1=-\alpha u\dfrac{\partial}{\partial x}, \qquad X_2=(-\alpha u-\gamma v)\dfrac{\partial}{\partial y}, \\ X_3=\alpha x\dfrac{\partial}{\partial x}+\beta y\dfrac{\partial}{\partial y}-(\alpha+\beta)z\dfrac{\partial}{\partial z}, \\ X_4=[(\alpha+\beta)u+\gamma v]\dfrac{\partial}{\partial z}, \qquad X_5=\gamma y\dfrac{\partial}{\partial y}-\gamma z\dfrac{\partial}{\partial z}. \end{cases}$$

The vector fields (21) are dependent, but linearly independent.

In this paper we shall use several times the method of finding the integral curve or orbit of a differentiable vector field X on the space $R^n(x^1,\dots,x^n)$.

Let be given a differentiable curve

$$(22)\qquad \begin{cases} \varphi : I \longrightarrow R^n(x^1,\dots,x^n), & I = (\alpha,\beta), \\ \varphi(t) = (x^1(t),\dots,x^n(t)), & \varphi(0) = x_0 = (x_o^1,\dots,x_0^1),\ t \in I, \end{cases}$$

and a differentiable vector field X on $R^n(x^1,\dots,x^n)$ defined on a neighbourhood $U \ni x_0$ such that

$$(23)\qquad X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}, \quad X^i(x^1,\dots,x^n).$$

The differentiable curve (22) is called an integral curve (or an orbit) of X if it is a solution of the following system of differential equations:

$$(24)\qquad \begin{cases} \dfrac{dx^i(t)}{dt} = X^i(x^1(t),\dots,x^n(t)), & i = 1,\dots,n, \\ \varphi(0) = x_0. \end{cases}$$

Therefore if $X_{x_0} \neq 0$, there exists an integral curve of X through x_0, ([4], p.41).

Calculating the one-parametr groups of the inner automorphisms of the Lie algebra $\mathfrak{g}$ by the method of orbits (24) successively for the vector fields (21), and presenting them in the matrix form as elements of the affine group of transformations, we shall receive by the multiplication the following form of the sought Lie group of the automorphisms of $\mathfrak{g}$:

$$(25)\quad G_1 = \left\{ \begin{bmatrix} e^{\alpha u} & 0 & 0 & \alpha x & 0 \\ 0 & e^{\beta u+\gamma v} & 0 & \beta y & \gamma y \\ 0 & 0 & e^{-(\alpha+\beta)u-\gamma v} & (\alpha+\beta)z & \gamma z \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{matrix} (x,y,z,u,v) \in \\ R^5(x,y,z,u,v) \end{matrix} \right\}.$$

We omit here the detailed calculations.

Thus, changing in the Lie algebra $\mathfrak{g} = \mathrm{Span}(e_1,e_2,e_4) \ominus \mathrm{Span}(e_3,e_5) = R^3 \ominus R^2$ with the Lie brackets of the form (18) only the numeration of two vectors: $e_3 \longrightarrow e_4$, and $e_4 \longrightarrow e_3$, we see that we have received for $\mathfrak{g}$ the result (14)d. In what follows it suffice to prove that the Lie group G_1 of the form (25) is isomorphic to the Lie group G of the form (14)d. For this purpose we introduce the following group mapping $\varphi : G \longrightarrow G_1$:

(26)

$$\varphi\left(\begin{bmatrix} e^{\alpha u} & 0 & 0 & x \\ 0 & e^{\beta u+\gamma v} & 0 & y \\ 0 & 0 & e^{-(\alpha+\beta)u-\gamma v} & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} e^{\alpha u} & 0 & 0 & \alpha x & 0 \\ 0 & e^{\beta u+\gamma v} & 0 & \beta y & \gamma y \\ 0 & 0 & e^{-(\alpha+\beta)u-\gamma v} & (\alpha+\beta)z & \gamma z \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We shall prove the following

LEMMA 3. *The mapping $\varphi : G \longrightarrow G_1$ defined by the formula (26) is a Lie group isomorphism, i.e., there are satisfied the following two conditions*

a) $\varphi : G \longrightarrow G_1$ *is an abstract group isomorphism,*
b) $\varphi : G \longrightarrow G_1$ *is a diffeomorphism of two differentiable manifolds.*

Proof. The verity of the condition a) follows immediately from the elementary group calculations. To prove the condition b) we shall use the Proposition 5.4.4 from ([2]. p.77). In order to obtain this we introduce two auxiliary group mappings of the additive group $R^5(x, y, z, u, v)$ into G, G_1, respectively:

$$(27) \qquad \begin{cases} \varphi_1 : R^5(x, y, z, u, v) \to G, & G \in GL(4, R), \\ \varphi_2 : R^5(x, y, z, u, v) \to G_1, & G_1 \in GL(5, R), \end{cases}$$

where the submanifolds (and subgroups) G, G_1 are determined by the formulas (14)d, and (25).

Analysing exactly the above mentioned Proposition 5.4.4 we see that there are satisfied all its assumptios by φ_1, φ_2. Thus, our mappings φ_1, φ_2 are diffeomorphisms of manifolds. From (26), and (27) we receive easily that $\varphi = \varphi_2 \circ \varphi_1^{-1}$, hence φ is a diffeomorphism, and Lemma 3 is completely proved. Thus, we have received the result (14)d, completely.

In the subcase (15)Ic we obtain the result (14)d.

In this subcase we receive from (6)b, and (8) the following Lie brackets:

$$(28) \qquad \begin{cases} [e_1, e_2] = 0, [e_1, e_3] = 0, [e_2, e_3] = 0, \\ [e_i, e_4] = \alpha_i e_i, \ i = 1, 2, 3, \ \alpha_i \neq 0, \ \alpha_1 + \alpha_2 + \alpha_3 = 0, \\ [e_j, e_5] = \ldots, \ j = 1, 2, 3, 4, \quad \text{have the same forms as in (16).} \end{cases}$$

Now, using the Jacobi identity we obtain for $[e_j, e_5]$, $j = 1, \ldots, 4$ the relations

$$(29) \qquad \begin{cases} [e_1, e_5] = t_{15}^1 e_1 + t_{15}^2 e_2 + t_{15}^3 e_3, \\ [e_2, e_5] = t_{15}^2 e_1 + t_{25}^2 e_2 + t_{25}^3 e_3, \\ [e_3, e_5] = t_{15}^3 e_1 + t_{25}^3 e_2 + t_{35}^3 e_3, \\ [e_4, e_5] = 0, \quad t_{15}^1 + t_{25}^2 + t_{35}^3 = 0, \end{cases}$$

and additionaly are fulfilled the following three conditions

$$(30) \qquad (\alpha_1 - \alpha_2) t_{15}^2 = 0, (\alpha_1 - \alpha_3) t_{15}^3 = 0, (\alpha_2 - \alpha_3) t_{25}^3 = 0.$$

Now, we introduce the following two endomorphisms

$$F(e_4), G(e_5) : \operatorname{Span}(e_1, e_2, e_3) \longrightarrow \operatorname{Span}(e_1, e_2, e_3)$$

defined on the basis $\{e_1, e_2, e_3\} \subset \mathfrak{h}$ in the following way

$$(31) \qquad F(e_4)(e_i) = [e_i, e_4], \quad G(e_5)(e_i) = [e_i, e_5], \quad i = 1, 2, 3.$$

From (27)–(30) it follows that the endomorphisms $F(e_4)$, and $G(e_5)$ are symmetric and commutative, hence on the basis of a suitable theorem from Linear Algebra there exist the orthonormal common eigen vectors $\{e_i'\} \subset \mathrm{Span}(e_1, e_2, e_3)$ of $F(e_4)$, and $G(e_5)$ such that there are fulfilled the following relations (we omit the prime ')

$$(32)\qquad \begin{cases} [e_i, e_4] = \alpha_i e_i, \ i = 1,2,3, \alpha_i \neq 0, \alpha_1 + \alpha_2 + \alpha_3 = 0, \\ [e_1, e_5] = \beta_i e_i, \ i = 1,2,3, \ \beta_1 + \beta_2 + \beta_3 = 0, \\ [e_j, e_k] = 0 \text{ otherwise.} \end{cases}$$

Taking into account the symmetry condition of the formulas (32) we shall change the basis of $\mathfrak{g}$ in the following way

$$(33)\qquad \begin{cases} e_i' = e_i, \quad i = 1,2,3, \quad e_4' = \frac{1}{\alpha}(\alpha_1 e_4 + \beta_1 e_5), \\ e_5' = \frac{1}{\alpha}(-\beta_1 e_4 + \alpha_1 e_5), \quad \alpha = \sqrt{\alpha_1^2 + \beta_1^2} \neq 0. \end{cases}$$

Substituting (33) into (32) we obtain after routine calculations the following forms of the Lie brackets (we omit the prime '):

$$(34)\qquad \begin{cases} [e_1, e_4] = \alpha e_1, \ [e_2, e_4] = \beta e_2, \ [e_3, e_4] = -(\alpha + \beta) e_3, \\ [e_1, e_5] = 0, \ [e_2, e_5] = \gamma e_2, \ [e_3, e_5] = -\gamma e_3, \\ [e_i, e_j] = 0 \text{ otherwise,} \end{cases}$$

where $\beta = \frac{1}{\alpha}(\alpha_1\alpha_2 + \beta_1\beta_2)$, $\gamma = \frac{1}{\alpha}(\alpha_1\beta_2 - \alpha_2\beta_1)$.

From (32)–(33) it follows that $\dim \mathfrak{g}' = 3$, hence the Lie brackets (34) represent the Lie algebra $\mathfrak{g}$ of type (14)d, see (18)–(25). We omit here the detailed calculations.

In the subcase (15)Id we receive the results (14)b, c, and (14)e.

Thus, from (8), and Lemma 1, (6)b, c it follows that the Lie algebra $\mathfrak{g}$ has the following Lie brackets:

$$(35)\qquad \begin{cases} [e_1, e_5] = a_1 e_1 + b_1 e_2 + c_1 e_3 + d_1 e_4, \\ [e_2, e_5] = b_1 e_1 + b_2 e_2 + c_2 e_3 + d_2 e_4, \\ [e_3, e_5] = c_1 e_1 + c_2 e_2 + c_3 e_3 + d_3 e_4, \\ [e_4, e_5] = d_1 e_1 + d_2 e_2 + d_4 e_3 + d_4 e_4, \\ [e_i, e_j] = 0 \text{ otherwise,} \quad a_1 + b_2 + c_3 + d_4 = 0. \end{cases}$$

For the symmetric endomorphism ad_{e_5} defined by formula (35) there exists a new orthonormal basis $\{e_1, e_2, e_3, e_4\} \subset \mathfrak{h}$ such that

$$(36)\qquad \begin{cases} [e_i, e_5] = \lambda_i e_i, \quad i = 1,2,3,4, \ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0, \\ [e_j, e_k] = 0 \text{ otherwise,} \ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 \neq 0. \end{cases}$$

On the page 407 we have supposed the noncommutativity of the sought Lie algebra $\mathfrak{g}$.

With respect to the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of ad_{e_5} we must distinquish three essential non-trivial possibilities.

For $\lambda_1, \lambda_2 \neq 0, \lambda_3, \lambda_4 = 0$ we receive the result (14)b. For $\lambda_1, \lambda_2, \lambda_3 \neq 0, \lambda_4 = 0$ we obtain the result (14)c. At last, for all $\lambda_i \neq 0, i = 1,2,3,4$ we get result (14)e. We omit all the detailed calculations.

The calculations and considerations presented above are characteristic for the next cases (II), (III), (IV).

Proof in the case (II). The sought Lie algebra $\mathfrak{g}$, and the corresponding Lie group can admit all the forms (14)a–(14)e.

Let $\{e_1, \ldots, e_5\}$ be an orthonormal basis of $\mathfrak{g}$ such that $\mathfrak{h} = \mathrm{Span}(e_1, e_2, e_3), \mathfrak{h}^\perp = \mathrm{Span}(e_4, e_5)$. Because on the basis of the formula (6)c of Lemma 1 the ideal $\mathfrak{h}$ is an unimodular Lie algebra, then from ([6], p. 18) it follows that

$$\mathfrak{h} = \mathfrak{sl}(2, R) \text{ or } \mathfrak{e}(1,1) \text{ or } R^3. \tag{37}$$

Supposing that the subalgebra $\mathfrak{h}^\perp$ is abelian or not , and repeating the considerations from ([6], pp. 19–20), and also using the previous method represented by the formulas (16)-(25), we get in the first subcase of (37) the result (14)a. In the second subcase of (37) we receive the results (14)b, and (14)d. In the third subcase of (37) for $\mathfrak{h}^\perp$ being abelian we receive the results: $\mathfrak{g} = R^5$, and (14)b, (14)c, (14)d. At last, supposing that $\mathfrak{h}^\perp$ is not abelian we receive the results:(14)b, (14)c, (14)e.

We omit here all the detailed calculations, because they are analogical as in the case (I).

Proof in the case (III). We get here all the results of type (14)a–(14)e as previously.

Let $\{e_1, \ldots, e_5\}$ be an orthonormal basis fo $\mathfrak{g}$ such that $\mathfrak{h} = \mathrm{Span}(e_1, e_2)$, $\mathfrak{h}^\perp = \mathrm{Span}(e_3, e_4, e_5)$. Supposing first that the Lie subalgebra $\mathfrak{h}$ is not unimodular ([8], p.309), i.e., for a suitable orthonormal basis $\{e_3, e_4, e_5\}$ of $\mathfrak{h}^\perp$ there are fulfilled the relations

$$\begin{cases} [e_3, e_4] = \alpha e_4 + \beta e_5, \ [e_3, e_5] = \gamma e_4 + \delta e_5, \\ [e_4, e_5] = 0, \ \alpha + \delta = 2, \end{cases} \tag{38}$$

then on the basis of (6), and (8) we shall receive the special cases of (14)b, (14)c, (14)d, and (14)e.

In the second place, supposing that $\mathfrak{h}^\perp$ is unimodular, i.e., of the type (37), then we obtain the results (14)a, (14)b, and special subcase of (14)e. We omit here all the detailed calculations.

Proof in the case (IV). We get results of types (14)a – (14)e. Let $\{e_1, \ldots, e_5\}$ be an orthonormal basis of $\mathfrak{g}$ such that $\mathfrak{h} = \mathrm{Span}(e_1), \mathfrak{h}^\perp = \mathrm{Span}(e_2, e_3, e_4, e_5), \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$, (a direct sum of two vector subspaces). From

Lemma 1, and formula (8) it follows that the sought Lie algebra $\mathfrak{g}$ has the following Lie brackets

$$(39)\quad \begin{cases} [e_1, e_i] = t^1_{1i} e_1, \ i = 2,3,4,5, \\ [e_2, e_3] = t^2_{23} e_2 + t^3_{23} e_3 + t^4_{23} e_4 + t^5_{23} e_5, \\ [e_2, e_4] = t^2_{24} e_2 + t^3_{24} e_3 + t^4_{24} e_4 + t^5_{24} e_5, \\ [e_2, e_5] = t^2_{25} e_2 + t^3_{25} e_3 + t^4_{25} e_4 + (t^1_{12} - t^3_{23} - t^4_{24}) e_5, \\ [e_3, e_4] = (t^3_{24} - t^4_{23}) e_2 + t^4_{34} e_4 + t^5_{34} e_5, \\ [e_3, e_5] = (t^3_{25} - t^5_{23}) e_2 + t^3_{35} e_3 + t^4_{35} e_4 + (t^1_{13} - t^2_{23} - t^4_{34}) e_5, \\ [e_4, e_5] = (t^4_{25} - t^5_{24}) e_2 + (t^4_{35} - t^5_{34}) e_3 + (-t^1_{15} - t^2_{25} - t^3_{35}) e_4 + \\ \qquad\qquad + (t^1_{14} + t^2_{24} + t^3_{34}) e_5. \end{cases}$$

Thus, from *(39)* it follows that there is true the following equivalence:

(40) $\mathfrak{h}^\perp$ is unimodular if and only if $t^1_{12} = t^1_{13} = t^1_{14} = t^1_{15} = 0$.

Now, we shall distinguish two subcases.

A) $\mathfrak{h}^\perp$ is *unimodular.*

The unimodular Lie subalgebra $\mathfrak{h}^\perp$ is an ideal of $\mathfrak{g}$.

Hence, the sought Lie algebra $\mathfrak{g}$ is an orthogonal direct sum of two ideals: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$, where $\mathfrak{h} = R$, and $\mathfrak{h}^\perp$ has one of the forms $(15)Ia - Id$.

Applying to $\mathfrak{h}^\perp$ successively the formulas (15), and (68)–(69), after the elementary analysis we obtain finally the results $(14)a$–$(14)c$. Also, we get the trivial result $\mathfrak{g} = R^5$. We omit the detailed calculations.

B) $\mathfrak{h}^\perp$ is not *unimodular.*

In this subcase the Lie algebra $\mathfrak{g}$ receives one of the forms $(14)b$, $(14)c$, $(14)e$, and a particular form of $(14)d$.

On the basis of ([8], p.318) for a finite dimensional Lie algebra $\mathfrak{g}$ is true the following

LEMMA 4. *The linear mapping*

$$(41)\qquad \mathrm{tr} \circ \mathrm{ad} : \mathfrak{g} \longrightarrow R$$

is a homomophism of Lie algebras.

Its kernel as an ideal

$$(42)\qquad \mathfrak{u} = \{x \in \mathfrak{g} : \mathrm{tr}(\mathrm{ad}(x)) = 0)\}$$

of an unimodular Lie algebra of $\mathfrak{g}$. If $\dim \mathfrak{g} = n$, then for $\mathfrak{g}$ nonunimodular $\dim \mathfrak{u} = n - 1$.

In the subcase B) there is satisfied the reletion (we denote the suitable new orthonormal basis of $\mathfrak{g}$ once again by $\{e_1, e_2, e_3, e_4, e_5\}$):

$$\mathfrak{h}^\perp = \mathfrak{u} \oplus \mathfrak{u}^\perp, \quad \text{(a direct sum of vector subspaces)}, \tag{43}$$

where $\mathfrak{u} = \mathrm{Span}(e_2, e_3, e_4), \mathfrak{u}^\perp = \mathrm{Span}(e_5)$.

On the basis of Lemma 1,3, and formulas (42), we see that for the unimodular ideal $\mathfrak{u}$ are satisfied both the conditions (4),(5), hence $\mathfrak{u}$ can have one of the forms (37). Thus, the endomorphism $ad_{e_5}|\mathfrak{u}^\perp$ is symmetric on $\mathfrak{u}$, and has the orthonormal eigenvectors $\{e'_2, e'_3, e'_4\} \subset \mathfrak{u}$.

We shall distinguish three subcases.

First of all, we present some general considerations.

Taking into account the formulas (39) and (42)-(43), then the Lie algebra $\mathfrak{g}$ has the following Lie brackets:

$$\begin{cases} [e_1, e_i] = t^1_{1i} e_1, \quad i = 2,3,4,5, \\ [e_2, e_3] = t^2_{23} e_2 + t^3_{23} e_3 + t^4_{23} e_4, \\ [e_2, e_4] = t^2_{24} e_2 + t^3_{24} e_3 + t^4_{24} e_4, \\ [e_3, e_4] = (t^3_{24} - t^4_{23}) e_2 + t^3_{34} e_3 + t^4_{34} e_4, \\ [e_2, e_5] = t^2_{25} e_2 + t^3_{25} e_3 + t^4_{25} e_4, \\ [e_3, e_5] = t^3_{25} e_2 + t^3_{35} e_3 + t^4_{35} e_4, \\ [e_4, e_5] = t^4_{25} e_2 + t^4_{35} e_3 + (t^1_{15} - t^2_{25} - t^3_{35}) e_4. \end{cases} \tag{44}$$

Additionally there are satisfied the following conditions:

$$t^1_{12} - t^3_{23} - t^4_{24} = 0, \quad t^1_{13} + t^2_{23} - t^4_{34} = 0, \quad t^1_{14} + t^2_{24} + t^3_{34} = 0. \tag{45}$$

Because on the unimodular ideal $\mathfrak{u} = \mathrm{Span}(e_2, e_3, e_4) \subset \mathfrak{h}^\perp$ all the $\mathrm{tr}(\mathrm{ad}_{e_i}) = 0, i = 2,3,4$, hence from (44) it follows that

$$t^4_{24} + t^3_{23} = 0, \quad t^4_{34} - t^2_{23} = 0, \quad t^3_{34} + t^2_{24} = 0. \tag{46}$$

From (45) – (46) we receive that

$$t^1_{12} = t^1_{13} = t^1_{14} = 0. \tag{47}$$

In the considered subcase B) the subalgebra $\mathfrak{h}^\perp$ is not unimodular hence there must be satisfied the condition: $\mathrm{tr}(\mathrm{ad}_{e_5}) = t^1_{15} \neq 0$.

Finally, combining the formulas (44)-(47) we see that the sought Lie algebra $\mathfrak{g}$ has the following Lie brackets:

$$(48)\quad \begin{cases} [e_1, e_5] = \delta e_1,\ \delta = t_{15}^1 \neq 0, \\ [e_2, e_3] = t_{23}^2 e_2 + t_{23}^3 e_3 + t_{23}^4 e_4, \\ [e_2, e_4] = t_{24}^2 e_2 + t_{24}^3 e_3 - t_{23}^3 e^4, \\ [e_3, e_4] = (t_{24}^3 - t_{23}^4) e_2 - t_{23}^2 e_3 + t_{23}^2 e_4, \\ [e_2, e_5] = t_{25}^2 e_2 + t_{25}^3 e_3 + t_{25}^4 e_4, \\ [e_3, e_5] = t_{25}^3 e_2 + t_{35}^3 e_3 + t_{35}^4 e_4, \\ [e_4, e_5] = t_{25}^4 e_2 + t_{35}^4 e_3 + (-\delta - t_{25}^2 - t_{35}^3) e_4, \\ [e_1, e_j] = 0, \quad j = 2, 3, 4. \end{cases}$$

We must underline that the forms of the Lie brackets (48) in the considered subcase B are true in each orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\} \subset \mathfrak{g}$ fulfilling the formula (43).

B_1). For the subcase $\mathfrak{u} = \mathrm{Span}(e_2, e_3, e_4) = R^3$ we obtain finally that the sought Lie algebra $\mathfrak{g}$ is of type (14)b or (14)c or (14)e.

B_2). In this subcase $\mathfrak{u} = \mathfrak{e}(1,1) = \mathrm{Span}(e_2, e_3, e_4)$, and on the basis of (48) we obtain finally the following Lie brackets

$$(49)\quad \begin{cases} [e_1, e_5] = \delta e_1, \quad [e_2, e_5] = -\frac{\delta}{2} e_2, \quad [e_3, e_5] = -\frac{\delta}{2} e_3, \delta \neq 0, \\ [e_2, e_4] = \lambda e_2, \quad [e_3, e_4] = -\lambda e_3, \lambda \neq 0, \quad [e_i, e_j] = 0 \text{ otherwise.} \end{cases}$$

Thus we have $\mathfrak{g} = R^3 \ominus R^2 = \mathrm{Span}(e_1, e_2, e_3) \ominus \mathrm{Span}(e_5, e_4)$, and this is a particular case of the result (14)d.

B_3). In this subcase $\mathfrak{u} = \mathfrak{sl}(2, R) = \mathrm{Span}(e_2, e_3, e_4)$, and have $[e_2, e_3] = \lambda_3 e_4, [e_3, e_4] = \lambda_1 e_2, [e_4, e_2] = \lambda_2 e_3, \lambda_1, \lambda_2 > 0, \lambda_3 = -\lambda_1 - \lambda_2$.

From the formula (48) we get finally a contraduction that $\delta = 0$.

Thus, the subcase B_3) is not possible.

The proof of Theorem 1 comes to an end.

Analysing exactly the proof of Theorem 1 we get the conclusion that from the theoretical point of view we can restrict our considerations only to the case (I) in (13).

It is true the following

THEOREM 2. *In each five-dimensional Lie algebra $\mathfrak{g}$ satisfying the assumptions of Theorem 1 there exists a four-dimensional ideal $\mathfrak{h} \subset \mathfrak{g}$.*

The proof follows immediately from the formulas (37), (38), (39)–(40), (44).

3. Algebras of derivations of five-dimensional Lie algebras (V,T)

Now, in the end, we shall determine the Lie algebra of derivations for five types of Lie algebras (V,T) presented by Theorem 1, see the proofs of Proposition 2.3, p.7, and Proposition 3.7, p.21, [6].

Let us calculate the Lie algebra $\mathfrak{k} = \{A \in \mathrm{End}(V) : A \cdot g = A \cdot T = 0\}$ where the endomorphisms A of V acts as derivations on the tensor algebra, $\mathcal{T}(V)$. Let us put

$$Ae_i = \sum_{j=1}^{5} a_{ji}e_j, \quad i = 1, \ldots, 5. \tag{50}$$

The condition $A \cdot g = 0$ means that the matrix a_{ij} is skew symmetric, and the relation $A \cdot T = 0$ means that

$$A(T(e_i, e_j)) = T(Ae_i, e_j) + T(e_i, Ae_j), \quad i, j, = 1, \ldots, 5. \tag{51}$$

The Lie algebra $\mathfrak{k}$ is generated by the following basic endomorphisms $A_{ij} \in \mathrm{End}(V), i \neq j$:

$$A_{ij}e_i = e_j,\ A_{ij}e_j = -e_i,\ A_{ij}e_k = 0,\ i, j \neq k, i, j, k = 1, \ldots, 5, \tag{52}$$

where the basis $\{e_i\} \subset V$ is orthonormal.

Now, we can formulate the following

THEOREM 3. *Taking into account successively the formulas* (14)*a*–(14)*e* *we receive that the sought Lie algebra* $\mathfrak{k}$ *of the derivations has the following forms*:

$$\left\{\begin{array}{l}
\text{a)}\left\{\begin{array}{l} \mathfrak{k} = \mathrm{Span}(A_{45}),\ \textit{for}\ \lambda_1 \neq \lambda_2, \\ \textit{or} \\ \mathfrak{k} = \mathrm{Span}(A_{12}, A_{45}),\ \textit{for}\ \lambda_1 = \lambda_2; \end{array}\right. \\
\text{b)}\ \ \mathfrak{k} = \mathrm{Span}(A_{45}); \\
\text{c)}\left\{\begin{array}{l} \mathfrak{k} = \{0\},\ \textit{for}\ \alpha_1, \alpha_2, \alpha_3 \neq,\ \textit{(all different}\ \alpha_i, i = 1,2,3), \\ \textit{or} \\ \mathfrak{k} = \mathrm{Span}(A_{12}),\ \textit{for}\ \alpha_1 = \alpha_2, \alpha_3 \neq; \end{array}\right. \\
\text{d)}\ \ \mathfrak{k} = \{0\}; \\
\text{e)}\left\{\begin{array}{l} \mathfrak{k} = \{0\}, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \neq, \\ \textit{or} \\ \mathfrak{k} = \mathrm{Span}(A_{12}),\ \textit{for}\ \alpha_1 = \alpha_2, \alpha_3, \alpha_4 \neq, \\ \textit{or} \\ \mathfrak{k} = \mathrm{Span}(A_{12}, A_{34}),\ \textit{for}\ \alpha_1 = \alpha_2, \alpha_3 = \alpha_4, \alpha_2 \neq \alpha_3, \\ \textit{or} \\ \mathfrak{k} = \mathrm{Span}(A_{12}, A_{13}, A_{23}),\ \textit{for}\ \alpha_1 = \alpha_2 = \alpha_3, \alpha_4 \neq\ . \end{array}\right.
\end{array}\right. \tag{53}$$

Proof. The reality of Theorem 3 follows immediately from the formulas (50)–(53), and (14)a–(14)e, after realizing the elementary calculations; we shall present here the proof only for the subcase a).

Thus, we have to deal with the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$ (orthogonal direkt sum of two ideals), where $\mathfrak{h} = \mathrm{Span}(e_1, e_2, e_3) = \mathfrak{sl}(2R)$, and $\mathfrak{k}^{\perp} = \mathrm{Span}(e_4, e_5) = R^2$, are represented by the Lie brackets:

$$(54)\qquad \begin{cases} [e_1, e_2] = \lambda_3 e_3, \quad [e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \\ [e_i, e_j] = 0 \ \text{ otherwise}, \ \lambda_1, \lambda_2 > 0, \lambda_3 = \lambda_1 - \lambda_2. \end{cases}$$

Applying the formula (51) successively to the Lie brackets $[e_1, e_2], [e_1, e_3]$, $[e_2, e_3]$ we obtain from (54), and (54) the following expressions:

$$(55)\qquad \begin{cases} (\lambda_i - \lambda_j) a_{ij} = 0, \ \ i, j = 1, 2, 3, \\ a_{45} \ \text{ is an arbitrary parametr}, \ \ a_{kl} = 0 \ \text{ otherwise}. \end{cases}$$

The remaining Lie brackets $[e_i, e_j]$ of type (54) does not give other conditions on a_{ij}.

From the formulas (54)–(55) we get for $\lambda_1 \neq \lambda_2$, and $\lambda_1 = \lambda_2$ successively the results $\mathfrak{k} = \mathrm{Span}(A_{ns})$, $\mathfrak{k} = \mathrm{Span}(A_{12}, A_{45})$, as expected in the subcase (53)a.

4. An application of the results of the previous sections

We shall apply the results obtained in the two previous sections to the classification of the five-dimensional Riemannian manifolds (M, g) admitting a homogenesous structure T of class T_2 in the case (C) of (xvi).

Let us consider the quadruplet (V, g, T, R), where the vector space V has the dimension $\dim V = 5, g$ is a positive definite scalar product on $V, \{e_1, e_2, e_3, e_4, e_5\}$ is the orthonormal basis of V, and $T \neq 0, R$ are the tensor fields of type $(1,2), (1,3)$, respectively, such that all the conditions (vii)–(xiii) are satisfied.

Considering the case (C), (xvi) we shall distinguish separately the two following subcases $(C_1), (C_2)$:

(56)
- (C_1) There is a 3-dimensional $\mathfrak{k}$-irreducible subspace $\mathrm{Span}(e_1, e_2, e_3) \subset V$, and $\mathfrak{k}$ acts trivially on the subspace $\mathrm{Span}(e_4, e_5) \subset V$;
- (C_2) There is a 3-dimensional $\mathfrak{k}$-irreducible subspace $\mathrm{Span}(e_1, e_2, e_3) \subset V$, and $\mathfrak{k}$ acts non-trivialy on the subspace $\mathrm{Span}(e_4, e_5) \subset V$.

In the subcase (C_1) the Lie algebra $\mathfrak{k}$ has the following form (see (52)):

$$\mathfrak{k} = \mathrm{Span}(A_{12}, A_{23}, A_{31}), \tag{57}$$

At first, we recall that for the tensor algebra $\mathcal{T}(V)$ the relation $A \cdot T = 0, A \subset \mathfrak{k}$ is equivalent to the following

$$A(T_X Y) = T_{AX} Y + T_X AY, \; X, Y \subset V, \quad A \subset \mathfrak{k}. \tag{58}$$

Acting by $A = A_{12}, A_{23}, A_{31}$ successively on (58), and taking into account (8), we get after routine but tedious calculations, the following form of the tensor $T_{e_i} e_j$:

$$\begin{cases} T_{e_1} e_2 = 0, \; T_{e_1} e_3 = 0, \; T_{e_2} e_3 = 0, \\ T_{e_1} e_4 = a e_1, \; T_{e_2} e_4 = a e_2, \; T_{e_3} e_4 = a e_3, \\ T_{e_1} e_5 = b e_1, \; T_{e_2} e_5 = b e_2, \; T_{e_3} e_5 = b e_3, \\ T_{e_4} e_5 = -3b e_4 + 3a e_5, \quad a, b \in R. \end{cases} \tag{59}$$

Omitting the trivial homogeneous structure $T = 0$ of class $\mathcal{T}_2$, we suppose in the sequel that $a^2 + b^2 > 0$.

Now, we introduce the new orthonormal basis $\{e_1', e_2', e_3', e_4', e_5'\}$ on V:

$$\begin{cases} e_i' = e_i, \quad i = 1, 2, 3, \; \rho = \sqrt{a^2 + b^2}, \\ e_4' = \frac{1}{\rho}(a e_4 + b e_5), \; e_5' = \frac{1}{\rho}(-b e_4 + a e_5). \end{cases} \tag{60}$$

Substituting (60) into (59) (and omitting the prim $'$) we get easily

$$\begin{cases} T_{e_1} e_4 = \rho e_1, \; T_{e_2} e_4 = \rho e_2, \; T_{e_3} e_4 = \rho e_3, \\ T_{e_4} e_5 = 3\rho e_5, \; T_{e_i} e_j = 0 \quad \text{otherwise}. \end{cases} \tag{61}$$

From (57) it follows that the curvature transformations have the following forms

$$\begin{cases} R_{e_i e_j} = a_{ij} A_{12} + b_{ij} A_{23} + c_{ij} A_{31}, \\ \text{where } a_{ij}, b_{ij}, c_{ij} \in R, \; i, j = 1, \ldots, 5. \end{cases} \tag{62}$$

Substituting (62) into the first and second reduced Bianchi identity (ix)-(x), we get after the elementary calculations that

$$a_{ij} = b_{ij} = c_{ij} = 0, \quad i, j = 1, \ldots, 5. \tag{63}$$

Hence, we receive

$$R_{e_i e_j} = 0, \quad i, j = 1, \ldots, 5, \text{ hence } \mathfrak{h} = \{0\} \subset \mathfrak{k}. \tag{64}$$

Now, applying the "Nomizu construction" (xv), on the basis of (61), and (62) we obtain the Lie algebra $\mathfrak{g} = V \oplus \mathfrak{k} = (V, -T)$ of the forms:

$$(65) \qquad \begin{cases} \mathfrak{g} = \mathrm{Span}(e_1, e_2, e_3, e_5) \oplus \mathrm{Span}(e_4) = R^4 \oplus R, \\ [e_1, e_4] = \rho e_1, \ [e_2, e_4] = \rho e_2, \ [e_3, e_4] = \rho e_3, \ \rho \neq 0, \\ [e_5, e_4] = -3\rho e_5, \ [e_i, e_j] = 0 \ \text{otherwise}. \end{cases}$$

The result presented by the formula (65) is exactly the result (14)e for $\alpha_1 = \alpha_2 = \alpha_3 = \rho, \alpha_4 = 3\rho, \rho \neq 0$. See the last result (53)e.

In the considered subcase [(C_1), formula (56)] we have proved the following

COROLLARY. *If in the quadruplet* (V, g, T, R) V *contains a 3-dimensional* $\mathfrak{k}$ — *irreducible subspace* $\mathrm{Span}(e_1, e_2, e_3)$, *and* $\mathfrak{k}$ *acts trivialy on the subspace* $\mathrm{Span}(e_4, e_5)$, *then the non-trivial homogeneous structure* T *of class* T_2 *determines the result* (14)e.

Repeating the above considerations and calculations for the subcase [(C_2), (56)], where $\mathfrak{k} = \mathrm{Span}(A_{12}, A_{23}, A_{31}, A_{45})$, then the Riemannian manifold (M, g) has only the trivial homogeneous structure $T = 0$ of class T_2.

References

[1] W. Ambrose and I. M. Singer, *On homogeneous Riemannian manifolds*, Duke Math. J. 25 (1958), 647–699.

[2] A. O. Barut and R. Rączka, *Theory of Group Representations and Applications*, Vol. 1, Polish Scientific Publishers, Warszawa, 1977.

[3] F. Brickell, R. Clark, *Differential Manifolds, An Introduction*, Van Nostrand Reinhold Company, London, 1970.

[4] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, Inc., New York, 1978.

[5] O. Kowalski, *Classification of generalized symmetric Riemannian spaces of dimension* $n \leq 5$, Rozpravy Českoslowenské Akad. Věd Řada Mat. 85 (8) (1975), 1–61.

[6] O. Kowalski and F. Tricerri, *Riemannian Manifolds of Dimension* $n \leq 4$ *Admitting a Homogeneous Structure of Class* T_2, Conferenze del Seminario di Matematica del' Università di Bari, (1987), 1–24.

[7] M. Kucharzewski, *Einige Bemerkungen über die linearen homogenen geometrischen Objekte erster Klasse*, Ann. Polon. Math. 16 (1967), 1–12.

[8] J. Milnor, *Curvature of left invariant metrics on Lie groups*, Adv. in Math. 21 (1976), 293–329.

[9] S. Sternberg, *Lectures on Differential Geometry*, Prentice Hall, Inc. Englewood Cliffss, New York, 1964.

[10] F. Tricerri and L. Vanhecke, *Homogeneous Structure on Riemannian Manifolds*, London Math. Soc. Lecture Note Series 83, Cambridge University Press, (1983), 1–125.

[11] W. Wojtyński, *Lie Groups and Lie Algebras*, Polish Scientific Publishers, Vol. 60, Warszawa, 1986.

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